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POINCARÉ SERIES FOR THE ALGEBRAS OF JOINT INVARIANTS AND COVARIANTS OF n QUADRATIC FORMS

We consider one of the fundamental objects of classical invariant theory, namely the Poincaré series for an algebra of invariants of Lie group SL_2. The first two terms of the Laurent series expansion of Poincaré series at the point $z = 1$ give us an important information about the structure of the algebra $\mathcal{I}_d$. It was derived by Hilbert for the algebra $\mathcal{I}_d = \mathbb{C}[V_d]^{SL_2}$ of invariants for binary $d-$form (by V_d we denote the vector space over $\mathbb{C}$ consisting of all binary forms homogeneous of degree d). Springer got this result, using explicit formula for the Poincaré series of this algebra. We consider this problem for the algebra of joint invariants $\mathcal{I}_{2n}=\mathbb{C}[\underbrace{V_2\oplus V_2\oplus \cdots \oplus V_2}_{n \text{ times}}]^{SL_2}$ and the algebra of joint covariants $\mathcal{C}_{2n}=\mathbb{C}[\underbrace{V_2\oplus V_2\oplus \cdots \oplus V_2}_{n \text{ times}} \oplus\mathbb{C}^2]^{SL_2}$ of n quadratic forms. We express the Poincaré series $\mathcal{P}(\mathcal{C}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{C}_{2n})_j\, z^j$ and $\mathcal{P}(\mathcal{I}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{I}_{2n})_j\, z^j$ of these algebras in terms of Narayana polynomials.

Also, for these algebras we calculate the degrees and asymptotic behavious of the degrees, using their Poincaré series.

Key words and phrases: classical invariant theory, invariants, Poincaré series, combinatorics.

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INTRODUCTION

Let V_2 be the complex vector space of quadratic binary forms endowed with the natural action of the special linear group SL_2. Consider the corresponding action of the group SL_2 on the algebras of polynomial functions $\mathbb{C}[nV_2]$ and $\mathbb{C}[nV_2 \oplus \mathbb{C}^2]$, where $nV_2 := \underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}}$.

Denote by $\mathcal{I}_{2n} = \mathbb{C}[nV_2]^{SL_2}$ and by $\mathcal{C}_{2n} = \mathbb{C}[nV_2 \oplus \mathbb{C}^2]^{SL_2}$ the corresponding algebras of invariant polynomial functions. In the language of classical invariant theory the algebras $\mathcal{I}_{2n}$ and $\mathcal{C}_{2n}$ are called the algebra of joint invariants and the algebra of joint covariants for the n quadratic binary forms respectively.

Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded complex algebra, $R_0 = \mathbb{C}$. Denote by

$$\mathcal{P}(R, z) = \sum_{j=0}^{\infty} \dim R_j z^j$$

its Poincaré series. Letting r be the transcendence degree of the quotient field of R over $\mathbb{C}$, the number

$$\deg(R) := \lim_{z\to 1}(1 - z)^r \mathcal{P}(R, z)$$

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is called the *degree of the algebra R*. The first two terms of the Laurent series expansion of $\mathcal{P}(R,z)$ at the point $z = 1$ have the following form

$$\mathcal{P}(R,z) = \frac{\deg(R)}{(1-z)^r} + \frac{\psi(R)}{(1-z)^{r-1}} + \cdots .$$

The numbers $\deg(R), \psi(R)$ are important characteristics of the algebra R. For instance, if R is an algebra of invariants of a finite group G then $\deg(R)^{-1}$ is order of the group G and $2\dfrac{\psi(R)}{\deg(R)}$ is the number of pseudo-reflections in G (see [3]).

Let V_d be the standard $(d+1)-$dimensional complex representation of SL_2. Consider the corresponding algebras of invariants $I_d := C[V_d]^{SL_2}$ and $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^{SL_2}$ be the corresponding algebra of invariants. Explicit formula for the degree of algebra of invariants for binary $d-$forms $\deg(\mathcal{I}_d)$ was derived by Hilbert in [4] and Springer in [8]. In [2] explicit formula for the degree of algebra of covariants for binary $d-$forms of $\deg(\mathcal{C}_d)$ was derived. For this purpose, in [8] and [2] authors used an explicit formula for the Poincaré series of those algebras.

The formal power series

$$\mathcal{P}(\mathcal{C}_{2n},z) = \sum_{j=0}^{\infty} \dim(\mathcal{C}_{2n})_j\, z^j \quad \text{and} \quad \mathcal{P}(\mathcal{I}_{2n},z) = \sum_{j=0}^{\infty} \dim(\mathcal{I}_{2n})_j\, z^j$$

are called the Poincaré series of the algebras $\mathcal{C}_{2n}$ and $\mathcal{I}_{2n}$. In the paper [1] the following expressions for the Poincaré series of those algebras was derived:

$$\mathcal{P}\mathcal{C}_{2n}(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n)_{n-k-i} z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}} \right),$$

$$\mathcal{P}\mathcal{I}_{2n}(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n)_{n-k-i} z^{2n-k-i-1}}{(1-z)^{n+i-1}(1-z^2)^{2n-k-i}} \right),$$

where $(n)_m := n(n+1)\cdots(n+m-1)$, $(n)_0 := 1$ denotes the shifted factorial.

In the present paper those formulas are reduced to the following forms:

$$\mathcal{P}(\mathcal{C}_{2n},z) = \frac{W_{n-1}(z^2)}{(1-z)^{3n-1}(1+z)^{2n-1}} \quad \text{and} \quad \mathcal{P}(\mathcal{I}_{2n},z) = \frac{W_{n-1}(z^2) - nzN_{n-1}(z^2)}{(1-z)^{3n-1}(1+z)^{2n-1}},$$

where

$$N_n(z) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} z^{k-1} \quad \text{and} \quad W_n(z) = \sum_{k=0}^{n} \binom{n}{k}^2 z^k$$

denotes the *Narayana polynomials* and the *Narayana polynomials of type B* respectively.

Also, the degrees of algebras $\mathcal{I}_{2n}$, $\mathcal{C}_{2n}$ and asymptotic behaviors of the degrees are calculated using the explicit expressions for the Poincaré series.

1 Combinatorial identities

Let us prove several auxiliary combinatorial identities.

Lemma 1. *Let m, n be positive integers. The following identities hold:*

$$(i)\quad \frac{W_{n-1}(z^2)}{(1-z)^a(1-z^2)^{2n-1}} = \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+a}}{(1-z)^{2n-k+a}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^a(1+z)^n} \right) \right),$$

$$(ii)\quad \frac{nzN_{n-1}(z^2)}{(1-z)^a(1-z^2)^{2n-1}} = \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k+a}}{(1-z)^{2n-k+a}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^a(1+z)^n} \right) \right).$$

Proof. We shall prove the relations by induction in a.

For $a = 0$ the statements follow immediately from the next identities (see [5]):

$$\sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1}}{(1-z^2)^{2n-k}} \right) = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k}}{(1-z^2)^{2n-1}},$$

$$\sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k}}{(1-z^2)^{2n-k}} \right) = \frac{\sum_{k=0}^{n-2} \binom{n-2}{k}\binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}}.$$

(i) Assume there is a non-negative m such that

$$\sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+m}}{(1-z)^{2n-k+m}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^m(1+z)^n} \right) \right) = \frac{\sum_{i=0}^{n-1} \binom{n-1}{i}^2 z^{2i}}{(1-z)^m(1-z^2)^{2n-1}}.$$

We must prove the formula (i) is true for $a = m+1$:

$$\sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k+m}}{(1-z)^{2n-k+m}} \frac{d^{n-k}}{dz^{n-k}} \frac{1}{z^{m+1}(1+z)^n} \right) = \frac{\sum_{i=0}^{n-1} \binom{n-1}{i}^2 z^{2i}}{(1-z)^{m+1}(1-z^2)^{2n-1}}.$$

That is,

$$(1-z) \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+m+1}}{(1-z)^{2n-k+m+1}} \frac{d^{n-k}}{dz^{n-k}} \frac{1}{z^{m+1}(1+z)^n} \right)$$
$$= \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+m}}{(1-z)^{2n-k+m}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^m(1+z)^n} \right) \right).$$

It sufficed to show that (we expanded the functions into the Taylor series about z)

$$\sum_{j=0}^{\min\{k,n-1\}} \sum_{i=0}^{k-j} \binom{n+k-j-1}{k} \binom{n+m+k-i-1}{k-j-i} (-1)^i \binom{n+i-1}{i} \binom{i-m}{j}$$
$$= \sum_{j=0}^{\min\{k,n-1\}} \sum_{i=0}^{k-j} \left(\binom{n+k-j-1}{k} \binom{n+m+k-i}{k-j-i} - \binom{n+k-j-2}{k-1} \binom{n+m+k-i-1}{k-j-i-1} \right)$$
$$\times (-1)^i \binom{n+i-1}{i} \binom{i-m-1}{j}.$$

Using following formulas

$$\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}, \binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1},$$

after some algebraic transformations we obtain the last equality.

The proof of (ii) is completely analogous to that of (i). □

2 The Poincaré series of the algebras of invariants and covariants

We use the derived above combinatorial identities to express the Poincaré series $\mathcal{P}(\mathcal{I}_{2n},z)$ and $\mathcal{P}(\mathcal{C}_{2n},z)$ in terms of Narayana polynomials.

Theorem 1. *The following formulas hold:*

$$(i)\quad \mathcal{P}(\mathcal{C}_{2n},z)=\frac{W_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}},$$
$$(ii)\quad \mathcal{P}(\mathcal{I}_{2n},z)=\frac{W_{n-1}(z^2)-nzN_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

Proof. (i) Note that

$$\begin{aligned}\mathcal{P}(\mathcal{C}_{2n},z)&=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(n)_i(n)_{n-k-i}z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}\right)\\&=\sum_{k=1}^{n}\frac{1}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{z^{3n-k-1}}{(1-z)^{3n-k}}\frac{d^{n-k}}{dz^{n-k}}\left(\frac{1}{(z(1+z))^n}\right)\right).\end{aligned}$$

Substituting n for a in Lemma 1 (i), we get

$$\sum_{k=1}^{n}\frac{1}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{z^{3n-k-1}}{(1-z)^{3n-k}}\frac{d^{n-k}}{dz^{n-k}}\left(\frac{1}{(z(1+z))^n}\right)\right)=\frac{W_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

(ii)

$$\begin{aligned}\mathcal{P}(\mathcal{I}_{2n},z)&=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(n)_i(n)_{n-k-i}z^{2n-k-i-1}}{(1-z)^{n+i-1}(1-z^2)^{2n-k-i}}\\&=\sum_{k=1}^{n}\frac{1}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{z^{3n-k-1}}{(1-z)^{3n-k-1}}\frac{d^{n-k}}{dz^{n-k}}\frac{1}{(z(1+z))^n}\right)\\&=\sum_{k=1}^{n}\frac{1}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{z^{3n-k-1}}{(1-z)^{3n-k}}\frac{d^{n-k}}{dz^{n-k}}\frac{1}{(z(1+z))^n}\right)\\&\quad-\sum_{k=1}^{n}\frac{1}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{z^{3n-k}}{(1-z)^{3n-k}}\frac{d^{n-k}}{dz^{n-k}}\frac{1}{(z(1+z))^n}\right).\end{aligned}$$

Substituting n for m in Lemma 1, we get

$$\mathcal{P}(\mathcal{I}_{2n},z)=\frac{W_{n-1}(z^2)-nzN_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

□

3 The degrees of the algebras of invariants and covariants

Let us calculate the degrees of the algebras of joint invariants and covariants of n quadratic binary forms using the formulas for the Poincaré series $\mathcal{P}(\mathcal{I}_{2n},z)$ and $\mathcal{P}(\mathcal{C}_{2n},z)$.

Theorem 2. *The following formulas hold*

$$(i) \quad \operatorname{tr}\deg_{\mathbb{C}} \mathcal{C}_{2n} = 3n-1,$$
$$(ii) \quad \operatorname{tr}\deg_{\mathbb{C}} \mathcal{I}_{2n} = 3n-3.$$

Proof. The transcendence degrees over $\mathbb{C}$ for the algebras $\mathcal{I}_{2n}, \mathcal{C}_{2n}$ is equal to order of the pole for $\mathcal{P}(\mathcal{I}_{2n}, z), \mathcal{P}(\mathcal{C}_{2n}, z)$ respectively, see [7]. Since $\frac{W_{n-1}(1)}{2^{2n-1}} \neq 0$. for all n then $\operatorname{tr}\deg_{\mathbb{C}} \mathcal{C}_{2n} = 3n-1$.

Note that

$$(W_{n-1}(z^2) - nzN_{n-1}(z^2))\,|_{z=1} = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 - n\sum_{k=1}^{n-1} \frac{1}{k}\binom{n-2}{k-1}\binom{n-1}{k-1} = 0,$$

$$(W_{n-1}(z^2) - nzN_{n-1}(z^2))'\,|_{z=1} = 2\sum_{k=1}^{n-1} k\binom{n-1}{k}^2 - n\sum_{k=1}^{n-1} \frac{2k-1}{k}\binom{n-2}{k-1}\binom{n-1}{k-1} = 0,$$

$$(W_{n-1}(z^2) - nzN_{n-1}(z^2))''\,|_{z=1} = \sum_{k=1}^{n-1} 2k(2k-1)\binom{n-1}{k}^2$$
$$- n\sum_{k=2}^{n-1} \frac{1}{k}(2k-1)(2k-2)\binom{n-1}{k-1}\binom{n-2}{k-1}\binom{2n-4}{n-2} \neq 0.$$

Thus, the function $(W_{n-1}(z^2) - nzN_{n-1}(z^2))$ has the pole of order 2 at $z = 1$. Let us remember that $\mathcal{P}(\mathcal{I}_{2n}, z) = \dfrac{W_{n-1}(z^2) - nzN_{n-1}(z^2)}{(1-z)^{3n-1}(1+z)^{2n-1}}$. This implies that $\operatorname{tr}\deg_{\mathbb{C}} \mathcal{I}_{2n} = 3n-3$. □

Note that the proof of previous Theorem is direct. Luna's Slice Theorem (see [6]) gives us more general result.

We know explicit forms for the Poincaré series for the algebras of joint invariants and covariants of n linear forms. Thus we can prove the following statement.

Theorem 3. *The degrees of the algebras of joint covariants and invariants of n quadratic binary forms are equal to*

$$(i) \quad \deg(\mathcal{C}_{2n}, z) = \frac{\binom{2n-2}{n-1}}{2^{2n-1}},$$
$$(ii) \quad \deg(\mathcal{I}_{2n}, z) = \frac{\binom{2n-4}{n-2}}{(n-1)2^{2n-1}}.$$

Proof. (i) Using Theorem 1 and Theorem 2, we have:

$$\deg(\mathcal{C}_{2n}) = \lim_{z=1}(1-z)^{3n-1}\mathcal{P}(\mathcal{C}_n, z) = \lim_{z=1}(1-z)^{3n-1}\frac{\sum\limits_{k=0}^{n-1}\binom{n-1}{k}^2 z^{2k}}{(1-z)^n(1-z^2)^{2n-1}} = \frac{\binom{2n-2}{n-1}}{2^{2n-1}}.$$

(ii) Similarly, we have

$$\deg(\mathcal{I}_{2n}) = \lim_{z=1}(1-z)^{3n-3}\mathcal{P}(\mathcal{I}_n, z) = \lim_{z=1}\frac{W_{n-1}(z^2) - nzN_{n-1}(z^2)}{(1-z)^2(1+z)^{2n-1}}$$
$$= \lim_{z=1}\frac{\left(W_{n-1}(z^2) - nzN_{n-1}(z^2)\right)''}{\left((1-z)^2(1+z)^{2n-1}\right)''} = \frac{\binom{2n-4}{n-2}}{(n-1)2^{2n-1}}.$$

□

Note that asymptotically, the Catalan numbers grow as

$$C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

It is easy to calculate asymptotic behaviours of the degrees of the algebras $\mathcal{I}_{2n}$ and $\mathcal{C}_{2n}$:

Corollary 1. *Asymptotic behaviours of the degrees of the algebras of joint invariants and covariants of n quadratic binary forms as $n \to \infty$ are follows*

$$\deg(\mathcal{I}_{2n}) \sim \frac{1}{8\sqrt{\pi n^3}} \quad \textit{and} \quad \deg(\mathcal{C}_{2n}) \sim \frac{1}{2\sqrt{\pi n}}.$$

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Ми розглядаємо одну з фундаментальних проблем класичної теорії інваріантів – дослідження ряду Пуанкаре алгебр інваріантів групи Лі SL_2. Відомо, що перші доданки розкладу ряду Пуанкаре в ряд Лорана в околі точки $z = 1$ несуть важливу інформацію про структуру цієї алгебри. Для алгебри $\mathcal{I}_d = \mathbb{C}[V_d]^{SL_2}$ інваріантів однієї бінарної форми вони були обчислені ще Гільбертом (тут V_d– комплексний $d+1-$ вимірний векторний простір бінарних форм степеня d). Пізніше цей же результат отримав Спрінгер, використовуючи явну формулу для ряду Пуанкаре алгебри $\mathcal{I}_d$. Розглядається аналогічна задача для алгебр спільних інваріантів $\mathcal{I}_{2n} = \mathbb{C}[\underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}}]^{SL_2}$ та спільних коваріантів $\mathcal{C}_{2n} = \mathbb{C}[\underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}} \oplus \mathbb{C}^2]^{SL_2}$ n квадратичних форм. Ми виразили ряди Пуанкаре $\mathcal{P}(\mathcal{C}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{C}_{2n})_j z^j$ та $\mathcal{P}(\mathcal{I}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{I}_{2n})_j z^j$ цих алгебр через поліноми Нараяна. Також ми обчислили степені цих алгебр та асиптотичну поведінку цих степенів, використовуючи ці ряди Пуанкаре.

Ключові слова і фрази: класична теорія інваріантів, інваріанти, ряди Пуанкаре, комбінаторика.