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On of solving nonlinear Noether integral-differential boundary value problems by the of Newton-Kantorovich method

Constructive conditions for the existence of a nonlinear Noether integral-differential boundary value problem are found. An iterative scheme with quadratic convergence is constructed to find the solution of a nonlinear integral-differential boundary value problem based on the modification of the Newton-Kantorovich method. In order to justify the quadratic convergence of the modified Newton-Kantorovich method in the case of an undefined system, the original conditions of convergence are proposed.

Знайдені конструктивні умови розв'язності нелінійної нетерової інтегрально-диференціальної крайової задачі. Для розв'язання нелінійної інтегрально-диференціальної крайової задачі на основі методу Ньютона-Канторовича побудовано ітераційну схему з квадратичною збіжністю. Особливістю узагальнення методу Ньютона-Канторовича є його застосування до недовизначених систем.

1. Statement of the problem. We are investigating the problem of constructing a solution [1, 2]

$$y(t) \in \mathbb{D}^2[a;b], \quad y'(t) \in \mathbb{L}^2[a;b]$$

nonlinear Noether ($n \neq p$) integral-differential system

$$y'(t) = A(t)y(t) + \Phi(t)\int_a^b F(y(s), y'(s), s)\, ds + f(t), \tag{1}$$

that satisfy the boundary condition

$$\ell y(\cdot) = \alpha, \quad \alpha \in \mathbb{R}^p. \tag{2}$$

We seek a solution of the Noetherian boundary value problem (1), (2) in a small neighborhood of solution

$$y_0(t) \in \mathbb{D}^2[a;b], \quad y_0'(t) \in \mathbb{L}^2[a;b]$$

of the generating problem

$$y_0'(t) = A(t)y_0(t) + f(t), \quad \ell y_0(\cdot) = \alpha. \tag{3}$$

Here

$$A(t) \in \mathbb{L}^2_{n\times n}[a;b] := \mathbb{L}^2[a;b] \otimes \mathbb{R}^{n\times n}, \quad \Phi(t) \in \mathbb{L}^2_{n\times m}[a;b],\ f(t) \in \mathbb{L}^2[a;b];$$

$\ell y(\cdot) : \ \mathbb{D}^2[a;b] \to \mathbb{R}^p$ —linear bounded vector functional defined in space $\mathbb{D}^2[a;b]$ n-dimensional absolutely continuous on a segment $[a,b]$ functions. Nonlinear vector-function $F(y(t), y'(t), t)$ twice continuously differentiable in the small neighborhood of the solution $y_0(t)$ generating boundary value problem (3), twice continuously differentiable with respect to $y_0'(t)$, and continuous in the independent variable t

on the segment $[a, b]$. The study of the boundary value problem (1), (2) continues the investigation of the linear integral-differential boundary value problem [2] in the case of a nonlinear integral vector-function $F(y, y', t)$.

2. The generalization of the Newton-Kantorovich theorem. We investigate the problem of finding the solution $z \in \mathbb{R}^n$ of the nonlinear equation

$$\varphi(z) = 0. \tag{4}$$

We assume that the function

$$\varphi(z) : \mathbb{R}^n \to \mathbb{R}^m, \quad m \neq n$$

is twice continuously differentiable with respect to z in some domain $\Omega \subseteq \mathbb{R}^n$. To construct an iteration scheme $\{z_k\}$, that converges to the solution $\tilde{z} \in \mathbb{R}^n$, we use the Newton method [3–5]. Interest in the use of the Newton-Kantorovich method is associated with its effective application in solving nonlinear equations, as well as in the theory of nonlinear oscillations [3–6], including in the theory of non-linear Noetherian boundary value problems [1, 7, 8].

Suppose an approximation z_k is found that is sufficiently close to an exact solution $\tilde{z}$ of the equation (4). We expand the function $\varphi(z)$ in a neighborhood of the exact solution

$$\varphi(\tilde{z}) = \varphi(z_k) + \varphi'(z_k, \varepsilon)\Big(\tilde{z} - z_k\Big) + R(\xi_k,\, \tilde{z} - z_k), \tag{5}$$

where

$$R(\xi_k,\, \tilde{z} - z_k) := \int_0^1 (1-s)\, d^2\varphi(\xi_k\,; \tilde{z} - z_k)\, ds.$$

Here ξ_k is a point lying between the points $\tilde{z}$ and z_k. In a small neighborhood of the exact solution we have the approximate equality

$$\varphi(z_k) + \varphi'(z_k)\Big(\tilde{z} - z_k\Big) \approx 0,$$

therefore, in order to find the next approximation of z_{k+1} to the exact solution, it is natural to put

$$\varphi(z_k) + \varphi'(z_k)\Big(z_{k+1} - z_k\Big) = 0, \tag{6}$$

whence under the condition

$$P_{J_k^*} = 0, \quad J_k := \varphi'(z_k) \in \mathbb{R}^{m \times n} \tag{7}$$

we find

$$z_{k+1} = z_k - J_k^+\, \varphi(z_k), \quad k = 0,\ 1,\ 2,\ \dots\ . \tag{8}$$

Here $P_{J_k^*} : \mathbb{R}^m \to \mathbb{N}(J_k^*)$ is an orthogonal projector of the matrix $J_k^* \in \mathbb{R}^{n \times m}$ and J_k^+ is the pseudoinverse Moore-Penrose matrix [1, 9]. Note that condition (7) is equivalent to the requirement of completeness of the rank matrix J_k and is possible only in case $m \leq n$. We show that the iteration scheme (8) converges to the exact solution $\tilde{z}$. Suppose that in the neighborhood of the exact solution $\tilde{z}$ there are inequalities

$$\left|\left|J_k^+\right|\right| \le \sigma_1(k), \quad \left|\left|d^2\varphi(\xi_k\,; \tilde{z} - z_k)\right|\right| \le \sigma_2(k) \cdot ||\tilde{z} - z_k||^2$$

and note that it follows from the equalities (5) and (6) that

$$\varphi'(z_k, \varepsilon)\Big(\tilde{z} - z_k\Big) = -R(\xi_k,\, \tilde{z} - z_k),$$

so

$$||\tilde{z} - z_{k+1}|| \le \left|\left|J_k^+\right|\right| \cdot \left|\left|R(\xi_k,\, \tilde{z} - z_k)\right|\right| \le \frac{\sigma_1(k)\sigma_2(k)}{2} \cdot ||\tilde{z} - z_k||^2.$$

Let there be a constant

$$\theta := \sup_{k\in N}\left\{\frac{\sigma_1(k)\sigma_2(k)}{2}\right\}.$$

In this case, there is an estimate

$$|\tilde{z} - z_{k+1}| \le \theta \cdot |\tilde{z} - z_k|^2,$$

which holds that if the iteration scheme (8) converges to the exact solution $\tilde{z}$ of the equation (4), then this convergence is quadratic. Let us find the condition for the convergence of the iteration scheme (8) to the exact solution $\tilde{z}$ of the equation (4). To do this, we make estimates

$$|\tilde{z} - z_1| \le \theta \cdot |\tilde{z} - z_0|^2, \quad |\tilde{z} - z_2| \le \theta \cdot |\tilde{z} - z_1|^2 \le \theta^{1+2} \cdot |\tilde{z} - z_0|^{2^2},$$

$$|\tilde{z} - z_3| \le \theta \cdot |\tilde{z} - z_2|^2 \le \theta^{1+2+2^2} \cdot |\tilde{z} - z_0|^{2^3},\ \dots\ ,$$

$$|\tilde{z} - z_k| \le \theta \cdot |\tilde{z} - z_{k-1}|^2 \le \theta^{1+2+2^2+\ \dots\ +2^{k-1}} \cdot |\tilde{z} - z_0|^{2^k},\ \dots\ .$$

So there's an inequality [5]

$$|\tilde{z} - z_k| \le \theta^{\frac{2^k-1}{2-1}} \cdot |\tilde{z} - z_0|^{2^k} = \frac{1}{\theta} \cdot \Bigg(\theta \cdot |\tilde{z} - z_0|\Bigg)^{2^k},$$

indicating the convergence of the iterative process (8) to an exact solution $\tilde{z}$ of the equation (4) under condition

$$\theta \cdot |\tilde{z} - z_0| < 1. \tag{9}$$

In practice, the last inequality can be replaced by the following one:

$$\theta \cdot |z_k - z_0| < 1, \quad k = 1,\ 2,\ \dots\ .$$

Lemma. *Suppose that for the equation (4) the following conditions are satisfied.*

1) *A non-linear vector-function $f(z) : \mathbb{R}^n \to \mathbb{R}^m$, twice continuously differentiable with respect to z in some region $\Omega \subseteq \mathbb{R}^n$, in a neighborhood of the point z_0 has a root z^*.*

2) In the neighborhood of the zeroth approximation $z_0 \in \Omega \subseteq \mathbb{R}^n$ there are inequalities

$$\left|\left|J_k^+\right|\right| \leq \sigma_1(k), \left|\left|d^2\varphi(\xi_k\, ; \tilde{z} - z_k)\right|\right| \leq \sigma_2(k) \cdot ||\tilde{z} - z_k||, \quad k = 0,\ 1,\ 2,\ \dots\ . \tag{10}$$

3) The following constant exists

$$\theta := \sup_{k \in N} \left\{ \frac{\sigma_1(k)\sigma_2(k)}{2} \right\}.$$

Then, under conditions (7) and (9), to find the solution z^* of equation (4) the iteration scheme (8) is applicable, and the rate of convergence of the sequence $\{z_k\}$ to the solution z^* of equation (4) is quadratic.

Example 1. *The iterative scheme (8) is approximate for finding the solution of the non-linear equation (4), where the vector-function is as follows:*

$$\varphi(u) := \begin{pmatrix} x + \sin y + \cos z \\ y + \sin z + \cos x \end{pmatrix}, \quad u := \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This vector-function $\varphi(u) : \mathbb{R}^3 \to \mathbb{R}^2$ is defined in any open domain $D \subset \mathbb{R}^3$ and is twice continuously differentiable with respect to z in the neighborhood $\Omega \subseteq D \subset \mathbb{R}^3$. We set

$$u_0 := \begin{pmatrix} -0,45 & -0,45 & -0,45 \end{pmatrix},$$

wherein

$$\text{rank}\,[\varphi'(u_0)] = 2,$$

besides

$$u_1 \approx \begin{pmatrix} -0,455\ 961 & -0,457\ 894 & -0,455\ 547 \end{pmatrix}^*,$$

and

$$\text{rank}\,[\varphi'(u_1)] = 2,$$

Then

$$\left|\left| [\varphi'(u_1)]^+ \right|\right|_\infty := \sigma_1(1) \approx 2,09\ 903, \qquad \left|\left| d^2\varphi(u_1) \right|\right|_\infty := \sigma_2(1) \approx 0,897\ 838.$$

In this case, the weakened condition (9)

$$\theta_1 \cdot ||u_1 - u_0||_\infty \approx 0,00743\ 856 \ll 1, \quad \theta_1 := \frac{\sigma_1(1)\sigma_2(1)}{2} \approx 0,942\ 293$$

is satisfied. Since the condition (9) is satisfied for the first step of the iteration scheme (8), we find

$$u_2 \approx \begin{pmatrix} -0,455\ 968\ 239\ 769\ 595 \\ -0,457\ 889\ 951\ 795\ 185 \\ -0,455\ 537\ 594\ 550\ 856 \end{pmatrix}.$$

Then

$$\operatorname{rank}\left[\varphi'(u_2)\right] = 2,$$

besides

$$\left|\left|\left[\varphi'(u_2)\right]^+\right|\right|_\infty := \sigma_1(2) \approx 2,099, \qquad \left|\left|d^2\varphi(u_2)\right|\right|_\infty := \sigma_2(2) \approx 0,897\ 835.$$

In this case, the weakened condition (9)

$$\theta_2 \cdot ||u_2 - u_0||_\infty \approx 0,00743\ 453 \ll 1,$$

is satisfied, where

$$\theta_2 := \frac{\sigma_1(2)\,\sigma_2(2)}{2} \approx 0,00743\ 453.$$

For the second step of the iteration scheme (8) the discrepancy of the obtained approximation

$$||\varphi(u_2)||_\infty \approx 3,69\ 679 \times 10^{-11}$$

is sufficiently big, so we find

$$u_3 \approx \begin{pmatrix} 0,455\ 968\ 239\ 730\ 150 \\ 0,457\ 889\ 951\ 789\ 936 \\ 0,455\ 537\ 594\ 568\ 580 \end{pmatrix}.$$

Then

$$\operatorname{rank}\left[\varphi'(u_3)\right] = 2,$$

besides

$$\left|\left|\left[\varphi'(u_3)\right]^+\right|\right|_\infty := \sigma_1(3) \approx 2,099, \qquad \left|\left|d^2\varphi(u_3)\right|\right|_\infty := \sigma_2(3) \approx 0,897\ 835.$$

In this case, the weakened condition (9)

$$\theta_3 \cdot ||u_3 - u_0||_\infty \approx 0,00743\ 453 \ll 1$$

is satisfied, where

$$\theta_3 := \frac{\sigma_1(3)\,\sigma_2(3)}{2} \approx 0,942\ 278.$$

For the third step of the iteration scheme (8) the discrepancy of the obtained approximation is

$$||\varphi(u_3)||_\infty \approx 0,$$

so it's natural to confine with this approximation.

The Lemma just proved generalizes the corresponding results [4–8] to the case of matrix J_k irreversibility and can be used in the theory of non-linear Noetherian boundary-value problems [1, 7, 8], in the theory of matrix boundary-value problems [10], and also in the theory of matrix linear differential-algebraic boundary value problem [11–15].

3. Construction of solutions of the integral-differential boundary value problem. Let's denote $X(t)$ normal $(X(a) = I_n)$ the fundamental matrix of the homogeneous part of the generating system (3). As you know [1], in a critical case

$$P_{Q^*} \neq 0, \quad Q := \ell X(\cdot) \in \mathbb{R}^{p\times n}$$

generating boundary value problem (3) is solvable if and only if the condition is fulfilled

$$P_{Q_d^*}\Big\{\alpha - \ell K[f(s)](\cdot)\Big\} = 0; \tag{11}$$

with r-parametric family of solutions of a problem (3)

$$y_0(t, c_r) = X_r(t)c_r + G[f(s); \alpha](t), \quad c_r \in \mathbb{R}^r$$

depicted by a generalized Green operator

$$G[f(s); \alpha](t) := X(t)Q^+\Big\{\alpha - \ell K[f(s)](\cdot)\Big\} + K[f(s)](t).$$

Here

$$K\Big[f(s)\Big](t) := X(t)\int_a^t X^{-1}(s)f(s)\,ds$$

is Green's operator of the Cauchy problem $y_0(a) = c$ for the generating system (3), $X_r(t)$ — $(n \times r)$ – dimensional matrix, formed from r-linear-independent columns of a normal fundamental matrix $X(t)$; matrix $P_{Q_d^*} \in \mathbb{R}^{d\times p}$, formed from d line-independent rows of orthoprojector matrix

$$P_{Q^*} : \mathbb{R}^p \to \mathbb{N}(Q^*).$$

In the critical case, boundary value problem (1), (2) has r-parametric system of solutions

$$y(t) = y_0(t, c_r) + x(t), \quad c_r \in \mathbb{R}^r.$$

To find a deviation

$$x(t) \in \mathbb{D}^2[a; b], \quad x'(t) \in \mathbb{L}^2[a; b]$$

from the generative solution $y_0(t, c_r)$ we get the boundary value problem

$$x'(t) = A(t)x(t) + \Phi(t)\int_a^b F(y(s), y'(s), s)\,ds, \tag{12}$$

$$\ell x(\cdot) = 0. \tag{13}$$

Deviation

$$x(t) = X(t)v + \Psi(t)u$$

from the generative solution $y_0(t, c_r)$ identify unknown steel

$$u := \int_a^b F(y(s), y'(s), s)\,ds \in \mathbb{R}^m, \quad v \in \mathbb{R}^m$$

and the matrix

$$\Psi(t) := K[\Phi(s)](t) \in \mathbb{D}^2_{n\times m}[a; b].$$

Solution of the integral-differential system (1) satisfies the boundary condition (2)

$$y(t) = y_0(t, c_r) + x(t), \quad x(t) = X(t)v + \Psi(t)u$$

can be found with the use of the equation

$$Q\,v + R\,u = 0, \quad R := \ell\,\Psi(\cdot) \in \mathbb{R}^{p\times m}. \tag{14}$$

Let's denote $P_\rho \in \mathbb{R}^{(m+n)\times\rho}$ matrix formed from ρ linearly independent columns of orthoprojector P_0 matrix

$$[\,Q;\ R\,] \in \mathbb{R}^{p\times(m+n)}.$$

Condition (14) satisfying vectors

$$v = P_1\,c_\rho, \quad u = P_2\,c_\rho, \quad c_\rho \in \mathbb{R}^\rho;$$

here

$$P_0 := \text{ col }(\,P_1,\,P_2\,), \quad P_1 \in \mathbb{R}^{n\times\rho}, \quad P_2 \in \mathbb{R}^{m\times\rho}.$$

To find the vector c_ρ, necessary to identify unknowns $u(c_\rho)$ and $v(c_\rho)$ we obtain a nonlinear equation (4); here

$$\varphi(c_\rho) := \varphi(u(c_\rho), v(c_\rho)) := u(c_\rho) -$$

$$-\int_a^b F(X(s)v + \Psi(s)u, A(s)X(s)v + \Phi(s)u, s)\,ds.$$

If for the obtained equation (4) the conditions of the Lemma are fulfilled, we find the unknown $u(c_\rho)$ та $v(c_\rho)$. Thus, the following theorem is proved.

Theorem. *In the critical case ($P_{Q^*} \neq 0$) generating boundary value problem (3) is solvable if and only if the condition is fulfilled (11); with r-parametric family of solutions of the generating problem (3) looks like*

$$y_0(t, c_r) = X_r(t)c_r + G[f(s);\alpha](t), \quad c_r \in \mathbb{R}^r.$$

The wanted solution

$$y(t) = y_0(t, c_r) + x(t), \quad x(t) = X(t)v + \Psi(t)u, \quad v = P_1\,c_\rho, \quad u = P_2\,c_\rho, \quad c_\rho \in \mathbb{R}^\rho$$

nonlinear integral-differential boundary value problem (1), (2) defines a vector c_ρ, which satisfies the nonlinear equation (4): $\varphi(c_\rho) = 0$, the conditions of solvability of which is determined by the Lemma. To find the vector c_ρ applicable iterative scheme (8), the rate of convergence of the sequence of approximations to the solution of the equation (4) quadratic.

In the partial case of the equation (4) is linear:

$$\varphi(c_\rho) := \mathcal{B}\,c_\rho + d, \quad \mathcal{B} \in \mathbb{R}^{m\times\rho},$$

the condition of solvability which is equivalent to demand

$$P_{\mathcal{B}^*} = 0. \tag{15}$$

Here $P_{\mathcal{B}^*}: \mathbb{R}^m \to \mathbb{N}(\mathcal{B}^*)$ — orthoprojector matrix $\mathcal{B}^*$. For terms (15) solution of the linear equation (4) looks like

$$c_\rho = P_J\, c_\mu - J^+ d, \quad c_\mu \in \mathbb{R}^\rho;$$

here $P_J: \mathbb{R}^\rho \to \mathbb{N}(J)$ — orthoprojector matrix $\mathcal{B}$.

Example 2. *The requirements of the Theorem satisfy the problem of constructing 2π-periodic solutions of a nonlinear integral-differential system*

$$y' = A(t)y + \Phi(t) \int_0^{2\pi} F(y(s), y'(s), s)\, ds + f(t); \tag{16}$$

here

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f(t) := \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad \Phi(t) := \begin{pmatrix} \cos^3 t & 0 \\ \cos^3 t & 0 \end{pmatrix},$$

in addition

$$F(y(t), y'(t), t) := y(t)(1 - y^*(t)y'(t)) \cos t.$$

For a generating periodic boundary value problem in the case of a system (16)

$$P_Q = P_{Q^*} = I_2 \neq 0,$$

therefore there is a critical case, and the condition of solvability (11) is fulfilled, with this two-parameter family of solutions of the generating problem

$$y_0(t, c_r) = X_r(t)c_r + G[f(s)](t), \quad c_r \in \mathbb{R}^2$$

depicted by a generalized Green operator

$$G[f(s)](t) = \begin{pmatrix} \sin t + \sin 3t \\ \cos t - \cos 3t \end{pmatrix}$$

and a normal fundamental matrix of the homogeneous part of the generating system (16)

$$X(t) = X_r(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Because

$$Q = 0, \quad R = -\frac{3\pi}{4} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

get matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

Periodic solution

$$y(t) = y_0(t, c_r) + x(t), \quad x(t) = X(t)v + \Psi(t)u, \quad v = P_1\, c_\rho, \quad u = P_2\, c_\rho, \quad c_\rho \in \mathbb{R}^3$$

nonlinear integral-differential system (16) defines a vector c_ρ, which satisfies the linear equation (4):

$$\varphi(c_\rho) := \mathcal{B}\, c_\rho + d;$$

here

$$\mathcal{B} = \frac{1}{8}\begin{pmatrix} -8\pi & 0 & \pi \\ -\pi & 8 & -8\pi \end{pmatrix},$$

in addition

$$d(c_r) = \frac{\pi}{32}\begin{pmatrix} 1 - 32\,c_1 + 4\,c_2 \\ -4\,(2 + c_1 + 8\,c_2) \end{pmatrix},\ c_r := \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The last equation is solvable due to the completeness of the matrix rank $\mathcal{B}$. Let's put it $c_\mu := 0$; the desired 2π-periodic solution of a nonlinear integral-differential system (16) looks like

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix};$$

here

$$y_1(t) = \frac{3\ 862\ 708\ \cos t}{1\ 383\ 908\ 045} + \frac{124\ 805\,c_1\cos t}{89\ 428\ 786} + \frac{12\ 394\ 757\,c_2\cos t}{1\ 110\ 179\ 952} +$$

$$+\frac{12\ 053\ 843\sin t}{539\ 822\ 396} + \frac{12\ 394\ 757\,c_1\sin t}{1\ 110\ 179\ 952} + \frac{11\ 712\ 929\,c_2\sin t}{131\ 138\ 704} + \frac{\sin 3t}{4},$$

$$y_2(t) = \frac{12\ 053\ 843\cos t}{539\ 822\ 396} + \frac{12\ 394\ 757\,c_1\cos t}{1\ 110\ 179\ 952} + \frac{11\ 712\ 929\,c_2\cos t}{131\ 138\ 704} - \frac{\cos 3t}{4} -$$

$$-\frac{3\ 862\ 708\sin t}{1\ 383\ 908\ 045} - \frac{124\ 805\,c_1\sin t}{89\ 428\ 786} - \frac{12\ 394\ 757\,c_2\sin t}{1\ 110\ 179\ 952}.$$

In a non-critical case ($P_{Q^*} = 0$) solvability condition (11) of integral-differential boundary value problem (1), (2) is satisfied for all inhomogeneities $f(t) \in \mathbb{L}^2[a;b]$ and $\alpha \in \mathbb{R}^p$. This is a typical case for Noether boundary-value problems [1, 2].

Consequence. *In a non-critical case ($P_{Q^*} = 0$) generating problem (3) has an r-parametric family of solutions*

$$y_0(t, c_r) = X_r(t)c_r + G[f(s);\alpha](t), \quad c_r \in \mathbb{R}^r.$$

Then, under condition (14) solutions

$$y(t) = y_0(t, c_r) + x(t), \quad x(t) = X(t)v + \Psi(t)u, \quad v = P_1\,c_\rho, \quad u = P_2\,c_\rho, \quad c_\rho \in \mathbb{R}^\rho$$

nonlinear Noether integral-differential boundary value problem (1), (2) defines a vector c_ρ, which satisfies the nonlinear equation (4): $\varphi(c_\rho) = 0$, the conditions of solvability of which is determined by the Lemma. To find the vector c_ρ applicable iterative scheme (8), the rate of convergence of the sequence of approximations to the solution of the equation (4) quadratic.

Example 3. *The requirements of the proved Consequence satisfy the problem of construction 2π-periodic solutions of the scalar integral-differential system*

$$y' = y + \Phi(t)\int_0^{2\pi} F(y(s), y'(s), s)\,ds + f(t); \tag{17}$$

here

$$f(t) := \sin t, \quad \Phi(t) := \cos t,$$

in addition

$$F(y(t), y'(t), t) := (1 - y^2(t)) \left(1 - y'^2(t)\right).$$

For a generating boundary-value problem in the case of a system (17) there is a non-critical case, with the unique solution

$$y_0(t) = G[f(s)](t);$$

here

$$G[f(s)](t) = -\frac{\sin t + \cos t}{2}$$

is generalized Green operator. Periodic solution

$$y(t) = y_0(t) + x(t), \quad x(t) = X(t)v + \Psi(t)u, \quad u = -2\,v, \quad v \in \mathbb{R}^1$$

nonlinear integral-differential system (17) determines the constant v, which satisfies the nonlinear equation (4):

$$\varphi(v) = -2v - \frac{17\,\pi}{16} + \frac{7\,v^2\pi}{2} - \pi v^4.$$

To find a constant v applicable iterative scheme (8), indeed: for terms

$$v_0 := -\frac{1}{2}$$

we get

$$J_0 = -2 - 3\pi \neq 0, \quad v_1 = -\frac{7\,\pi}{8 + 12\,\pi},$$

with

$$J_1 = -\frac{256 + 1152\,\pi + 2512\,\pi^2 + 3216\,\pi^3 + 1421\,\pi^4}{16\,(2 + 3\,\pi)^3},$$

$$\sigma_1(1) \approx 0,0\ 894\ 281,$$

in addition

$$\varphi(v_1) \approx 0,0894\ 281.$$

Similarly

$$\sigma_2(1) \approx 13,2612, \quad \theta_1 \approx 0,285\ 343 < 1,$$

hence the condition of convergence of the iterative scheme (8) to the exact solution of the equation (4) the first step is complete. At the second step of the iterative scheme (8) we get

$$v_2 = \frac{\pi}{w}(\,4352 + 26\ 112\,\pi + 69\ 728\,\pi^2 + 91\ 680\,\pi^3 + 39\ 525\,\pi^4),$$

with

$$J_2 \approx -11,1795 \neq 0, \quad \sigma_1(2) \approx 0,0894\ 495,$$

in addition

$$\varphi(v_2) \approx 2,70\ 456 \times 10^{-7}.$$

Here

$$w := 16(2+3\,\pi)(256+1152\,\pi+2512\,\pi^2+3216\,\pi^3+1421\pi^4).$$

Similarly

$$\sigma_2(2) \approx 13,2685, \quad \theta_2 \approx 0,285\ 449 < 1,$$

hence the condition of convergence of the iterative scheme (8) to the exact solution of the equation (4) the first step is complete. On the third step of the iterative scheme (8)we get

$$v_3 \approx -\frac{149\ 078\ 559}{309\ 925\ 526},$$

with

$$\sigma_1(3) \approx 0,0894\ 496, \quad \sigma_2(3) \approx 13,2685.$$

Because

$$\theta_3 \approx 0,285\ 449 < 1,$$

condition of convergence of the iterative scheme (8) to the exact solution of the equation (4) the third step is completed; in addition

$$\varphi(v_3) \approx 3,99\ 680 \times 10^{-15}.$$

Thus, an approximation to a periodic solution is obtained

$$y(t) \approx -\frac{160\ 106\ 464\ \cos t}{163\ 205\ 053} - \frac{20\ 751\ 201\ \sin t}{1\ 092\ 981\ 631}$$

integral-differential system (17). To evaluate the accuracy of the approximation found to the periodic solution of the integral-differential system (17) define an deviation

$$\left|\left| y' - y - \Phi(t) \int_0^{2\pi} F(y(s), y'(s), s)\, ds - f(t) \right|\right|_{C[0;2\pi]} \approx 3,44\ 169 \times 10^{-15}.$$

In addition, we note the periodicity of the approximation obtained.

In the case of insolubility of a nonlinear integral-differential boundary value problem (1), (2) it can be regularize the same way [16–19]. We also note that the scheme proposed in the paper for the study of a nonlinear integral-differential boundary value problem (1), (2) can be transferred to the integral-differential boundary-value problem with delay [1, 20, 21].

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