Таким чином, коли позначити через $P_{\pm}$ проектори на $G_{\pm}$, то для фіксованої послідовності $\{u_n, n\in\mathbb{Z}\}\subset l_1(\mathbb{B})$ при виконанні умов наслідку, у формулах (6)–(8) матимемо $u_m^- = P_- u_m, u_m^+ = P_+ u_m$ для кожного $m\in\mathbb{Z}$. Тому із (6)–(8) випливає, що

$$\|x_1\| \le \sum_{\upsilon=-\infty}^{0}\left\|B_-^{|\upsilon|}\right\|\|P_-\|\|u_\upsilon\| + \sum_{\upsilon=1}^{\infty}\left\|A_+^{-\upsilon}\right\|\|P_+\|\|u_\upsilon\|, \quad (19)$$

$$\|x_n\| \le \sum_{k=0}^{n-1}\left\|A_-^{n-1-k}\right\|\|P_-\|\|u_k\| + \sum_{\upsilon=-\infty}^{-1}\left\|A_-^{n-1}\right\|\left\|B_-^{|\upsilon|}\right\|\|P_-\|\|u_\upsilon\| + \sum_{\upsilon=n}^{\infty}\left\|A_+^{n-1-\upsilon}\right\|\|P_+\|\|u_\upsilon\|, \quad n\ge 2, \quad (20)$$

$$\|x_n\| \le \sum_{\upsilon=-\infty}^{n-1}\left\|B_-^{|\upsilon|+n-1}\right\|\|P_-\|\|u_\upsilon\| + \sum_{\upsilon=n}^{0}\left\|B_+^{|\upsilon|+n-1}\right\|\|P_+\|\|u_\upsilon\| + \sum_{\upsilon=1}^{\infty}\left\|B_+^{n-1}\right\|\left\|A_+^{-\upsilon}\right\|\|P_+\|\|u_\upsilon\|, \quad n\le 0. \quad (21)$$

Внаслідок (19)–(21) і збіжності рядів (5) справджується включення $\{x_n, n\in\mathbb{Z}\}\in l_1(\mathbb{B})$. Отже, виконуються усі умови теореми 2. Наслідок доведено.

ВИСНОВКИ. Отримано необхідні та достатні умови існування єдиного сумовного розв'язку різницевого рівняння (1). Отримані результати узагальнюють і доповнюють для рівняння (1) твердження теореми 7 з [4, с. 25].

СПИСОК ВИКОРИСТАНИХ ДЖЕРЕЛ

1. Баскаков А. Г., Пастухов А. И. Спектральный анализ оператора взвешенного сдвига с неограниченными операторными коэффициентами // Сиб. мат. журнал. – 2001. – 42, №6. – С. 1231–1243.
2. Городній М. Ф. l_p -розв'язки одного різницевого рівняння в банаховому просторі // Укр. матем. журн. – 2003. – 55, №3. – С. 425–430.
3. Городний М. Ф. Ограниченные и периодические решения одного разностного уравнения и его стохастического аналога в банаховом пространстве // Укр. матем. журн. – 1991. – 43, №1. – С. 42–46.
4. Дороговцев А. Я. Периодические и стационарные режимы бесконечномерных детерминированных и стохастических динамических систем. – К.: Вища шк., 1992. – 319 с.
5. Ким В. С. Об условиях существования ограниченных решений разностного уравнения в банаховом пространстве // Диффер. уравнения. – 1967. – 3, №12. – С. 2151–2160.
6. Слюсарчук В. Е. Обратимость линейных неавтономных разностных операторов в пространстве ограниченных на $\mathbb{Z}$ функций // Мат. заметки. – 1985. – 37, вып. 5. – С. 662–666.
7. Хенри Д. Геометрическая теория полулинейных параболических уравнений. – М.: Мир, 1985. – 376 с.

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О СУММИРУЕМЫХ РЕШЕНИЯХ ОДНОГО РАЗНОСТНОГО УРАВНЕНИЯ

Исследуется вопрос о существовании единственного суммируемого решения одного разностного уравнения с переменным операторным коэффициентом в банаховом пространстве.

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ON THE SUMMABLE SOLUTIONS OF ONE DIFFERENCE EQUATION

We study the problem of existence of the unique summable solution of the difference equation with variable operator coefficient in Banach space.

УДК 517.946

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THE INTEGRAL REPRESENTATION OF THE SOLUTION OF BOUNDARY VALUE PROBLEM FOR SYSTEM OF EVOLUTIONARY PARABOLIC EQUATIONS IN PIECE-HOMOGENEOUS POLAR AXIS WITH SOFT LIMITS

By means of method of Laplace integral transform in combination with the method of Cauchy functions the integral representation of exact analytical solution of mixed problem for the system of evolutionary parabolic equations modeled by hybrid differential operator of Bessel-Euler-(Kontorovich-Lebedev) in the piece-homogeneous polar axis $r\ge 0$ with soft limits is obtained.

INTRODUCTION. Parabolic boundary value problems in homogeneous environments (simply connected domains) make up a significant theoretical and practical interest because they are used in the mathematical modeling of various processes and phenomena of natural science, technique, various technologies [1, 15].

In the last decades much attention is given to the investigation of parabolic boundary value problems in nonhomogeneous media [3, 16]. In this case, the coefficients of equations are piecewise continuous or, in particular, piecewise constant. For these problems it is difficult to apply classical method of separation of variables [18]. But for rather wide class of similar problems in the construction of exact solutions the method of integral or hybrid integral transforms [2, 6, 7, 11] can be effectively used.

In theoretical studies and applied problems most often the differential operators of 2nd order are used, in particular it is Fourier differential operator [17]

$$F = \frac{d^2}{dr^2},$$

Euler differential operator [10]

$$B_\alpha^* = r^2 \frac{d^2}{dr^2} + (2\alpha+1) r \frac{d}{dr} + \alpha^2 ,$$

Bessel differential operator [13]

$$B_{\nu,\alpha} = \frac{d^2}{dr^2} + \frac{2\alpha+1}{r}\frac{d}{dr} - \frac{\nu^2-\alpha^2}{r^2} ,$$

Legendre differential operator [5]

$$\Lambda_{(\mu)} = \frac{d^2}{dr^2} + cthr \frac{d}{dr} + \frac{1}{4} - \frac{\mu^2}{sh^2 r},$$

generalized Legendre differential operator [5]

$$\Lambda_{(\mu)} = \frac{d^2}{dr^2} + cthr \frac{d}{dr} + \frac{1}{4} + \frac{1}{2}\left(\frac{\mu_1^2}{1-chr} + \frac{\mu_2^2}{1+chr}\right), \quad \mu = (\mu_1, \mu_2)$$

and Kontorovich-Lebedev differential operator [12]

$$B_\alpha = r^2 \frac{d^2}{dr^2} + (2\alpha+1) r \frac{d}{dr} + \alpha^2 - \lambda^2 r^2 .$$

If $\theta(x)$ is the Heaviside step function [19], and L_j is one of listed differential operators, then we can always create the hybrid differential operator that corresponds to the geometric structure of piecewise homogeneous domain.

For example, for the piecewise homogeneous interval $(R_0, R_1) \cup (R_1, R_2) \cup (R_2, R)$ it is possible to create hybrid differential operator

$$M = \theta(r-R_0)\theta(R_1-r)a_1^2 L_1 + \theta(r-R_1)\theta(R_2-r)a_2^2 L_2 + \theta(r-R_2)\theta(R-r)a_3^2 L_3 ; \; a_j^2 = const .$$

It is obvious that operator L_1 is defined in the interval (R_0, R_1), operator L_2 is defined in the interval (R_1, R_2), and operator L_3 is defined in the interval (R_2, R).

It is clear, that if we change the order of operators L_1, L_2, L_3 we get other hybrid differential operator.

We propose in this paper the exact solution of the initial-boundary value problem for a system of evolution equations of parabolic type, modeled by hybrid differential operator of Bessel-Euler-(Kontorovich-Lebedev) on piecewise homogeneous polar axis $(0, R_1) \cup (R_1, R_2) \cup (R_2, +\infty)$ with soft limits.

FORMULATION OF THE PROBLEM. We consider the problem of the structure of bounded in the set $D_2^+ = \left\{(t,r) : t \in (0;+\infty); \;\; r \in I_2^+ = (0, R_1) \cup (R_1, R_2) \cup (R_2, +\infty)\right\}$ solution for separate system of partial differential equations of the parabolic type of 2nd order

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + \gamma_1^2 u_1(t,r) - a_1^2 B_{\upsilon,\alpha_1}[u_1(t,r)] = f_1(t,r), \quad r \in (0, R_1) , \\ &\frac{\partial u_2}{\partial t} + \gamma_2^2 u_2(t,r) - a_2^2 B_{\alpha_2}^*[u_2(t,r)] = f_2(t,r), \quad r \in (R_1, R_2) , \\ &\frac{\partial u_3}{\partial t} + \gamma_3^2 u_3(t,r) - a_3^2 B_{\alpha_3}[u_3(t,r)] = f_3(t,r), \quad r \in (R_2, +\infty) , \end{aligned} \tag{1}$$

with initial conditions

$$\begin{gathered} u_1(t,r)\big|_{t=0} = g_1(r), \quad r \in (0, R_1); \qquad u_2(t,r)\big|_{t=0} = g_2(r), \quad r \in (R_1, R_2) ; \\ u_3(t,r)\big|_{t=0} = g_3(r), \quad r \in (R_2, \infty) , \end{gathered} \tag{2}$$

boundary conditions

$$\lim_{r\to 0} \frac{\partial^k u_1(t,r)}{\partial r^k} = 0; \quad \lim_{r\to +\infty} \frac{\partial^k u_3(t,r)}{\partial r^k} = 0; \quad k = 0,1 \tag{3}$$

and conjugate conditions

$$\left(L_{j1}^k\left[u_k(t,r)\right] - L_{j2}^k\left[u_{k+1}(t,r)\right]\right)\Big|_{r=R_k} = \omega_{jk}(t); \quad j,k = 1,2 , \tag{4}$$

here $B_{\upsilon,\alpha_1} = \frac{\partial^2}{\partial r^2} + \frac{2\alpha_1+1}{r}\frac{\partial}{\partial r} - \frac{\upsilon^2-\alpha_1^2}{r^2}$ is Bessel differential operator, $B_{\alpha_2}^* = r^2 \frac{\partial^2}{\partial r^2} + (2\alpha_2+1) r \frac{\partial}{\partial r} + \alpha_2^2$ is Euler differential operator, $B_{\alpha_3} = r^2 \frac{\partial^2}{\partial r^2} + (2\alpha_3+1) r \frac{\partial}{\partial r} + \alpha_3^2 - \lambda^2 r^2$ is Kontorovich-Lebedev differential operator.

In equations (4) there are the differential operators

$$L_{jm}^k = \left(\alpha_{jm}^k + \delta_{jm}^k \frac{\partial}{\partial t}\right)\frac{\partial}{\partial r} + \beta_{jm}^k + \gamma_{jm}^k \frac{\partial}{\partial t}; \quad j,m,k = 1,2.$$

We assume that conditions on the coefficients are true: $\alpha_{jm}^k \ge 0$, $\delta_{jm}^k \ge 0$, $\beta_{jm}^k \ge 0$, $\gamma_{jm}^k \ge 0$; $c_{j1,k} = \alpha_{2j}^k \beta_{1j}^k - \alpha_{1j}^k \beta_{2j}^k$; $c_{11,k} \cdot c_{21,k} > 0$; $c_{j2,k} \equiv \delta_{2j}^k \gamma_{1j}^k - \delta_{1j}^k \gamma_{2j}^k = 0$, $\alpha_{11}^k \gamma_{21}^k - \alpha_{21}^k \gamma_{11}^k = \beta_{11}^k \delta_{21}^k - \beta_{21}^k \delta_{11}^k$, $\alpha_{12}^k \gamma_{22}^k - \alpha_{22}^k \gamma_{12}^k = \beta_{12}^k \delta_{22}^k - \beta_{22}^k \delta_{12}^k$, $j,k = 1,2$.

Remark 1. The presence of the differentiation operator in time in the conjugate conditions we interpret based on physical reasons about heat waves as softness of boundary of domain on reflection of waves.

Remark 2. In the case of hard boundary of domain on reflection of waves ($\delta_{jm}^k = \gamma_{jm}^k = 0$), we have mixed problem with classic conjugate conditions, solution of which is obtained from the solution of the problem (1)–(4) as a particular case.

THE MAIN PART. To solve the problem (1)–(4) let's attract integral Laplace transform relative to the variable t on the assumption that the given functions $f_j(t,r)$ $(j=\overline{1,3})$, $\omega_{jk}(t)$ $(j,k=1,2)$ and the required function $u(t,r)=\{u_1(t,r);u_2(t,r);u_3(t,r)\}$ are the originals by Laplace [9].

In the Laplace image for the problem (1)–(4) it is corresponds the boundary value problem to construct solution which is bounded in the set I_2^+ for the separate system of differential equations of Bessel, Euler and Kontorovich-Lebedev for modified functions

$$
\begin{aligned}
&(B_{\upsilon,\alpha_1} - q_1^2)u_1^*(p,r) = -F_1^*(p,r), \qquad r\in(0,R_1),\\
&(B_{\alpha_2}^* - q_2^2)u_2^*(p,r) = -F_2^*(p,r), \qquad r\in(R_1,R_2),\\
&(B_{\alpha_3} - q_3^2)u_3^*(p,r) = -F_3^*(p,r), \qquad r\in(R_2,+\infty)
\end{aligned}
\tag{5}
$$

with boundary conditions

$$
\lim_{r\to 0}\frac{d^k u_1^*(p,r)}{dr^k}=0;\quad \lim_{r\to+\infty}\frac{d^k u_3^*(p,r)}{dr^k}=0;\quad k=0,1 \tag{6}
$$

and conjugate conditions

$$
\left[\left(\overline{\alpha}_{j1}^k\frac{d}{dr}+\overline{\beta}_{j1}^k\right)u_k^*(p,r)-\left(\overline{\alpha}_{j2}^k\frac{d}{dr}+\overline{\beta}_{j2}^k\right)u_{k+1}^*(p,r)\right]\Bigg|_{r=R_k}=\overline{\omega}_{jk}^*(p);\quad j,k=1,2. \tag{7}
$$

In the equalities (5)–(6) we accept the denotation:

$q_j^2 = a_j^{-2}(p+\gamma_j^2)^{1/2}$, $F_j^*(p,r)=\left[f_j^*(p,r)+g_j(r)\right]a_j^{-2}$, $\overline{\alpha}_{jm}^k=\alpha_{jm}^k+p\delta_{jm}^k$, $\overline{\beta}_{jm}^k=\beta_{jm}^k+p\gamma_{jm}^k$,

$\overline{\omega}_{jk}^*(p)=\omega_{jk}^*(p)+\psi_{jk}$; $\psi_{jk}=[\delta_{j1}^k g_k'(R_k)+\gamma_{j1}^k g_k(R_k)]-[\delta_{j2}^k g_{k+1}'(R_k)+\gamma_{j2}^k g_{k+1}(R_k)]$; $u_j^*(p,r)=\int\limits_0^\infty u_j(t,r)e^{-pt}dt$,

$f_j^*(p,r)=\int\limits_0^\infty f_j(t,r)e^{-pt}dt$, $\omega_{jk}^*(p)=\int\limits_0^\infty \omega_{jk}(t)e^{-pt}dt$, $p=\sigma+is$, $\sigma>\sigma_0$,

here σ_0 is the abscissa of convergence of the Laplace integral $s\in(-\infty,+\infty)$.

We can assume that $\psi_{jk}=0,\ \ j,k=1,2$. Otherwise, let's proceed to the new initial data $\overline{g}_j(r)$ using formulas:

$$
g_1(r)=\overline{g}_1(r)+b_1,\ \ r\in(0,R_1),\qquad g_2(r)=\overline{g}_2(r)+a_2r+b_2,\ \ r\in(R_1,R_2),
$$
$$
g_3(r)=\overline{g}_3(r)+b_3,\ \ r\in(R_2,+\infty);\ \ a_2>0,\ \ b_j>0,\ \ j=\overline{1,3}.
$$

Let's find the values a_2 and b_j $(j=\overline{1,3})$ from the equalities

$$
\begin{aligned}
&\gamma_{j1}^1 b_1 - [(\delta_{j2}^1+\gamma_{j2}^1R_1)a_2+\gamma_{j2}^1 b_2]=\psi_{j1},\ \ j=1,2;\\
&(\delta_{j1}^2+\gamma_{j1}^2R_2)a_2+\gamma_{j1}^2 b_2-\gamma_{j2}^2 b_3=\psi_{j2},\ \ j=1,2.
\end{aligned}
\tag{8}
$$

We have got inhomogeneous algebraic linear system of four equations for four unknown quantities b_1, a_2, b_2, b_3. If the conditions on the coefficients are satisfied, than system (8) has a unique solution that can be obtained by Cramer rules [8]:

$$
\begin{aligned}
b_1 &= -\frac{\psi_{11}}{\Delta}[\gamma_{22}^1\overline{a}_{12}^2-\delta_{21}^1\overline{a}_{22}^2+\gamma_{22}^1\overline{a}_{22}^2(R_2-R_1)]+\frac{\psi_{21}}{\Delta}[\gamma_{12}^1\overline{a}_{12}^2-\delta_{12}^1\overline{a}_{22}^2+\gamma_{12}^1\overline{a}_{22}^2(R_2-R_1)],\\
a_2r+b_2 &= -\frac{\psi_{11}}{\Delta}\gamma_{21}^1[\overline{a}_{12}^2+\overline{a}_{22}^2(R_2-r)]+\frac{\psi_{21}}{\Delta}\gamma_{11}^1[\overline{a}_{12}^2+\overline{a}_{22}^2(R_2-r)]+\\
&\quad+\frac{\psi_{12}}{\Delta}\gamma_{22}^2[\overline{a}_{21}^1-\overline{a}_{22}^1(r-R_1)]-\frac{\psi_{22}}{\Delta}\gamma_{12}^2[\overline{a}_{21}^1-\overline{a}_{22}^1(r-R_1)],\\
b_3 &= \frac{\psi_{12}}{\Delta}[\overline{a}_{21}^1\gamma_{21}^2-\overline{a}_{22}^1\delta_{21}^2-\overline{a}_{22}^1\gamma_{21}^2(R_2-R_1)]+\frac{\psi_{22}}{\Delta}[\overline{a}_{22}^1\delta_{11}^2-\overline{a}_{21}^1\gamma_{11}^2+\overline{a}_{22}^1\gamma_{11}^2(R_2-R_1)],
\end{aligned}
\tag{9}
$$

here

$$
\overline{a}_{11}^k=\delta_{11}^k\delta_{22}^k-\delta_{21}^k\delta_{12}^k,\qquad \overline{a}_{21}^k=\gamma_{11}^k\delta_{22}^k-\gamma_{21}^k\delta_{12}^k,\ \ k=1,2;\ \overline{a}_{12}^k=\delta_{11}^k\gamma_{22}^k-\delta_{21}^k\gamma_{12}^k,\qquad \overline{a}_{22}^k=\gamma_{11}^k\gamma_{22}^k-\gamma_{21}^k\gamma_{12}^k,
$$
$$
\Delta=\overline{a}_{21}^1\overline{a}_{22}^2-\overline{a}_{22}^1\overline{a}_{12}^2-\overline{a}_{22}^1\overline{a}_{22}^2(R_2-R_1)\neq 0.
$$

Be note that the numbers ψ_{jk} appear as a result of soft lines of conjugation $r=R_k$ on wave reflection, such as heat which arise during distributing of the initial temperature.

The fundamental system of solutions for Bessel differential equation $(B_{\upsilon,\alpha_1}-q_1^2)v=0$ is formed by modified Bessel functions of the 1st kind $I_{\upsilon,\alpha_1}(q_1r)$ and the 2nd kind $K_{\upsilon,\alpha_1}(q_1r)$ [13]; the fundamental system of solutions for Euler differential equation

$(B^*_{\alpha_2}-q_2^2)v=0$ is formed by functions $r^{-\alpha_2-q_2}$ and $r^{-\alpha_2+q_2}$ [14]; the fundamental system of solutions for Kontorovich-Lebedev differential equation $(B_{\alpha_3}-q_3^2)v=0$ is formed by modified Bessel functions $I_{q_3,\alpha_3}(\lambda r)$ and $K_{q_3,\alpha_3}(\lambda r)$ [12].

The presence of fundamental system of solutions allows us to construct the solution of boundary value problem (5)–(7) by the method of Cauchy functions [17] by the formulas

$$u_1^*(p,r)=A_1I_{\upsilon,\alpha_1}(q_1r)+\int_0^{R_1}E_1^*(p,r,\rho)F_1^*(p,\rho)\rho^{2\alpha_1+1}d\rho,$$

$$u_2^*(p,r)=A_2r^{-\alpha_2-q_2}+B_2r^{-\alpha_2+q_2}+\int_{R_1}^{R_2}E_2^*(p,r,\rho)F_2^*(p,\rho)\rho^{2\alpha_2-1}d\rho, \tag{10}$$

$$u_3^*(p,r)=B_3K_{q_3,\alpha_3}(\lambda r)+\int_{R_2}^{\infty}E_3^*(p,r,\rho)F_3^*(p,\rho)\rho^{2\alpha_3-1}d\rho,$$

here $E_j^*(p,r,\rho)$ are the Cauchy functions:

$$E_j^*(p,r,\rho)\big|_{r=\rho+0}-E_j^*(p,r,\rho)\big|_{r=\rho-0}=0,\quad j=\overline{1,3}, \tag{11}$$

$$\frac{dE_j^*(p,r,\rho)}{dr}\bigg|_{r=\rho+0}-\frac{dE_j^*(p,r,\rho)}{dr}\bigg|_{r=\rho-0}=-\rho^{-(2\alpha_j+1)},\quad 2\alpha_j+1>0.$$

Let's suppose that the Cauchy function $E_1^*(p,r,\rho)$ is defined by the formula:

$$E_1^*(p,r,\rho)=\begin{cases}\overset{-}{E}{}_1^*\equiv C_1I_{\upsilon,\alpha_1}(q_1r) & ,\quad 0<r<\rho<R_1,\\ \overset{+}{E}{}_1^*\equiv C_2I_{\upsilon,\alpha_1}(q_1r)+D_2K_{\upsilon,\alpha_1}(q_1r), & 0<\rho<r<R_1.\end{cases}$$

Properties (11) of Cauchy function give us algebraic system of two equations:

$$(C_2-C_1)I_{\upsilon,\alpha_1}(q_1\rho)+D_2K_{\upsilon,\alpha_1}(q_1\rho)=0,$$

$$(C_2-C_1)I'_{\upsilon,\alpha_1}(q_1\rho)+D_2K'_{\upsilon,\alpha_1}(q_1\rho)=-(q_1\rho^{2\alpha_1+1})^{-1},$$

whence we obtain the ratio:

$$C_2-C_1=-q_1^{2\alpha_1}K_{\upsilon,\alpha_1}(q_1\rho),\quad D_2=q_1^{2\alpha_1}I_{\upsilon,\alpha_1}(q_1\rho). \tag{12}$$

Let's complement the equalities (12) by algebraic equation

$$\left(\bar{\alpha}_{11}^1\frac{d}{dr}+\bar{\beta}_{11}^1\right)\overset{+}{E}{}_1^*\Big|_{r=R_1}=0:\ U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)C_2+U^{12}_{\upsilon,\alpha_1;11}(q_1R_1)D_2=0. \tag{13}$$

From system of equations (12), (13) we find that

$$C_1=[U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)]^{-1}q_1^{2\alpha_1}\Psi^{1*}_{\upsilon,\alpha_1;11}(q_1R_1,q_1\rho),$$

$$\Psi^{1*}_{\upsilon,\alpha_1;11}(q_1R_1,q_1\rho)=U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)K_{\upsilon,\alpha_1}(q_1\rho)-U^{12}_{\upsilon,\alpha_1;11}(q_1R_1)I_{\upsilon,\alpha_1}(q_1\rho),$$

$$U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)=\left(\frac{\upsilon-\alpha_1}{R_1}\bar{\alpha}_{11}^1+\bar{\beta}_{11}^1\right)I_{\upsilon,\alpha_1}(q_1R_1)+\bar{\alpha}_{11}^1q_1^2R_1I_{\upsilon+1,\alpha_1+1}(q_1R_1),$$

$$U^{12}_{\upsilon,\alpha_1;11}(q_1R_1)=\left(\frac{\upsilon-\alpha_1}{R_1}\bar{\alpha}_{11}^1+\bar{\beta}_{11}^1\right)K_{\upsilon,\alpha_1}(q_1R_1)-\bar{\alpha}_{11}^1q_1^2R_1K_{\upsilon+1,\alpha_1+1}(q_1R_1).$$

So Cauchy function $E_1^*(p,r,\rho)$ is defined and due to symmetry about the diagonal $r=\rho$ has the structure:

$$E_1^*(p,r,\rho)=\frac{q_1^{2\alpha_1}}{U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)}\begin{cases}I_{\upsilon,\alpha_1}(q_1r)\Psi^{1*}_{\upsilon,\alpha_1;11}(q_1R_1,q_1\rho), & 0<r<\rho<R_1,\\ I_{\upsilon,\alpha_1}(q_1\rho)\Psi^{1*}_{\upsilon,\alpha_1;11}(q_1R_1,q_1r), & 0<\rho<r<R_1.\end{cases} \tag{14}$$

Let Cauchy function $E_2^*(p,r,\rho)$ is defined by the formula:

$$E_2^*(p,r,\rho)=\begin{cases}\overset{-}{E}{}_2^*\equiv C_1r^{-\alpha_2-q_2}+D_1r^{-\alpha_2+q_2}, & R_1<r<\rho<R_2,\\ \overset{+}{E}{}_2^*\equiv C_2r^{-\alpha_2-q_2}+D_2r^{-\alpha_2+q_2}, & R_1<\rho<r<R_2.\end{cases}$$

Properties (11) of Cauchy function give us algebraic system of equations

$$(C_2-C_1)\rho^{-\alpha_2-q_2}+(D_2-D_1)\rho^{-\alpha_2+q_2}=0,$$

$$(C_2-C_1)(\alpha_2+q_2)\rho^{-\alpha_2-q_2}+(D_2-D_1)(\alpha_2-q_2)\rho^{-\alpha_2+q_2}=\rho^{-2\alpha_2},$$

whence we obtain the ratio

$$C_2-C_1=(2q_2)^{-1}\rho^{-\alpha_2+q_2},\qquad D_2-D_1=-(2q_2)^{-1}\rho^{-\alpha_2-q_2}. \tag{15}$$

Let's complement the equalities (15) by algebraic equations

$$\left(\bar\alpha_{12}^1\frac{d}{dr}+\bar\beta_{12}^1\right)\overset{-}{E}{}_2^*\Big|_{r=R_1}=0:\ \ Z_{\alpha_2;12}^{11}(q_2,R_1)C_1+Z_{\alpha_2;12}^{12}(q_2,R_1)D_1=0\,, \tag{16}$$

$$\left(\bar\alpha_{11}^2\frac{d}{dr}+\bar\beta_{11}^2\right)\overset{+}{E}{}_2^*\Big|_{r=R_2}=0:\ \ Z_{\alpha_2;11}^{21}(q_2,R_2)C_2+Z_{\alpha_2;11}^{22}(q_2,R_2)D_2=0\,.$$

Because of (15) algebraic system (16) takes the form

$$Z_{\alpha_2;12}^{11}(q_2,R_1)C_1+Z_{\alpha_2;12}^{12}(q_2,R_1)D_1=0\,, \tag{17}$$

$$Z_{\alpha_2;11}^{21}(q_2,R_2)C_1+Z_{\alpha_2;11}^{22}(q_2,R_2)D_1=\frac{1}{2q_2}\Psi_{\alpha_2;11}^{2*}(q_2,\rho)\,.$$

Hence we find that

$$C_1=-\frac{Z_{\alpha_2;12}^{12}(q_2,R_1)\Psi_{\alpha_2;11}^{2*}(q_2,\rho)}{2q_2\Delta_{\alpha_2;11}(q_2;R_1,R_2)},\ \ D_1=\frac{Z_{\alpha_2;12}^{11}(q_2,R_1)\Psi_{\alpha_2;11}^{2*}(q_2,\rho)}{2q_2\Delta_{\alpha_2;11}(q_2;R_1,R_2)}\,.$$

So Cauchy function $E_2^*(p,r,\rho)$ is defined and due to symmetry about the diagonal $r=\rho$ has the structure:

$$E_2^*(p,r,\rho)=-\frac{1}{2q_2\Delta_{\alpha_2;11}(q_2;R_1,R_2)}\begin{cases}\Psi_{\alpha_2;12}^{1*}(q_2,r)\Psi_{\alpha_2;11}^{2*}(q_2,\rho), & R_1<r<\rho<R_2,\\ \Psi_{\alpha_2;12}^{1*}(q_2,\rho)\Psi_{\alpha_2;11}^{2*}(q_2,r), & R_1<\rho<r<R_2.\end{cases} \tag{18}$$

In formulas (16)–(18) there are functions:

$$Z_{\alpha;jk}^{m1}(q,R_m)=[(\bar\beta_{jk}^m-\alpha R_m^{-1}\bar\alpha_{jk}^m)-\bar\alpha_{jk}^m qR_m^{-1}]R_m^{-\alpha-q}\,,\ \ Z_{\alpha;jk}^{m2}(q,R_m)=[(\bar\beta_{jk}^m-\alpha R_m^{-1}\bar\alpha_{jk}^m)+\bar\alpha_{jk}^m qR_m^{-1}]R_m^{-\alpha+q}\,,$$

$$\Psi_{\alpha;jk}^{m*}(q,r)=Z_{\alpha;jk}^{m2}(q,R_m)r^{-\alpha_2-q}-Z_{\alpha;jk}^{m1}(q,R_m)r^{-\alpha_2+q},\qquad j,k=1,2\,,$$

$$\Delta_{\alpha_2;jk}(q_2;R_1,R_2)=Z_{\alpha_2;j2}^{11}(q_2,R_1)Z_{\alpha_2;k1}^{22}(q_2,R_2)-Z_{\alpha_2;j2}^{12}(q_2,R_1)Z_{\alpha_2;k1}^{21}(q_2,R_2)\,.$$

Let's suppose that the Cauchy function $E_3^*(p,r,\rho)$ is defined by the formula:

$$E_3^*(p,r,\rho)=\begin{cases}\overset{-}{E}{}_3^*\equiv C_1I_{q_3,\alpha_3}(\lambda r)+D_1K_{q_3,\alpha_3}(\lambda r), & R_2<r<\rho<\infty,\\ \overset{+}{E}{}_3^*\equiv \qquad\qquad\qquad D_2K_{q_3,\alpha_3}(\lambda r), & R_2<\rho<r<\infty.\end{cases}$$

Properties (11) of Cauchy function give us algebraic system of equations

$$-C_1I_{q_3,\alpha_3}(\lambda\rho)+(D_2-D_1)K_{q_3,\alpha_3}(\lambda\rho)=0\,,$$

$$-C_1I'_{q_3,\alpha_3}(\lambda\rho)+(D_2-D_1)K'_{q_3,\alpha_3}(\lambda\rho)=-(\lambda\rho^{2\alpha_3+1})^{-1}\,,$$

whence we obtain the ratio:

$$C_1=\lambda^{2\alpha_3}K_{q_3,\alpha_3}(\lambda\rho),\ \ D_2-D_1=\lambda^{2\alpha_3}I_{q_3,\alpha_3}(\lambda\rho)\,. \tag{19}$$

Let's complement the equalities (19) by algebraic equation

$$\left(\bar\alpha_{12}^2\frac{d}{dr}+\bar\beta_{12}^2\right)\overset{-}{E}{}_3^*\Big|_{r=R_2}=0:\ \ U_{q_3,\alpha_3;12}^{12}(\lambda R_2)C_1+U_{q_3,\alpha_3;12}^{22}(\lambda R_2)D_1=0\,. \tag{20}$$

From algebraic system of equations (19), (20) we find that

$$D_2=-\left(U_{q_3,\alpha_3;12}^{22}(\lambda R_2)\right)^{-1}\lambda^{2\alpha_3}\Psi_{q_3,\alpha_3;12}^{2*}(\lambda R_2,\lambda\rho)\,.$$

So Cauchy function $E_3^*(p,r,\rho)$ is defined and due to symmetry about the diagonal $r=\rho$ has the structure:

$$E_3^*(p,r,\rho)=-\frac{\lambda^{2\alpha_3}}{U_{q_3,\alpha_3;12}^{22}(\lambda R_2)}\begin{cases}K_{q_3,\alpha_3}(\lambda\rho)\Psi_{q_3,\alpha_3;12}^{2*}(\lambda R_2,\lambda r), & R_2<r<\rho<+\infty,\\ K_{q_3,\alpha_3}(\lambda r)\Psi_{q_3,\alpha_3;12}^{2*}(\lambda R_2,\lambda\rho), & R_2<\rho<r<+\infty.\end{cases} \tag{21}$$

The conjugate conditions (7) to determine the values A_1,A_2,B_2,B_3 give us algebraic inhomogeneous system of four equations:

$$U_{\upsilon,\alpha_1;j1}^{11}(q_1R_1)A_1-Z_{\alpha_2;j2}^{11}(q_2,R_1)A_2-Z_{\alpha_2;j2}^{12}(q_2,R_1)B_2=\bar\omega_{j1}^*(p)+\delta_{j2}G_{12}^*,\ \ j=1,2\,,$$

$$Z_{\alpha_2;j1}^{21}(q_2,R_2)A_2+Z_{\alpha_2;j1}^{22}(q_2,R_2)B_2-U_{q_3,\alpha_3;j2}^{22}(\lambda R_2)B_3=\bar\omega_{j2}^*(p)+\delta_{j2}G_{23}^*\,. \tag{22}$$

In system (22) there are functions

$$G_{12}^*=\frac{c_{11}^*}{R_1^{2\alpha_1+1}}\int_0^{R_1}\frac{I_{\upsilon,\alpha_1}(q_1\rho)}{U_{\upsilon,\alpha_1;11}^{11}(q_1R_1)}F_1^*(p,\rho)\rho^{2\alpha_1+1}d\rho+\frac{c_{21}^*}{R_1^{2\alpha_2+1}}\int_{R_1}^{R_2}\frac{\Psi_{\alpha_2;11}^{2*}(q_2,\rho)}{\Delta_{\alpha_2;11}(q_2;R_1,R_2)}F_2^*(p,\rho)\rho^{2\alpha_2-1}d\rho\,,$$

$$G_{23}^*=-\frac{c_{12}^*}{R_2^{2\alpha_2+1}}\int_{R_1}^{R_2}\frac{\Psi_{\alpha_2;12}^{1*}(q_2,\rho)}{\Delta_{\alpha_2;11}(q_2;R_1,R_2)}F_2^*(p,\rho)\rho^{2\alpha_2-1}d\rho+\frac{c_{22}^*}{R_2^{2\alpha_3+1}}\int_{R_2}^{\infty}\frac{K_{q_3,\alpha_3}(\lambda\rho)}{U_{q_3,\alpha_3;12}^{22}(\lambda R_2)}F_3^*(p,\rho)\rho^{2\alpha_3-1}d\rho$$

and Kronecker symbol $\delta_{j2}(\delta_{12}=0,\delta_{22}=1)$.

Let us introduce functions for consideration:

$$A^{\alpha_2}_{\upsilon,\alpha_1;j}(p)=U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)\Delta_{\alpha_2;2j}(q_2;R_1,R_2)-U^{11}_{\upsilon,\alpha_1;21}(q_1R_1)\Delta_{\alpha_2;1j}(q_2;R_1,R_2);$$

$$B_{(\alpha);j}(p)=U^{22}_{q_3,\alpha_3;22}(\lambda R_2)\Delta_{\alpha_2;j1}(q_2;R_1,R_2)-U^{22}_{q_3,\alpha_3;12}(\lambda R_2)\Delta_{\alpha_2;j2}(q_2;R_1,R_2);$$

$$\Theta^{\alpha_2}_{\upsilon,\alpha_1;1}(p,r)=U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)\Psi^{1*}_{\alpha_2;22}(q_2,r)-U^{11}_{\upsilon,\alpha_1;21}(q_1R_1)\Psi^{1*}_{\alpha_2;12}(q_2,r),\quad j=1,2;$$

$$\Theta_{(\alpha);2}(p,r)=U^{22}_{q_3,\alpha_3;12}(\lambda R_2)\Psi^{2*}_{\alpha_2;21}(q_2,r)-U^{22}_{q_3,\alpha_3;22}(\lambda R_2)\Psi^{2*}_{\alpha_2;11}(q_2,r).$$

Let's assume that the condition of unique solvability of boundary value problem (5)-(7) is satisfied: for $p=\sigma+is$ with $\operatorname{Re}p=\sigma>\sigma_0$, here σ_0 is the abscissa of convergence of the Laplace integral and $\operatorname{Im}p=s\in(-\infty,+\infty)$, determinant of algebraic system (22) is different from zero:

$$\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)\equiv U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)B_{(\alpha);2}(p)-U^{11}_{\upsilon,\alpha_1;21}(q_1R_1)B_{(\alpha);1}(p)=$$
$$=U^{22}_{q_3,\alpha_3;22}(\lambda R_2)A^{\alpha_2}_{\upsilon,\alpha_1;1}(p)-U^{22}_{q_3,\alpha_3;12}(\lambda R_2)A^{\alpha_2}_{\upsilon,\alpha_1;2}(p)\neq 0,\quad (\alpha)=(\alpha_2,\alpha_3). \tag{23}$$

Let's define the main solutions of boundary value problem (5)–(7):

1) Green's functions generated by inhomogeneity of the conjugate conditions (7)

$$R^{(\alpha),1*}_{\upsilon,\alpha_1;11}(p,r)=\frac{B_{(\alpha);2}(p)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}I_{\upsilon,\alpha_1}(q_1r),\quad R^{(\alpha),1*}_{\upsilon,\alpha_1;21}(p,r)=-\frac{B_{(\alpha);1}(p)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}I_{\upsilon,\alpha_1}(q_1r),$$

$$R^{(\alpha),1*}_{\upsilon,\alpha_1;12}(p,r)=-\frac{2q_2c^*_{21}}{R_1^{2\alpha_2+1}}\frac{U^{22}_{q_3,\alpha_3;22}(\lambda R_2)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}I_{\upsilon,\alpha_1}(q_1r),\quad R^{(\alpha),1*}_{\upsilon,\alpha_1;22}(p,r)=\frac{2q_2c^*_{21}}{R_1^{2\alpha_2+1}}\frac{U^{22}_{q_3,\alpha_3;12}(\lambda R_2)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}I_{\upsilon,\alpha_1}(q_1r),$$

$$R^{(\alpha),2*}_{\upsilon,\alpha_1;11}(p,r)=-\frac{U^{11}_{\upsilon,\alpha_1;21}(q_1R_1)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\Theta_{(\alpha);2}(p,r),\quad R^{(\alpha),2*}_{\upsilon,\alpha_1;21}(p,r)=\frac{U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\Theta_{(\alpha);2}(p,r),$$

$$R^{(\alpha),2*}_{\upsilon,\alpha_1;12}(p,r)=-\frac{U^{22}_{q_3,\alpha_3;22}(\lambda R_2)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\Theta^{\alpha_2}_{\upsilon,\alpha_1;1}(p,r),\quad R^{(\alpha),2*}_{\upsilon,\alpha_1;22}(p,r)=\frac{U^{22}_{q_3,\alpha_3;12}(\lambda R_2)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\Theta^{\alpha_2}_{\upsilon,\alpha_1;1}(p,r),$$

$$R^{(\alpha),3*}_{\upsilon,\alpha_1;11}(p,r)=-\frac{2q_2c^*_{12}}{R_2^{2\alpha_2+1}}\frac{U^{11}_{\upsilon,\alpha_1;21}(q_1R_1)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}K_{q_3,\alpha_3}(\lambda r),\quad R^{(\alpha),3*}_{\upsilon,\alpha_1;21}(p,r)=\frac{2q_2c^*_{12}}{R_2^{2\alpha_2+1}}\frac{U^{11}_{\upsilon,\alpha_1;11}(q_1R_1)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}K_{q_3,\alpha_3}(\lambda r),$$

$$R^{(\alpha),3*}_{\upsilon,\alpha_1;12}(p,r)=\frac{A^{\alpha_2}_{\upsilon,\alpha_1;2}(p)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}K_{q_3,\alpha_3}(\lambda r),\quad R^{(\alpha),3*}_{\upsilon,\alpha_1;22}(p,r)=-\frac{A^{\alpha_2}_{\upsilon,\alpha_1;1}(p)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}K_{q_3,\alpha_3}(\lambda r);$$

2) the influence functions generated by the inhomogeneity of system (5)

$$H^{(\alpha)*}_{\upsilon,\alpha_1;11}(p,r,\rho)=\frac{q_1^{2\alpha_1}}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\begin{cases}I_{\upsilon,\alpha_1}(q_1r)[B_{(\alpha);2}(p)\Psi^{1*}_{\upsilon,\alpha_1;11}(q_1R_1,q_1\rho)-B_{(\alpha);1}(p)\Psi^{1*}_{\upsilon,\alpha_1;21}(q_1R_1,q_1\rho)], & 0<r<\rho<R_1\\ I_{\upsilon,\alpha_1}(q_1\rho)[B_{(\alpha);2}(p)\Psi^{1*}_{\upsilon,\alpha_1;11}(q_1R_1,q_1r)-B_{(\alpha);1}(p)\Psi^{1*}_{\upsilon,\alpha_1;21}(q_1R_1,q_1r)], & 0<\rho<r<R_1\end{cases},$$

$$H^{(\alpha)*}_{\upsilon,\alpha_1;12}(p,r,\rho)=\frac{c^*_{21}(p)}{R_1^{2\alpha_2+1}}\frac{I_{\upsilon,\alpha_1}(q_1r)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\Theta_{(\alpha);2}(p,\rho),\quad H^{(\alpha)*}_{\upsilon,\alpha_1;13}(p,r,\rho)=\frac{2q_2c^*_{21}}{R_1^{2\alpha_2+1}}\frac{c^*_{22}}{R_2^{2\alpha_3+1}}\frac{I_{\upsilon,\alpha_1}(q_1r)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}K_{q_3,\alpha_3}(\lambda\rho),$$

$$H^{(\alpha)*}_{\upsilon,\alpha_1;21}(p,r,\rho)=\frac{c^*_{11}(p)}{R_1^{2\alpha_1+1}}\frac{I_{\upsilon,\alpha_1}(q_1\rho)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\Theta_{(\alpha);2}(p,r),$$

$$H^{(\alpha)*}_{\upsilon,\alpha_1;22}(p,r,\rho)=\frac{1}{2q_2\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\begin{cases}\Theta^{\alpha_2}_{\upsilon,\alpha_1;1}(p,r)\Theta_{(\alpha);2}(p,\rho), & R_1<r<\rho<R_2,\\ \Theta^{\alpha_2}_{\upsilon,\alpha_1;1}(p,\rho)\Theta_{(\alpha);2}(p,r), & R_1<\rho<r<R_2,\end{cases}$$

$$H^{(\alpha)*}_{\upsilon,\alpha_1;23}(p,r,\rho)=\frac{c^*_{22}(p)}{R_2^{2\alpha_3+1}}\frac{\Theta^{\alpha_2}_{\upsilon,\alpha_1;1}(p,r)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}K_{q_3,\alpha_3}(\lambda\rho),\quad H^{(\alpha)*}_{\upsilon,\alpha_1;31}(p,r,\rho)=\frac{c^*_{11}(p)}{R_1^{2\alpha_1+1}}\frac{2q_2c^*_{12}}{R_2^{2\alpha_2+1}}\times$$

$$\times\frac{I_{\upsilon,\alpha_1}(q_1\rho)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}K_{q_3,\alpha_3}(\lambda r),\quad H^{(\alpha)*}_{\upsilon,\alpha_1;32}(p,r,\rho)=\frac{c^*_{12}}{R_2^{2\alpha_2+1}}\frac{K_{q_3,\alpha_3}(\lambda r)}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\Theta^{\alpha_2}_{\upsilon,\alpha_1;1}(p,\rho),$$

$$H^{(\alpha)*}_{\upsilon,\alpha_1;33}(p,r,\rho)=\frac{\lambda^{2\alpha_3}}{\Delta^{(\alpha)}_{\upsilon,\alpha_1}(p)}\begin{cases}K_{q_3,\alpha_3}(\lambda\rho)[A^{\alpha_2}_{\upsilon,\alpha_1;2}(p)\Psi^{2*}_{q_3,\alpha_3;12}(\lambda R_2,\lambda r)-A^{\alpha_2}_{\upsilon,\alpha_1;1}(p)\Psi^{2*}_{q_3,\alpha_3;22}(\lambda R_2,\lambda r)], & R_2<r<\rho<\infty\\ K_{\upsilon,\alpha_3}(q_3r)[A_{(\alpha);2}(p)\Psi^{2*}_{\upsilon,\alpha_3;12}(q_3R_2,q_3\rho)-A^{\alpha_2}_{\upsilon,\alpha_1;1}(p)\Psi^{2*}_{q_3,\alpha_3;22}(\lambda R_2,\lambda\rho)], & R_2<\rho<r<\infty\end{cases}.$$

As a result of unique of solvability of algebraic system (22), substituting the obtained values of variables A_1, A_2, B_2 and B_3 by Cramer rules in equations (10) and a number of elementary transformations, we have the unique solution of boundary value problem (5)–(7):

$$u_j^*(p,r) = \sum_{m,k=1}^{2} R^{(\alpha),j*}_{\upsilon,\alpha_1;mk}(p,r)\bar{\omega}^*_{mk}(p) + \int_0^{R_1} H^{(\alpha)*}_{\upsilon,\alpha_1;j1}(p,r,\rho)F_1^*(p,\rho)\rho^{2\alpha_1+1}d\rho +$$

$$+\int_{R_1}^{R_2} H^{(\alpha)*}_{\upsilon,\alpha_1;j2}(p,r,\rho)F_2^*(p,\rho)\rho^{2\alpha_2-1}d\rho + \int_{R_2}^{+\infty} H^{(\alpha)*}_{\upsilon,\alpha_1;j3}(p,r,\rho)F_3^*(p,\rho)\rho^{2\alpha_3-1}d\rho, \quad j=\overline{1,3}. \tag{24}$$

Returning to the original in formulas (24), we get the only exact solution of mixed parabolic conjugation problem (1)–(4):

$$u_j(t,r) = \sum_{m,k=1}^{2}\int_0^t R^{(\alpha),j}_{\upsilon,\alpha_1;mk}(t-\tau,r)\bar{\omega}_{mk}(\tau)d\tau + \int_0^t\int_0^{R_1} H^{(\alpha)}_{\upsilon,\alpha_1;j1}(t-\tau,r,\rho)a_1^{-2}[f_1(\tau,\rho)+g_1(\rho)\delta_+(\tau)]\rho^{2\alpha_1+1}d\rho d\tau +$$

$$+\int_0^t\int_{R_1}^{R_2} H^{(\alpha)}_{\upsilon,\alpha_1;j2}(t-\tau,r,\rho)[f_2(\tau,\rho)+g_2(\rho)\delta_+(\tau)]a_2^{-2}\rho^{2\alpha_2-1}d\rho d\tau + \int_0^t\int_{R_2}^{\infty} H^{(\alpha)}_{\upsilon,\alpha_1;j3}(t-\tau,r,\rho)[f_3(\tau,\rho)+g_3(\rho)\delta_+(\tau)]\times$$

$$\times a_3^{-2}\rho^{2\alpha_3-1}d\rho d\tau, \quad j=\overline{1,3}, \tag{25}$$

here $\delta_+(\tau)$ is Dirac delta function, that is focused in the point $\tau=+0$ [19].

In formulas (25) we apply the Green's functions

$$R^{(\alpha);j}_{\upsilon,\alpha_1;mk}(t,r) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} R^{(\alpha);j*}_{\upsilon,\alpha_1;mk}(p,r)e^{pt}dp, \quad m,k=1,2, \quad j=\overline{1,3} \tag{26}$$

and influence functions

$$H^{(\alpha)}_{\upsilon,\alpha_1;jk}(t,r,\rho) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} H^{(\alpha)*}_{\upsilon,\alpha_1;jk}(p,r,\rho)e^{pt}dp, \quad j,k=\overline{1,3}. \tag{27}$$

Let's find expressions for the Green's functions $R^{(\alpha);j}_{\upsilon,\alpha_1;mk}(t,r)$ and influence functions $H^{(\alpha)}_{\upsilon,\alpha_1;jk}(t,r,\rho)$ that are convenient for use in theoretical studies and applications.

Branch points $p=-\gamma_1^2$, $p=-\gamma_2^2$, $p=-\gamma_3^2$ and $p=\infty$ are the particular points of functions $R^{(\alpha);j*}_{\upsilon,\alpha_1;mk}(p,r)$ and $H^{(\alpha)*}_{\upsilon,\alpha_1;jk}(p,r,\rho)$. Let's put $q_j = ib_j \equiv ia_j^{-1}(\beta^2+k_j^2)^{1/2}$, $k_j^2 \ge 0$, $j=\overline{1,3}$. We obtain: $p+\gamma_j^2 = (\beta^2+k_j^2)$; $p=-(\beta^2+k_j^2+\gamma_j^2) = \; =-(\beta^2+\gamma^2)$, here $\gamma^2=\max\{\gamma_1^2;\gamma_2^2;\gamma_3^2\}$. If $\gamma^2=\gamma_1^2>0$, then $k_1^2=0$, $k_2^2=\gamma_1^2-\gamma_2^2\ge 0$, $k_3^2=\gamma_1^2-\gamma_3^2\ge 0$; if $\gamma^2=\gamma_2^2>0$, then $k_1^2=\gamma_2^2-\gamma_1^2\ge 0$, $k_2^2=0$, $k_3^2=\gamma_2^2-\gamma_3^2\ge 0$; if $\gamma^2=\gamma_3^2>0$, then $k_1^2=\gamma_3^2-\gamma_1^2\ge 0$, $k_2^2=\gamma_3^2-\gamma_2^2\ge 0$, $k_3^2=0$.

Let's use the known functional relations:

$$U^{11}_{\upsilon,\alpha_1;j1}(ib_1R_1) = u^{11}_{\upsilon,\alpha_1;j1}(b_1R_1)\exp[\frac{\pi i}{2}(\upsilon-\alpha_1)], \;\; U^{12}_{\upsilon,\alpha_1;j1}(ib_1R_1) = -\frac{\pi i}{2}\exp[-\frac{\pi i}{2}(\upsilon+\alpha_1)]\times$$

$$\times[u^{11}_{\upsilon,\alpha_1;j1}(b_1R_1) - iu^{12}_{\upsilon,\alpha_1;j1}(b_1R_1)]; \;\; \Psi^{1*}_{\upsilon,\alpha_1;jk}(ib_1R_1, ib_1r) = -\frac{\pi}{2}e^{-\pi i\alpha_1}\psi^1_{\upsilon,\alpha_1;jk}(b_1R_1,b_1r) =$$

$$= -\frac{\pi}{2}e^{-\pi i\alpha_1}[u^{11}_{\upsilon,\alpha_1;jk}(b_1R_1)N_{\upsilon,\alpha_1}(b_1r) - u^{12}_{\upsilon,\alpha_1;jk}(b_1R_1)J_{\upsilon,\alpha_1}(b_1r)];$$

$$\Delta_{\alpha_2;jk}(ib_2;R_1,R_2) = 2i\delta_{\alpha_2;jk}(b_2;R_1,R_2) \equiv 2i[Y^{11}_{\alpha_2;j2}(b_2,R_1)Y^{22}_{\alpha_2;k1}(b_2,R_2) - Y^{12}_{\alpha_2;j2}(b_2,R_1)Y^{21}_{\alpha_2;k1}(b_2,R_2)];$$

$$A^{\alpha_2}_{\upsilon,\alpha_1;j}\left(e^{\pi i}(\beta^2+\gamma^2)\right) = 2i\exp[\frac{\pi i}{2}(\upsilon-\alpha_1)][u^{11}_{\upsilon,\alpha_1;11}(b_1R_1)\delta_{\alpha_2;2j}(b_2;R_1,R_2) - u^{11}_{\upsilon,\alpha_1;21}(b_1R_1)\delta_{\alpha_2;1j}(b_2;R_1,R_2)] \equiv 2ie^{\frac{\pi i}{2}(\upsilon-\alpha_1)}a^{\alpha_2}_{\upsilon,\alpha_1;j}(\beta);$$

$$U^{22}_{ib_3,\alpha_3;j2}(\lambda R_2) = \frac{\pi}{sh(\pi b_3)}X^{22}_{\alpha_3;j2}(\lambda R_2,b_3) \equiv \frac{\pi}{sh(\pi b_3)}\left[\left(\tilde{\alpha}^2_{j2}\frac{d}{dr} + \tilde{\beta}^2_{j2}\right)D_{\alpha_2}(\lambda r,b_3)\Big|_{r=R_2}\right].$$

As a result of elementary calculations we get that

$$\Delta^{(\alpha)}_{\upsilon,\alpha_1}\left(e^{\pi i}(\beta^2+\gamma^2)\right) = \frac{2i\pi}{sh(\pi b_3)}e^{\frac{\pi i}{2}(\upsilon-\alpha_1)}[X^{22}_{\alpha_3;22}(\lambda R_2,b_3)a^{\alpha_2}_{\upsilon,\alpha_1;1}(\beta) - X^{22}_{\alpha_3;12}(\lambda R_2,b_3)a^{\alpha_2}_{\upsilon,\alpha_1;2}(\beta)] \equiv \frac{2\pi i}{sh(\pi b_3)}e^{\frac{\pi i}{2}(\upsilon-\alpha_1)}\delta^{(\alpha)}_{\upsilon,\alpha_1}(\beta).$$

If $\delta^{(\alpha)}_{\nu,\alpha_1}(\beta)\ne 0$, then functions $R^{(\alpha);j}_{\nu,\alpha_1;mk}(t,r)\equiv 0$ and $H^{(\alpha)}_{\nu,\alpha_1;jk}(t,r,\rho)\equiv 0$ according to formulas (26) and (27). Functions $u_j(t,r)\equiv 0$ according to formulas (25). So, the solution of the parabolic problem $u(t,r)=\{u_1(t,r);u_2(t,r);u_3(t,r)\}\equiv 0$, which is impossible. It follows that $\delta^{(\alpha)}_{\nu,\alpha_1}(\beta)=0$. We obtained the transcendental equation for calculating the eigenvalues of the hybrid differential operator $M^{(\alpha)}_{\nu,\alpha_1} = \theta(r)\theta(R_1-r)a_1^2B_{\nu,\alpha_1} + \theta(r-R_1)\theta(R_2-r)a_2^2B^*_{\alpha_2} + \theta(r-R_2a_3^2)B_{\alpha_3}$.

Let's determine values and functions:

$$a_1^2\sigma_1 = \frac{c_{11,1}c_{11,2}}{c_{21,1}c_{21,2}}\frac{R_1^{2\alpha_2+1}R_2^{2\alpha_3+1}}{R_1^{2\alpha_1+1}R_2^{2\alpha_2+1}},\ a_2^2\sigma_2 = \frac{c_{11,2}R_2^{2\alpha_3+1}}{c_{21,2}R_2^{2\alpha_2+1}},\ a_3^2\sigma_3 = 1,\ V^{(\alpha)}_{\nu,\alpha_1;1}(r,\beta_n) = q_{\alpha_2;1}(\beta_n)X^{22}_{\alpha_3;12}(\lambda R_2, b_{3n})J_{\nu,\alpha_1}(b_{1n}r),$$

$$q_{\alpha_2;1} = \frac{c_{21,1}b_{2n}}{R_1^{2\alpha_2+1}},\ V^{(\alpha)}_{\nu,\alpha_1;2}(r,\beta_n) = X^{22}_{\alpha_3;12}(\lambda R_2, b_{3n})\left[u^{11}_{\nu,\alpha_1;11}(b_{1n}R_1)\ \psi^1_{\alpha_2;22}(b_{2n},r) - u^{11}_{\nu,\alpha_1;21}(b_{1n}R_1)\psi^1_{\alpha_2;12}(b_{2n},r)\right];$$

$$\psi^1_{\alpha_2;j2}(b_{2n},r) = Y^{12}_{\alpha_2;j2}(b_{2n}R_1)r^{-\alpha_2}\cos(b_{2n}\ln r) - Y^{11}_{\alpha_2;j2}(b_{2n}R_1)r^{-\alpha_2}\sin(b_{2n}\ln r);\ V^{(\alpha)}_{\nu,\alpha_1;3}(r,\beta_n) = -a^{\alpha_2}_{\nu,\alpha_1;1}(\beta_n)D_{\alpha_3}(\lambda r, b_{3n}),$$

$$b_{jn} = a_j^{-1}(\beta_n^2 = k_j^2)^{1/2},\quad j=\overline{1,3}.$$

Let us introduce to consideration the weight function

$$\sigma(r) = \theta(r)\theta(R_1 - r)\sigma_1 r^{2\alpha_1+1} + \theta(r-R_1)\theta(R_2-r)\sigma_2 r^{2\alpha_2-1} + \theta(r-R_2)\sigma_3 r^{2\alpha_3-1},$$

spectral function $V^{(\alpha)}_{\nu,\alpha_1}(r,\beta_n) = \theta(r)\theta(R_1-r)V^{(\alpha)}_{\nu,\alpha_1;1}(r,\beta_n) + \theta(r-R_1)\theta(R_2-r)V^{(\alpha)}_{\nu,\alpha_1;2}(r,\beta_n) + \theta(r-R_2)V^{(\alpha)}_{\nu,\alpha_1;3}(r,\beta_n)$ and its generalized squared norm [4]

$$\left\|V^{(\alpha)}_{\nu,\alpha_1}(r,\beta_n)\right\|_1^2 \equiv \int_0^\infty \left[V^{(\alpha)}_{\nu,\alpha_1}(r,\beta_n)\right]^2\sigma(r)dr + \Theta_2(\beta_n,\beta_n) = \ell^{-\frac{\pi i}{2}(\nu-\alpha_1)}c^{-1}_{21,2}a^{\alpha_2}_{\nu,\alpha_1;1}(\beta_n)sh(\pi b_{3n})\times$$

$$\times X^{22}_{\alpha_3;12}(\lambda R_2, b_{3n})\left[-\frac{1}{2\pi i}\frac{d}{d\rho}\left(\Delta^{(\alpha)}_{\nu,\alpha_1}(\rho)\right)\right]\Bigg|_{\rho=-\left(\beta_m^2+\gamma^2\right)}.$$

According to the generalized theorem of the development [9] we have that

$$H^{(\alpha)}_{\nu,\alpha_1;jk}(t,r,\rho) = \sum_{n=1}^\infty \ell^{-(\beta_n^2+\gamma^2)t}\frac{V^{(\alpha)}_{\nu,\alpha_1;j}(r,\beta_n)V^{(\alpha)}_{\nu,\alpha_1;k}(\rho,\beta_n)}{\left\|V^{(\alpha)}_{\nu,\alpha_1}(r,\beta_n)\right\|_1^2}\sigma_k a_k^2;\quad j,k=\overline{1,3},$$

$$R^{(\alpha);j,k}_{\nu,\alpha_1;m2}(t,r) = d_k\sum_{n=1}^\infty \ell^{-(\beta_n^2+\gamma^2)t}\frac{Z^{(\alpha);k}_{\nu,\alpha_1;m2}(\beta_n)V^{(\alpha)}_{\nu,\alpha_1;j}(r,\beta_n)}{\left\|V^{(\alpha)}_{\nu,\alpha_1}(r,\beta_n)\right\|_1^2};\quad m,k=1,2;\ j=\overline{1,3},$$

here $d_k = a_k^2\sigma_k R_k^{2\alpha_k+1}\cdot c^{-1}_{11,k}$, $Z^{(\alpha);k}_{\nu,\alpha_1;m2}(\beta_n) = \left(\tilde{\alpha}^k_{m2}\frac{d}{dr} + \tilde{\beta}^k_{m2}\right)V^{(\alpha)}_{\nu,\alpha_1;k+1}(r,\beta_n)\Big|_{r=R_k}$; $m,k=1,2.$

Integral representation of solution of mixed parabolic problem of conjugation under consideration according to formulas (25) takes the form

$$u_j(t,r) = \sum_{k=1}^2\int_0^t\left[R^{(\alpha);j,k}_{\nu,\alpha_1;12}(t-\tau,r)\omega_{2k}(\tau) - R^{(\alpha);j,k}_{\nu,\alpha_1;22}(t-\tau,r)\omega_{1k}(\tau)\right]d\tau + \sum_{k=1}^2\left[R^{(\alpha);j,k}_{\nu,\alpha_1;12}(t,r)\psi_{2k} - R^{(\alpha);j,k}_{\nu,\alpha_1;22}(t,r)\psi_{1k}\right]+$$

$$+\int_0^t\int_0^{R_1}\left(\sum_{n=1}^\infty \ell^{-(\beta_n^2+\gamma^2)(t-\tau)}S^{(\alpha)}_{\nu,\alpha_1;j}(r,\beta_n)V^{(\alpha)}_{\nu,\alpha_1;1}(\rho,\beta_n)\right)\left[f_1(\tau,\rho)+g_1(\rho)\delta_+(\tau)\right]\sigma_1\rho^{2\alpha_1+1}d\rho d\tau +$$

$$+\int_0^t\int_{R_1}^{R_2}\left(\sum_{n=1}^\infty \ell^{-(\beta_n^2+\gamma^2)(t-\tau)}S^{(\alpha)}_{\nu,\alpha_1;j}(r,\beta_n)V^{(\alpha)}_{\nu,\alpha_1;2}(\rho,\beta_n)\right)\left[f_2(\tau,\rho)+g_2(\rho)\delta_+(\tau)\right]\sigma_2\rho^{2\alpha_2-1}d\rho d\tau +$$

$$+\int_0^t\int_{R_2}^{\infty}\left(\sum_{n=1}^\infty \ell^{-(\beta_n^2+\gamma^2)(t-\tau)}S^{(\alpha)}_{\nu,\alpha_1;j}(r,\beta_n)V^{(\alpha)}_{\nu,\alpha_1;3}(\rho,\beta_n)\right)\left[f_3(\tau,\rho)+g_3(\rho)\delta_+(\tau)\right]\sigma_3\rho^{2\alpha_3-1}d\rho d\tau,\quad j=\overline{1,3},$$

here $S^{(\alpha)}_{\nu,\alpha_1;j}(r,\beta_n) = V^{(\alpha)}_{\nu,\alpha_1;j}(r,\beta_n)\left(\left\|V^{(\alpha)}_{\nu,\alpha_1}(r,\beta_n)\right\|_1^2\right)^{-1}$, $j=\overline{1,3}.$

We get the following theorem as the summary of the above results.

Theorem. Let us suppose that the next conditions are true:

1) functions $f_j(t,r)$ and $\omega_{jk}(t)$ are originals by Laplace on variable t;

2) functions $f_j(t,r)$ and $g_j(r)$ satisfy the conjugate conditions;

3) functions $f(t,r) = \{f_1(t,r), f_2(t,r), f_3(t,r)\}$ and $g(r) = \{g_1(r), g_2(r), g_3(r)\}$ are bounded, continuous, absolutely summable with the weight function $\sigma(r)$ and have the bounded variation in the set I_2^+;

4) function $F(t,r) = \left\{\frac{\partial}{\partial r}B_{\nu,\alpha_1}[f_1(t,r)], \frac{\partial}{\partial r}B^*_{\alpha_2}[f_2(t,r)], \frac{\partial}{\partial r}B_{\alpha_3}[f_3(t,r)]\right\}$ is continuously differentiable by t and continuous by r in the set D_2^+. Then in the class of functions $u(t,r) = \{u_1(t,r), u_2(t,r), u_3(t,r)\}$, which are continuously differentiable by t and continuously differentiable by r twice in the set D_2^+ and satisfy conditions 1), 3), parabolic mixed problem of conjugation (1)–(4) has unique bounded solution, which is determined by formula (28).

CONCLUSIONS. By means of method of Laplace integral transform in combination with the method of Cauchy functions the integral representation of exact analytical solution of mixed problem of conjugation for the separate system of evolutionary parabolic equations of the second order, which is modelled by hybrid differential operator of Bessel-Euler-(Kontorovich-Lebedev) in the piece-homogeneous polar axis $(0,R_1)\cup(R_1,R_2)\cup(R_2,+\infty)$ with soft limits is obtained. Analytical expressions for main solutions (Green's functions and influence functions) that are convenient for use in theoretical studies and applications are also obtained.

REFERENCES

1. Городецький В. В. Граничні властивості гладких у шарі розв'язків рівнянь параболічного типу / В. В. Городецький. – Чернівці, 1998.
2. Громик А. П., Конет І. М., Ленюк М. П. Температурні поля в кусково-однорідних просторових середовищах. – Кам'янець-Подільський, 2011.
3. Дейнека В. С., Сергиенко И. В., Скопецкий В. В. Модели и методы решения задач с условиями сопряжения. – К., 1998.
4. Комаров Г. М., Ленюк М. П., Мороз В. В. Скінченні гібридні внтегральні перетворення, породжені диференціальними рівняннями другого порядку. – Чернівці, 2001.
5. Конет І. М., Ленюк М. П. Інтегральні перетворення типу Мелера-Фока. – Чернівці, 2002.
6. Конет І. М., Ленюк М. П. Стаціонарні та нестаціонарні температурні поля в циліндрично-кругових областях. – Чернівці, 2001.
7. Конет І. М., Ленюк М. П. Температурні поля в кусково-однорідних циліндричних областях. – Чернівці, 2004.
8. Курош А. Г. Курс высшей алгебры. – М., 1971.
9. Лаврентьев М. А., Шабат Б. В. Методы теории функций комплексного переменного. – М., 1987.
10. Ленюк М. П. Інтегральні перетворення, породжені диференціальним оператором Ейлера другого порядку. – Чернівці, 2012.
11. Ленюк М. П. Температурні поля в плоских кусково-однорідних ортотропних областях. – К., 1997.
12. Ленюк М. П., Міхалевська Г. І. Інтегральні перетворення типу Конторовича-Лєбєдєва. – Чернівці, 2002.
13. Ленюк М. П., Шинкарик М. І. Гібридні інтегральні перетворення (Фур'є. Бесселя, Лежандра). Частина 1. – Тернопіль, 2004.
14. Ленюк М. П., Шинкарик М. І. Гібридні інтегральні перетворення (Фур'є, Ейлера, Бесселя, Лежандра). Частина 2. – Тернопіль, 2011.
15. Матійчук М. І. Параболічні та еліптичні крайові задачі з особливостями. – Чернівці, 2003.
16. Сергиенко И. В., Скопецкий В. В., Дейнека В. С. Математическое моделирование и исследование процессов в неоднородных средах. – К., 1991.
17. Степанов В. В. Курс дифференциальных уравнений. – М., 1959.
18. Тихонов А. Н., Самарский А. А. Уравнения математической физики. – М., 1972.
19. Шилов Г. Е. Математический анализ. Второй специальный курс. – М., 1965.

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ІНТЕГРАЛЬНЕ ЗОБРАЖЕННЯ РОЗВ'ЯЗКУ КРАЙОВОЇ ЗАДАЧІ ДЛЯ СИСТЕМИ ЕВОЛЮЦІЙНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ НА КУСКОВО-ОДНОРІДНІЙ ПОЛЯРНІЙ ОСІ З М'ЯКИМИ МЕЖАМИ

Методом інтегрального перетворення Лапласа у поєднанні з методом функцій Коші одержано інтегральне зображення точного аналітичного розв'язку мішаної задачі для системи еволюційних параболічних рівнянь, змодельованих гібридним диференціальним оператором Бесселя-Ейлера-(Конторовича-Лєбєдєва) на кусково-однорідній полярній осі $r \geq 0$ з м'якими межами.

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ИНТЕГРАЛЬНОЕ ПРЕДСТАВЛЕНИЕ РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ ДЛЯ СИСТЕМЫ ЭВОЛЮЦИОННЫХ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ НА КУСОЧНО-ОДНОРОДНОЙ ПОЛЯРНОЙ ОСИ С МЯГКИМИ ГРАНИЦАМИ

Методом интегрального преобразования Лапласа в сочетании с методом функций Коши получено интегральное представление точного аналитического решения смешанной задачи для системы эволюционных параболических уравнений, смоделированных гибридным дифференциальным оператором Бесселя-Эйлера-(Конторовича-Лебедева) на кусочно-однородной полярной оси $r \geq 0$ с мягкими границами.

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УМОВНА ІНВАРІАНТНІСТЬ (1+1) – ВИМІРНОГО РІВНЯННЯ РЕАКЦІЇ-ДИФУЗІЇ-КОНВЕКЦІЇ ВІДНОСНО АЛГЕБР ГАЛІЛЕЯ

Знайдено всі рівняння реакції-дифузії-конвекції розмірності (1+1), які є умовно інваріантними відносно алгебри Галілея без оператора маси, та алгебри Галілея з оператором маси

ВСТУП. В основі класичної механіки лежить принцип відносності Галілея. В (1+1)-вимірному просторі йому відповідають перетворення Галілея вигляду

$$t' = t,\ x' = x - vt. \tag{1}$$

Виконавши для формул (1) перепозначення $t \to x_0,\ x \to x_1,\ v \to \theta$, одержимо

$$x_0' = x_0,\ x_1' = x_1 - \theta x_0. \tag{2}$$

Перетворення (2), породжуються операторами Галілея

$$G_1 = x_0\partial_1 + x_1 Q_1, \tag{3}$$

$$G_2 = x_0\partial_1 + Q_2, \tag{4}$$

де $Q_i = (\alpha_i u + \beta_i)\partial_u$, α_i, β_i – певні константи.

Оператор (3) разом з операторами $\partial_0 = \dfrac{\partial}{\partial_{x_0}}, \partial_1 = \dfrac{\partial}{\partial_{x_1}}, Q_1$, утворюють одну з реалізацій алгебри Галілея з оператором маси, яку ми позначимо

$$AG^M(1,1) = <\partial_0, \partial_1, G_1, Q_1>. \tag{5}$$