

# A new way to construct 1-singular Gelfand-Tsetlin modules

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**ABSTRACT.** We present a simplified way to construct the Gelfand-Tsetlin modules over  $\mathfrak{gl}(n, \mathbb{C})$  related to a 1-singular GT-tableau defined in [6]. We begin by reframing the classical construction of generic Gelfand-Tsetlin modules found in [3], showing that they form a flat family over generic points of  $\mathbb{C}^{\binom{n}{2}}$ . We then show that this family can be extended to a flat family over a variety including generic points and 1-singular points for a fixed singular pair of entries. The 1-singular modules are precisely the fibers over these points.

## 1. Introduction

A constant theme throughout the work of Sergei Ovsienko was the notion of Gelfand-Tsetlin subalgebras and modules introduced for  $\mathfrak{sl}(3, \mathbb{C})$  in [2], then more generally for  $\mathfrak{gl}(n, \mathbb{C})$  in [1], finally arriving at the more general definition of Harish-Chandra algebras and modules in [3]. He returned several times to the subject and its applications to representation theory, as can be seen in the articles [7–12, 17, 19, 20]. There were also many independent developments such as the construction of quantized versions of Gelfand-Tsetlin modules [18], the study of Gelfand-Tsetlin algebras and modules over orthogonal Lie algebras [14, 15], and the construction

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of several new families of infinite dimensional Gelfand-Tsetlin modules, including applications to the classification of irreducible modules over  $\mathfrak{gl}(n, \mathbb{C})$  [4–6], just to name a few. It is a pleasure to contribute to the continuing development of this subject.

The notion of a Gelfand-Tsetlin module (see Definition 2) has its origin in the classical article [13], where I. Gelfand and M. Tsetlin gave a presentation of all the finite dimensional irreducible representations of  $\mathfrak{gl}(n, \mathbb{C})$  in terms of certain tableaux, which have come to be known as *Gelfand-Tsetlin tableaux*, or GT-tableaux for short. The action of  $\mathfrak{gl}(n, \mathbb{C})$  over a tableau is given by rational functions in its entries, and the poles of these rational functions form a nowhere dense set in  $\mathbb{C}^{\binom{n}{2}}$ ; the formulas for this action are known as the Gelfand-Tsetlin formulas. Starting from this observation, Yu. Drozd, Ovsienko and V. Futorny introduced a large family of infinite dimensional  $\mathfrak{gl}(n, \mathbb{C})$ -modules in [3]. These modules have a basis parametrized by Gelfand-Tsetlin tableaux with complex coefficients such that no pattern is a pole for the rational functions appearing in the GT-formulae (such tableaux are called *generic*, hence the name “generic Gelfand-Tsetlin module”).

In [6], Futorny, D. Grantcharov and L. E. Ramírez built new GT-modules starting from a 1-singular tableaux (see Definition 1). These 1-singular modules are given by generators and relations, and the action of  $\mathfrak{gl}(n, \mathbb{C})$  is given by new explicit formulas; the proof that these formulas indeed define an action of  $\mathfrak{gl}(n, \mathbb{C})$  requires long calculations. We present a new approach, which consist of first building a “universal generic GT-module” (Definition 2) and then finding a “universal 1-singular module” (Definition 3) as a submodule. Thus the  $\mathfrak{gl}(n, \mathbb{C})$ -module structure is built in by construction on the singular modules, and the formulas for this action are deduced from those of the generic case.

Let us be a little more explicit. The set of poles  $P$  of the original Gelfand-Tsetlin formulae can be described as the union of all integral translates of a certain finite hyperplane arrangement centered at the origin in  $\mathbb{C}^{\binom{n}{2}}$ . Let  $X$  be the complement of  $P$ , and let  $A$  be the algebra of regular functions defined over  $X$ , so for every point  $x \in X$  there is a generic Gelfand-Tsetlin module  $V_x$ . We show that these modules form a flat family over  $X$  by constructing a free  $A$ -module  $V_A$  over which  $\mathfrak{gl}(n, \mathbb{C})$  acts by  $A$ -linear operators, such that  $V_A \otimes_A \mathbb{C}_x \cong V_x$  for each  $x \in X$  as  $\mathfrak{gl}(n, \mathbb{C})$ -modules, see Theorem 2. Now fix two entries  $(k, i)$  and  $(k, j)$ , let  $H \subset P$  be the set of tableaux with  $v_{k,i} = v_{k,j}$  and no other critical entries, and let  $B$  be the algebra of regular functions over  $Y = X \cup H$ . In Theorem 3 we show that the flat family over  $X$  extends to a flat family over  $Y$  by finding a full  $B$ -lattice  $V_B \subset V_A$  which is stable by

the action of  $\mathfrak{gl}(n, \mathbb{C})$ . From this we immediately get for each  $x \in X$  that  $V_x \cong V_B \otimes_B \mathbb{C}_x$  as  $\mathfrak{gl}(n, \mathbb{C})$ -modules, while for  $y \in H$  we obtain a  $\mathfrak{gl}(n, \mathbb{C})$ -module  $V_y = V_B \otimes_B \mathbb{C}_y$ . We finally show that  $V_y$  is isomorphic to the corresponding 1-singular module constructed by Futorny, Grantcharov and Ramírez, and recover the action of the Gelfand-Tsetlin subalgebra of  $U(\mathfrak{gl}(n, \mathbb{C}))$  on it.

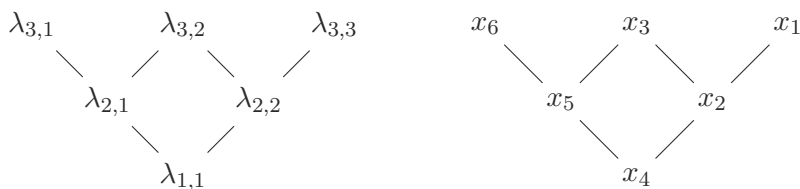
## 2. Gelfand-Tsetlin tableaux

We begin by quoting some results from [6, section 3]. We direct the reader to that article for proofs or references.

Fix  $n \in \mathbb{N}_{\geq 2}$ , and set  $N = \frac{n(n+1)}{2}$ . A *Gelfand-Tsetlin tableau* of size  $n$  (or GT-tableau of size  $n$ , for short) is an array with  $N$  complex entries of the form

$$\begin{array}{cccccc}
 v_{n,1} & v_{n,2} & \cdots & v_{n,n-1} & v_{n,n} & \\
 & v_{n-1,1} & & \cdots & v_{n-1,n-1} & \\
 & & \ddots & & \ddots & \\
 & & & v_{2,1} & v_{2,2} & \\
 & & & & v_{1,1} & 
 \end{array}$$

The set of all GT-tableaux can be identified with  $\mathbb{C}^N$  by indexing the entries of a point  $v \in \mathbb{C}^N$  by  $\{(k, i) \mid 1 \leq i \leq k \leq n\}$ . We fix one particular enumeration which will be useful later. Start by  $\lambda_{n,n}$ , which we denote by  $x_1$ ; then, looking at all entries with second coordinate  $n - 1$ , we enumerate them by taking  $x_2 = \lambda_{n-1,n-1}$ , then  $x_3 = \lambda_{n,n-1}$ ; next we take the elements with second coordinate  $n - 2$  starting by  $x_4 = \lambda_{n-2,n-2}$  and moving in the northwest direction. Explicitly, setting  $\varphi(i, j) = (i - j + 1) + \frac{(n-j)(n-j+1)}{2}$  we write  $x_{\varphi(i,j)} = \lambda_{i,j}$ . The following figure shows the enumeration corresponding to  $n = 3$ .



We will denote by  $T(v)$  the tableau corresponding to  $v \in \mathbb{C}^N$ .

**Definition 1.** We say that a point  $v \in \mathbb{C}^N$ , or the corresponding tableau  $T(v)$ , is:

- *integral* if  $v \in \mathbb{Z}^N$ ;
- *standard* if for all  $1 \leq i < j \leq k \leq n$  the difference  $v_{k,i} - v_{k-1,i} \in \mathbb{Z}_{\geq 0}$  and  $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{>0}$ ;
- *generic* if for all  $1 \leq i < j \leq k < n$  the difference  $v_{k,i} - v_{k,j}$  is not in  $\mathbb{Z}$ ;
- *singular* if it is a non-generic tableau, i.e. there is a pair of entries in the same row differing by an integer; if there is exactly one such pair then we say that the tableau is *1-singular*;
- *critical* if there are two entries in the same row which are equal; if there is exactly one such pair then we say the tableau is *1-critical*.

The enumeration of the  $\lambda_{k,i}$  was chosen so all the equations defining standard tableaux are of the form  $x_t - x_s > 0$  or  $x_t - x_s \geq 0$ . Drawing a graph analogous to the one shown above with the  $x_i$  as vertices, it is clear that we only get equations  $x_t - x_s$  with  $t > s$  and  $x_t$  and  $x_s$  joined by an edge.

We denote the set of standard points by  $\mathbb{C}_{\text{std}}^N$ , and the set of generic points by  $\mathbb{C}_{\text{gen}}^N$ . Any pair of two entries differing by an integer will be called a *singular pair*, and if they are equal they will be called a *critical pair*.

### 3. Gelfand-Tsetlin modules

In this section, we recall the general notions of Gelfand-Tsetlin algebras and modules, and review the classical construction of generic Gelfand-Tsetlin modules due to Drozd, Futorny and Ovsienko. Our proof follows the outline given by V. Mazorchuk in [16, Theorem 19], but we believe it is worthwhile to include it since it allows us to introduce and become familiar with the main ingredients of our re-imagining of the construction of [6].

Recall we have fixed  $n \in \mathbb{N}$  and set  $N = \frac{n(n+1)}{2}$ . Let  $\Lambda = \mathbb{C}[\lambda_{i,j} \mid 1 \leq i \leq j \leq n]$  be a polynomial algebra in  $N$  variables, and set  $\Lambda_m = \mathbb{C}[\lambda_{m,k} \mid 1 \leq k \leq m]$  for each  $1 \leq m \leq n$ . The group  $S_m$  acts on  $\Lambda_m$  permuting the variables in the obvious way. This induces an action of the group  $G = S_n \times S_{n-1} \times \cdots \times S_1$  on  $\Lambda$ . Let

$$\gamma_{m,k} = \sum_{i=1}^m (\lambda_{m,i} + m - 1)^k \prod_{i \neq j} \left( 1 - \frac{1}{\lambda_{m,i} - \lambda_{m,j}} \right).$$

Although it is not obvious, the  $\gamma_{m,k}$  are algebraically independent polynomials and  $\Lambda_m^{S_m} = \mathbb{C}[\gamma_{m,k} \mid 1 \leq k \leq m]$ .

For each  $m \in \mathbb{N}$  set  $U_m = U(\mathfrak{gl}(m, \mathbb{C}))$ . We denote by  $Z_m \subset U_m$  the center of  $U_m$ . Also we write  $U$  for  $U_n$ . We get a chain of inclusions  $U_1 \subset U_2 \subset \dots \subset U_n$  induced by the maps sending standard generators  $E_{i,j} \in \mathfrak{gl}(m, \mathbb{C})$  to the corresponding  $E_{i,j} \in \mathfrak{gl}(m+1, \mathbb{C})$ . The algebra  $Z_m$  is a polynomial algebra on the generators

$$c_{m,k} = \sum_{(i_1, \dots, i_k) \in [m]^k} E_{i_1, i_2} E_{i_2, i_3} \cdots E_{i_k, i_1} \quad 1 \leq k \leq m,$$

and there is an embedding  $Z_m \rightarrow \Lambda_m$  given by  $c_{m,k} \mapsto \gamma_{m,k}$ . We write  $\Gamma = \mathbb{C}[c_{m,k} \mid 1 \leq k \leq m \leq n] \subset U$ , which is the algebra generated by  $\bigcup_{k=1}^n Z_k$ . The  $c_{m,k}$  are algebraically independent and hence  $\Gamma$  is isomorphic to a polynomial algebra in  $N$  generators. Thus  $\Gamma$  is isomorphic to  $\Lambda^G$ .

**Definition 2** ([3, section 2.1]). The algebra  $\Gamma$  is called the *Gelfand-Tsetlin subalgebra* of  $U(\mathfrak{gl}(n, \mathbb{C}))$ . A finitely generated  $U$ -module is called a *Gelfand-Tsetlin module* if

$$M = \bigoplus_{\mathfrak{m} \in \text{Specm } \Gamma} M(\mathfrak{m}),$$

where  $M(\mathfrak{m}) = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \geq 0\}$ .

### 3.1. Irreducible representations of $\mathfrak{gl}(n, \mathbb{C})$

For each  $1 \leq i \leq k \leq n$  set

$$p_{k,i}^\pm(\lambda) = \prod_{j=1}^{k \pm 1} (\lambda_{k,i} - \lambda_{k \pm 1, j}); \quad q_{k,i}(\lambda) = \prod_{j \neq i} (\lambda_{k,i} - \lambda_{k,j}).$$

$$|\lambda|_k = \lambda_{k,1} + \lambda_{k,2} + \dots + \lambda_{k,k}.$$

The following is a classical result due to Gelfand and Tsetlin.

**Theorem 1** ([13]). *Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a dominant integral weight of  $\mathfrak{gl}(n, \mathbb{C})$ , and let*

$$V(\lambda) = \langle T(v) \mid v \in \mathbb{C}_{\text{std}}^N \text{ and } v_{n,1} = \lambda_1, v_{n,2} = \lambda_2 - 1, \dots, v_{n,n} = \lambda_n - n + 1 \rangle_{\mathbb{C}}$$

(by convention, if  $v$  is non-standard then  $T(v) = 0$  in  $V(\lambda)$ ). The vector space  $V(\lambda)$  can be endowed with a  $\mathfrak{gl}(n, \mathbb{C})$ -module structure, with the

action of the canonical generators given by

$$\begin{aligned}
 E_{k,k+1}T(v) &= -\sum_{i=1}^k \frac{p_{k,i}^+(v)}{q_{k,i}(v)} T(v + \delta^{k,i}), \\
 E_{k+1,k}T(v) &= \sum_{i=1}^k \frac{p_{k,i}^-(v)}{q_{k,i}(v)} T(v - \delta^{k,i}), \\
 E_{k,k}T(v) &= (|v|_k - |v|_{k-1} + k - 1)T(v),
 \end{aligned}$$

where  $\delta^{k,i}$  is the element of  $\mathbb{Z}^N$  with a 1 in position  $(k, i)$  and 0's elsewhere. Furthermore, for each  $1 \leq k \leq m \leq n$  we have  $c_{m,k}T(v) = \gamma_{m,k}(v)T(v)$ .

It is immediate to check that with this structure,  $V(\lambda)$  is an irreducible finite dimensional representation of maximal weight  $\lambda$ , so this theorem provides an explicit presentation of all finite dimensional simple  $\mathfrak{gl}(n, \mathbb{C})$ -modules. The formulas for the action of the generators of  $\mathfrak{gl}(n, \mathbb{C})$  in the previous theorem are known as the *Gelfand-Tsetlin formulas*.

### 3.2. Generic Gelfand-Tsetlin modules

Let  $\mathbb{Z}_0^N \subset \mathbb{Z}^N$  be the set of vectors of  $\mathbb{C}^N$  with integral entries such that  $v_{n,i} = 0$  for all  $1 \leq i \leq n$ . For  $v \in \mathbb{C}^N$  set

$$V(T(v)) = \langle T(v + z) \mid z \in \mathbb{Z}_0^N \rangle_{\mathbb{C}}.$$

We now quote [3, section 2.3].

**Theorem 2.** *Suppose  $v \in \mathbb{C}_{\text{gen}}^N$ . Then the vector space  $V(T(v))$  can be endowed with the structure of a  $\mathfrak{gl}(n, \mathbb{C})$ -module with the action of the canonical generators given by the Gelfand-Tsetlin formulas. Furthermore, for each  $1 \leq k \leq m \leq n$  we have  $c_{m,k}T(w) = \gamma_{m,k}(w)T(w)$ , so  $V(T(v))$  is a Gelfand-Tsetlin module.*

We will now reprove Theorem 2. We begin with a technical lemma. Recall that we have identified  $\mathbb{C}^N$  with  $\text{Specm } \Lambda$ , so we identify rational functions over  $\mathbb{C}^N$  with elements of the field  $\mathbb{C}(\lambda_{k,i} \mid 1 \leq i \leq k \leq n)$ .

**Lemma 1.** *Let  $W \subset \mathbb{Z}_0^N$  be a nonempty finite set and let  $S_W = \bigcap_{w \in W} \mathbb{C}_{\text{std}}^N - w$ . Let  $F \in \mathbb{C}(\lambda_{k,i} \mid 1 \leq i \leq k \leq n)$  be a rational function without poles in  $S_W$ . If  $F(v) = 0$  for all  $v \in S_W$ , then  $F = 0$ .*

*Proof.* Explicitly,  $v$  lies in  $S_W$  if and only if

$$\begin{aligned}
 v_{k,i} - v_{k-1,i} + \min_{w \in W} \{w_{k,i} - w_{k-1,i}, 0\} + 1 &\in \mathbb{Z}_{>0}, \\
 v_{k-1,i} - v_{k,i+1} + \min_{w \in W} \{w_{k-1,i} - w_{k,i+1}\} &\in \mathbb{Z}_{>0},
 \end{aligned}$$

for all  $1 \leq i \leq n$ . With the enumeration fixed in the previous section, and writing  $x_{\varphi(i,j)}$  for  $\lambda_{i,j}$ , these inequalities are all of the form  $x_t - x_s - r_{t,s} \in \mathbb{Z}_{>0}$  with  $t > s$  and  $r_{t,s} \in \mathbb{Z}$  (notice that not all pairs  $(t, s)$  appear). Thus fixing  $x_1(v) = 0$ , we can build recursively an element  $v \in S_W$ , in particular  $S_W$  is not empty.

For each  $1 \leq s \leq n$  let  $w_s \in \mathbb{Z}^N$  be such that  $x_t(w_s) = 1$  if  $t \geq s$ , and  $x_t(w_s) = 0$  if  $t < s$ . Now  $x \in S_W$  implies  $x + rw_s \in S_W$  for all  $r \in \mathbb{N}$ , and applying this to the element  $v \in S_W$  we found before, we get that  $v + \sum_{s=1}^N v + r_s w_s \in S_W$  whenever  $r_s \in \mathbb{N}$ , and each of these points is a zero of  $F$ .

For each  $1 \leq s \leq n$  let  $e_s \in \mathbb{Z}^N$  be such that  $x_s(e_s) = 1$  and  $x_t(e_s) = 0$ , and let  $C : \mathbb{C}^N \rightarrow \mathbb{C}^N$  be the affine transformation defined by  $C(0) = v$ ,  $C(e_s) = w_s$ . Then  $F \circ C$  is a rational function without poles in  $\mathbb{N}^N$  and furthermore  $F \circ C(\mathbb{N}^n) = 0$ . This implies that  $F \circ C = 0$ , which in turn implies  $F = 0$ . □

Let  $A$  be the algebra of regular functions defined over  $\mathbb{C}_{\text{gen}}^N$ . The algebra  $A$  contains all the rational functions appearing in the Gelfand-Tsetlin formulas. Consider the action of  $\mathbb{Z}^N$  over  $\mathbb{C}^N$ , given by  $v^z = v + z$ . This induces an action on  $A$ , given by  $(z \cdot f)(v) = f(v^z)$ . We will sometimes write  $f(\lambda^z)$  for  $(z \cdot f)$ .

**Proposition 1.** *Let  $V_A$  be the free  $A$ -module with basis  $\{T(z) \mid z \in \mathbb{Z}_0^N\}$ . The  $A$ -module  $V_A$  can be endowed with the structure of a  $U$ -module with the action of the canonical generators given by*

$$E_{k,k+1}T(z) = - \sum_{i=1}^k \frac{p_{k,i}^+(\lambda^z)}{q_{k,i}(\lambda^z)} T(z + \delta^{k,i});$$

$$E_{k+1,k}T(z) = \sum_{i=1}^k \frac{p_{k,i}^-(\lambda^z)}{q_{k,i}(\lambda^z)} T(z - \delta^{k,i});$$

$$E_{k,k}T(z) = (|\lambda^z|_k - |\lambda^z|_{k-1} + k - 1)T(z).$$

Furthermore, for each  $1 \leq k \leq m \leq n$ , we have  $c_{m,k}T(z) = \gamma_{m,k}(\lambda^z)T(z)$ .

*Proof.* Let  $F$  be the free  $\mathbb{C}$ -algebra generated by  $X_{k,k+1}, X_{k+1,k}$  for  $1 \leq k < n$ , and  $X_{k,k}$  for  $1 \leq k \leq n$ ; there is an obvious map  $\varphi : F \rightarrow U$ . Replacing  $E$  by  $X$  in the formulas in the statement, we certainly get an  $F$ -module structure on  $V_A$ , so there is an algebra map  $F \rightarrow \text{End}_A V_A$ , and we must show that this map factors through  $U$ .

Let  $a \in F$ . Then for each  $w \in \mathbb{Z}_0^N$  there exists a rational function  $f_{a,w} \in A$  such that

$$aT(z) = \sum_{w \in \mathbb{Z}_0^N} f_{a,w}(\lambda^z)T(z + w)$$

for all  $z \in \mathbb{Z}_0^N$ . The sum is of course over a finite subset of  $\mathbb{Z}_0^N$ , which we call  $W$ . By Lemma 1 the rational function  $f_{a,w}$  is determined by its values on  $S_W = \bigcap_{w \in W} \mathbb{C}_{\text{std}}^N - w$ .

Fix  $v \in S_W$  and let  $\lambda = (v_{n,1}, v_{n,1} + 1, \dots, v_{n,n} + n - 1)$ . Notice that  $\lambda$  is the  $n$ -th row of  $v + x$  for all  $x \in S_W$  since these are all elements of  $\mathbb{Z}_0^N$ , and this in turn implies that  $\lambda$  is an integral dominant weight of  $\mathfrak{gl}(n, \mathbb{C})$  and we can consider the representation  $V(\lambda)$  as defined in Theorem 1. By construction the set  $\{T(v + w) \mid w \in W\}$  consists of nonzero linearly independent elements of  $V(\lambda)$ , and by construction

$$\varphi(a)T(v) = \sum_{w \in W} f_{a,w}(v)T(v + w).$$

Thus if  $\varphi(a) = 0$  then  $f_{a,w}(v) = 0$  for all  $v \in S_W$ , so  $f_a(-, w) = 0$ . This implies that the formulas in the statement indeed define a  $U$ -module structure on  $V_A$ . Furthermore, if  $\varphi(a) = c_{m,k}$  for some  $k$  and  $m$  then  $f_{a,w}(v) = 0$  for all  $w \neq 0$  and  $f_{a,0}(v) = \gamma_{m,k}(v)$ , so again by Lemma 1 the action of the  $c_{m,k}$  is the one given in the statement.  $\square$

*Proof of Theorem 2.* If  $v \in \mathbb{C}_{\text{gen}}^N$  then the map  $f \in A \mapsto f(v) \in \mathbb{C}$  is well defined, and induces a one-dimensional representation of  $A$  which we denote  $\mathbb{C}_v$ . Now  $V_A \otimes_A \mathbb{C}_v$  is a  $U$ -module, with the action of  $U$  induced by its action on  $V_A$ , and furthermore it is isomorphic to  $V(T(v))$  as  $\mathbb{C}$ -vector space through the assignation  $1 \otimes_A T(z) \mapsto T(v + z)$ . Thus  $V(T(v))$  gets a  $U$ -module structure, which by construction is given by the Gelfand-Tsetlin formulas.  $\square$

#### 4. The construction of 1-singular Gelfand-Tsetlin modules

We will give a new construction of Futorny, Grantcharov and Ramirez’s 1-singular Gelfand-Tsetlin modules, which will follow the same spirit as the construction of generic Gelfand-Tsetlin modules presented in the previous section. Fix  $k, i, j \in \mathbb{N}$  with  $1 \leq i < j \leq k < n$ . From now a 1-critical point, will be a critical point  $v$  whose only critical pair is  $v_{k,i}, v_{k,j}$ . We set  $x = \lambda_{k,i}, y = \lambda_{k,j}$ .



Let  $B \subset A$  be the algebra of consisting of functions in  $A$  without poles in the set of 1-critical points; for example  $1/(x - y)$  lies in  $A$  but not in  $B$ . The idea behind our construction is finding a  $B$ -lattice  $V_B \subset V_A$  which is stable by the action of  $U$ . Once we have found this lattice, the construction of the  $U$ -module associated to a 1-singular point  $v$  can go as before: take  $\mathbb{C}_v$  to be the 1-dimensional  $B$ -module associated to  $v$  and set  $V(T(v)) = V_B \otimes_B \mathbb{C}_v$ , which inherits its  $U$ -module structure from  $V_B$ .

Let  $I = \{(r, s) \mid 1 \leq s \leq r \leq n\}$ , and let  $\tau : I \rightarrow I$  be the involution that interchanges  $(k, i)$  and  $(k, j)$ , while leaving all other elements of  $I$  fixed. By abuse of notation, we also denote by  $\tau$  the linear transformation  $\tau : \mathbb{C}^N \rightarrow \mathbb{C}^N$  given by  $\tau(\delta^{r,s}) = \delta^{\tau(r,s)}$  for all  $(r, s) \in I$ , and also the algebra automorphism  $\tau : B \rightarrow B$  induced by the assignation  $\tau(\lambda_{r,s}) = \lambda_{\tau(r,s)}$ . In each case  $\tau^2$  is the identity map.

**Definition 3.** For each  $z \in \mathbb{Z}_0^N$  we define

$$S(z) = \frac{T(z) + T(\tau(z))}{2}, \quad A(z) = \frac{T(z) - T(\tau(z))}{2(x - y)}.$$

We define  $V_B \subset A$  to be the  $B$ -module generated by  $\{S(z), A(z) \mid z \in \mathbb{Z}_0^N\}$ .

As stated above, we will show that  $V_B$  is stable by the action of  $U$ . Notice that  $T(z) = S(z) + (x - y)A(z)$ , so  $T(z) \in V_B$  for any  $z \in \mathbb{Z}_0^N$ . Notice also that  $S(\tau(z)) = S(z)$  and  $A(\tau(z)) = -A(z)$ ; in particular if  $z = \tau(z)$  then  $S(z) = T(z)$  and  $A(z) = 0$ . It follows that  $aT(z) + bT(\tau(z)) = (a + b)S(z) + (a - b)(x - y)A(z)$  for all  $a, b \in A$ .

Let  $D : B \rightarrow B$  be the differential operator  $\frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)$ . The following lemma can be verified by direct calculations.

**Lemma 2.** Let  $f \in B$ .

- 1) The rational function  $\frac{f - \tau \cdot f}{x - y}$  lies in  $B$ , and is equal to  $2D(f)$ .
- 2) For each  $z \in \mathbb{Z}^N$  we have  $\tau \cdot f(\lambda^z) = (\tau \cdot f)(\lambda^{\tau(z)})$ .
- 3) We have  $\tau \cdot p_{t,s}^\pm = p_{\tau(t,s)}^\pm$  and  $\tau \cdot q_{t,s} = q_{\tau(t,s)}$ .

**Theorem 3.** The  $B$ -module  $V_B$  is a  $U$ -submodule of  $V_A$ .

*Proof.* We will check by direct inspection that  $V_B$  is stable by the action of the canonical generators of  $\mathfrak{gl}(n, \mathbb{C})$ . First, since  $|\lambda^z|_t = |\lambda^{\tau(z)}|_t$  for all  $1 \leq t \leq n$ , both  $T(z)$  and  $T(\tau(z))$  are eigenvectors of  $E_{t,t}$  of eigenvalue  $|\lambda^z|_t - |\lambda^z|_{t-1} + t - 1$ , so

$$\begin{aligned} E_{t,t}S(z) &= (|\lambda^z|_t - |\lambda^z|_{t-1} + t - 1)S(z), \\ E_{t,t}A(z) &= (|\lambda^z|_t - |\lambda^z|_{t-1} + t - 1)A(z), \end{aligned}$$

and  $V_B$  is not only stable by the action of the  $E_{t,t}$  with  $1 \leq t \leq n$ , but also a weight module.

Let us study the action of  $E_{t,t+1}$  on  $S(z)$ . By definition we get

$$E_{t,t+1}S(z) = -\frac{1}{2} \sum_{s=1}^t \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} T(z + \delta^{t,s}) + \frac{p_{t,s}^+(\lambda^{\tau(z)})}{q_{t,s}(\lambda^{\tau(z)})} T(\tau(z) + \delta^{t,s}).$$

Now since  $T(\tau(z) + \delta^{t,s}) = T(\tau(z + \delta^{\tau(t,s)}))$ , we can rewrite this last expression as

$$\begin{aligned} & -\frac{1}{2} \sum_{s=1}^t \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} T(z + \delta^{t,s}) + \frac{p_{\tau(t,s)}^+(\lambda^{\tau(z)})}{q_{\tau(t,s)}(\lambda^{\tau(z)})} T(\tau(z + \delta^{t,s})) \\ & = -\frac{1}{2} \sum_{s=1}^t \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} + \frac{p_{\tau(t,s)}^+(\lambda^{\tau(z)})}{q_{\tau(t,s)}(\lambda^{\tau(z)})} \right) S(z + \delta^{t,s}) \\ & \quad + \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} - \frac{p_{\tau(t,s)}^+(\lambda^{\tau(z)})}{q_{\tau(t,s)}(\lambda^{\tau(z)})} \right) (x - y) A(z + \delta^{t,s}). \end{aligned}$$

By definition  $1/q_{t,s}(\lambda^z) \in B$  unless  $(t, s) \in \{(k, i), (k, j)\}$  and  $z = \tau(z)$ . Hence the coefficients in the equation above lie in  $B$ , except perhaps when  $z = \tau(z)$  and  $t = k$ , in which case it is not clear that the coefficients of  $S(z + \delta^{k,i}) = S(z + \delta^{k,j})$  and  $A(z + \delta^{k,i}) = -A(z + \delta^{k,j})$  lie in  $B$ . Set  $q_{k,i}^*(\lambda^z) = q_{k,i}(\lambda^z)/(x - y)$ , so the inverse of  $q_{k,i}^*(\lambda^z)$  lies in  $B$ . The coefficient of  $A(z + \delta^{k,i})$  can be written as

$$2 \left( \frac{p_{k,i}^+(\lambda^z)}{q_{k,i}^*(\lambda^z)} + \frac{p_{k,j}^+(\lambda^z)}{q_{k,j}^*(\lambda^z)} \right) \in B,$$

where the factor 2 arises from the fact that  $A(z + \delta^{k,i}) = -A(z + \delta^{k,j})$ . Now considering the coefficient of  $S(z + \delta^{k,i})$ , we may use Lemma 2 to rewrite it as

$$\begin{aligned} & 2 \left( \frac{p_{k,i}^+(\lambda^z)}{q_{k,i}(\lambda^z)} + \frac{p_{k,j}^+(\lambda^{\tau(z)})}{q_{k,j}(\lambda^{\tau(z)})} \right) \\ & = \frac{2}{(x - y)} \left( \frac{p_{k,i}^+(\lambda^z)}{q_{k,i}^*(\lambda^z)} - \tau \cdot \frac{p_{k,i}^+(\lambda^z)}{q_{k,i}^*(\lambda^z)} \right) = 4D \left( \frac{p_{k,i}^+(\lambda^z)}{q_{k,i}^*(\lambda^z)} \right). \end{aligned}$$

Thus  $E_{t,t+1}S(z) \in V_B$  for all  $t$  and all  $z$ . A similar analysis shows that  $E_{t+1,t}S(z) \in V_B$ .

Following the same reasoning we find that

$$E_{t,t+1}A(z) = -\frac{1}{2} \sum_{s=1}^t \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} - \frac{p_{\tau(t,s)}^+(\lambda^{\tau(z)})}{q_{\tau(t,s)}(\lambda^{\tau(z)})} \right) \frac{1}{x-y} S(z + \delta^{t,s}) + \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} + \frac{p_{\tau(t,s)}^+(\lambda^{\tau(z)})}{q_{\tau(t,s)}(\lambda^{\tau(z)})} \right) A(z + \delta^{t,s}).$$

If  $z = \tau(z)$  then  $A(z) = 0$ , so we only have to consider the case  $z \neq \tau(z)$ , in which case the formula above equals

$$-\frac{1}{2} \sum_{s=1}^t 2D \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} \right) S(z + \delta^{t,s}) + \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} + \frac{p_{\tau(t,s)}^+(\lambda^{\tau(z)})}{q_{\tau(t,s)}(\lambda^{\tau(z)})} \right) A(z + \delta^{t,s}).$$

Since all coefficients lie in  $B$ , we see that  $E_{t,t+1}A(z) \in V_B$ . A similar formula can be obtained for  $E_{t+1,t}A(z) \in V_B$ , which completes the proof that  $V_B$  is stable by the action of  $U$ .  $\square$

The formulas obtained in the proof of Theorem 3 can be misleading. For example, notice that for  $t \neq k-1, k$  the coefficient of each  $A(z + \delta^{t,s})$  in  $E_{t,t+1}S(z)$  is zero. A case by case analysis reveals much simpler formulas, but we postponed for the sake of brevity and because we will add one further simplification. Since we are going to study the specialization of  $V_B$  at 1-critical points, we might as well consider the coefficients not in  $B$  but in  $B' = B/(x - y)$ . Reduction modulo  $(x - y)$  of course simplifies the formulas.

**Definition 4.** Set  $\bar{V} = B' \otimes_B V_B$ . We refer to  $\bar{V}$  as the *universal 1-singular Gelfand-Tsetlin module*. Given a 1-singular point  $v \in \mathbb{C}^N$  with  $v_{k,i} = v_{k,j}$ , we denote by  $\mathbb{C}_v$  the one-dimensional representation of  $B'$  induced by  $v$ , and by  $V(T(v))$  the  $U$ -module  $\bar{V} \otimes_{B'} \mathbb{C}_v = V \otimes_B \mathbb{C}_v$ .

We write  $\bar{S}(z)$  and  $\bar{A}(z)$  for the images of  $S(z)$  and  $A(z)$  in  $\bar{V}$ , respectively. Also, we denote by  $\bar{p}_{t,s}^\pm, \bar{q}_{t,s}$  the images of  $p_{(t,s)}^\pm, q_{t,s}$  in  $B'$ , respectively. First, notice that if  $t \neq k$  or  $z \neq \tau(z)$  the definitions imply that

$$\frac{p_{t,s}^\pm(\lambda^z)}{q_{t,s}(\lambda^z)} \equiv \frac{p_{\tau(t,s)}^\pm(\lambda^{\tau(z)})}{q_{\tau(t,s)}(\lambda^{\tau(z)})} \pmod{(x - y)}.$$

Applying reduction modulo  $(x - y)$  to the formulas obtained in the proof of Theorem 3, we obtain that if  $t \neq k$  or  $z \neq \tau(z)$  then

$$E_{t,t+1}\bar{S}(z) = - \sum_{s=1}^t \frac{\bar{p}_{t,s}^+(\lambda^z)}{\bar{q}_{t,s}(\lambda^z)} \bar{S}(z + \delta^{t,s}),$$

$$E_{t+1,t}\bar{S}(z) = \sum_{s=1}^t \frac{\bar{p}_{t,s}^-(\lambda^z)}{\bar{q}_{t,s}(\lambda^z)} \bar{S}(z - \delta^{t,s}).$$

Writing as before  $q_{k,r}^*(\lambda^z) = q_{k,r}(\lambda^z)/(x - y)$ , and setting  $\bar{D}(f) = \overline{D(f)}$  we also obtain formulas

$$E_{k,k+1}\bar{S}(z) = - \sum_{s \neq i,j} \frac{\bar{p}_{k,s}^+(\lambda^z)}{\bar{q}_{k,s}(\lambda^z)} \bar{S}(z + \delta^{k,s})$$

$$- 2\bar{D} \left( \frac{p_{k,i}^+(\lambda^z)}{q_{k,i}^*(\lambda^z)} \right) \bar{S}(z + \delta^{k,i}) - 2 \left( \frac{\bar{p}_{k,i}^+(\lambda^z)}{\bar{q}_{k,i}^*(\lambda^z)} \right) \bar{A}(z + \delta^{k,i});$$

$$E_{k+1,k}\bar{S}(z) = \sum_{s \neq i,j} \frac{\bar{p}_{k,s}^-(\lambda^z)}{\bar{q}_{k,s}(\lambda^z)} \bar{S}(z - \delta^{k,s})$$

$$+ 2\bar{D} \left( \frac{p_{k,i}^-(\lambda^z)}{q_{k,i}^*(\lambda^z)} \right) \bar{S}(z - \delta^{k,i}) + 2 \left( \frac{\bar{p}_{k,i}^-(\lambda^z)}{\bar{q}_{k,i}^*(\lambda^z)} \right) \bar{A}(z - \delta^{k,i}).$$

For the sake of comparison with the original construction, we point out that this formulas can be summarized as

$$E_{t,t+1}S(z) = - \sum_{s=1}^t \bar{D} \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}^*(\lambda^z)} \right) \bar{S}(z + \delta^{t,s}) + \left( \frac{\bar{p}_{t,s}^+(\lambda^z)}{\bar{q}_{t,s}(\lambda^z)} \right) \bar{A}(z + \delta^{t,s})$$

$$E_{t+1,t}S(z) = \sum_{s=1}^t \bar{D} \left( \frac{p_{t,s}^-(\lambda^z)}{q_{t,s}^*(\lambda^z)} \right) \bar{S}(z + \delta^{t,s}) + \left( \frac{\bar{p}_{t,s}^-(\lambda^z)}{\bar{q}_{t,s}(\lambda^z)} \right) \bar{A}(z + \delta^{t,s})$$

In the case of  $\bar{A}(z)$ , reduction modulo  $(x - y)$  of the formulas already found gives

$$E_{t,t+1}\bar{A}(z) = - \sum_{s=1}^t \bar{D} \left( \frac{p_{t,s}^+(\lambda^z)}{q_{t,s}(\lambda^z)} \right) \bar{S}(z + \delta^{t,s}) + \left( \frac{\bar{p}_{t,s}^+(\lambda^z)}{\bar{q}_{t,s}(\lambda^z)} \right) \bar{A}(z + \delta^{t,s}),$$

$$E_{t+1,t}\bar{A}(z) = \sum_{s=1}^t \bar{D} \left( \frac{p_{t,s}^-(\lambda^z)}{q_{t,s}(\lambda^z)} \right) \bar{S}(z - \delta^{t,s}) + \left( \frac{\bar{p}_{t,s}^-(\lambda^z)}{\bar{q}_{t,s}(\lambda^z)} \right) \bar{A}(z - \delta^{t,s})$$

With this formulas, the following theorem is clear.

**Theorem 4.** *Let  $v \in \mathbb{C}^N$  be a 1-singular point with  $v_{k,i} = v_{k,j}$ . Then the  $U$ -module  $V(T(v)) = V_{B'} \otimes_{B'} \mathbb{C}_v$  is isomorphic to the 1-singular Gelfand-Tsetlin module defined in [6, Theorem 4.11] through the map  $1 \otimes_{B'} \bar{S}(z) \mapsto T(v+z), 1 \otimes_{B'} \bar{A}(z) \mapsto DT(v+z)$ .*

We finish by studying the action of the Gelfand-Tsetlin algebra on  $\bar{V}$ . As before, we first consider its action on  $V_B$ , and in that case we get

$$c_{m,t}S(z) = \gamma_{m,t}(\lambda^z)S(z); \quad c_{m,t}A(z) = \gamma_{m,t}(\lambda^z)A(z) \quad (m \neq k),$$

and

$$c_{k,t}S(z) = \frac{\gamma_{k,t}(\lambda^z) + \gamma_{k,t}(\lambda^{\tau(z)})}{2}S(z) + (x-y)\frac{\gamma_{k,t}(\lambda^z) - \gamma_{k,t}(\lambda^{\tau(z)})}{2}A(z);$$

$$c_{k,t}A(z) = \frac{\gamma_{k,t}(\lambda^z) + \gamma_{k,t}(\lambda^{\tau(z)})}{2}A(z) + \frac{\gamma_{k,t}(\lambda^z) - \gamma_{k,t}(\lambda^{\tau(z)})}{2(x-y)}S(z).$$

Now  $\gamma_{t,s}$  is a symmetric polynomial, which implies that  $\tau \cdot \gamma_{t,s} = \gamma_{t,s}$ . Thus reducing modulo  $x-y$  and using item 1) of Lemma 2 we obtain

$$c_{m,t}\bar{S}(z) = \gamma_{m,t}(\lambda^z)\bar{S}(z); \quad c_{m,t}\bar{A}(z) = \gamma_{m,t}(\lambda^z)\bar{A}(z); \quad (m \neq k),$$

and

$$c_{k,t}\bar{S}(z) = \bar{\gamma}_{k,t}(\lambda^z)\bar{S}(z); \quad c_{k,t}\bar{A}(z) = \gamma_{k,t}(\lambda^z)\bar{A}(z) + \bar{D}(\gamma_{k,t}(\lambda^z))\bar{S}(z).$$

From this we recover [6, Corollary 4.13].

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