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REMOVABLE ISOLATED SINGULARITIES FOR SOLUTIONS OF ANISOTROPIC EVOLUTION p-LAPLACIAN EQUATION

In this paper we study nonnegative weak solutions of a quasilinear parabolic equations in a divergent form, one of the model cases of which is anisotropic evolution p-Laplacian equation. New pointwise estimates near an isolated singularity and sufficient condition of the removability of isolated singularity were obtained.

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1. Introduction and main results.

In this article we study solutions of the quasilinear parabolic equation in the divergent form

$$u_t - div\, A(x,t,u,\nabla u) = b(x,t,u,\nabla u), \quad (x,t) \in \Omega_T \setminus (x_0, 0), \tag{1}$$

satisfying a initial condition

$$u(x,0) = 0 \qquad x \in \Omega \setminus \{x_0\}, \tag{2}$$

where Ω is a bounded domain in R^n, $n \geq 3$, $x_0 \in \Omega$, $\Omega_T := \Omega \times (0,T)$, $0 < T < \infty$.

Throughout the article, we suppose that the functions $A : \Omega_T \times R \times R^n \longrightarrow R^n$ and $b : \Omega_T \times R \times R^n \longrightarrow R^n$ are such that $A(\cdot,\cdot,u,\varsigma)$, $b(\cdot,\cdot,u,\varsigma)$ are Lebesque measurable for all $u \in R, \varsigma \in R^n$, and $A(x,t,\cdot,\cdot)$, $b(x,t,\cdot,\cdot)$ are continuous for almost all $(x,t) \in \Omega_T$, $A = (a_1, a_2, ...a_n)$. We also assume that the following structure conditions are satisfied:

$$\sum_{i=1}^{n} a_i(x,t,u,\varsigma)\varsigma_i \geqslant \nu_1 \sum_{i=1}^{n} |\varsigma_i|^{p_i},$$

$$|a_i(x,t,u,\varsigma)| \leqslant \nu_2 \left(\sum_{j=1}^{n} |\varsigma_j|^{p_j} \right)^{1-\frac{1}{p_i}}, \quad i = \overline{1,n}, \tag{3}$$

$$|b(x,t,u,\varsigma)| \leqslant \nu_2 \sum_{i=1}^{n} |\varsigma_i|^{p_i\left(1-\frac{1}{p}\right)}$$

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with some positive constants ν_1, ν_2.

We further suppose that the following conditions are satisfied:

$$2n/(n+1) < p_1 \leqslant p_2 \leqslant ....p_n, \quad \max_{1\leqslant i \leq n} < 2 + \frac{k}{n}, \tag{4}$$

where some p_i can be less than 2 (so called "singular" case), the other p_i can be greater than 2 (so called "degenerate" case), and

$$k = n(p-2) + p, \quad \frac{1}{p} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}. \tag{5}$$

Let us denote by $D(r), r > 0$ the following set

$$D(r) = \left\{(x,t) \in \Omega_T : \sum_{i=1}^{n}\left(\frac{|x_i - x_i^{(0)}|}{r^{k_i}}\right)^{p_i} + \frac{t}{r^k} \leq 1\right\}, \tag{6}$$

where

$$k_i = \frac{p + n(p-p_i)}{p_i}. \tag{7}$$

We formulate the removability result in the term of behavior of the function

$$M(r) = \operatorname{ess\,sup}\{| u(x,t) | : (x,t) \in D(R_0) \setminus D(r)\}, \tag{8}$$

where R_0 is some sufficiently small fixed positive number such that $D(R_0) \subset \Omega_T$. It follows from [1] that $M(r)$ is finite number for any $r > 0$.

Let us remind the reader the definition of a weak solution to (1), (2).

Definition 1. We will write $V_p(\Omega_T)$ for the class of functions $\varphi \in C(0,T,L_2(\Omega)) \cap L_p(0,T;W^{1,\overline{p}}(\Omega))$ with $\sum\limits_{i=1}^{n}\iint\limits_{\Omega_T}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i} dxdt < \infty$ and $\overset{o}{V}_p(\Omega_T) = \{\varphi \in V_p(\Omega_T) : \varphi|_{(0,T)\times\partial\Omega} = 0\}$, where $W^{1,\overline{p}}(\Omega) = \left\{\varphi \in W^{1,1}(\Omega) : \frac{\partial\varphi}{\partial x_i} \in L_{p_i}(\Omega),\ i = \overline{1,n}\right\}$.

Definition 2. We say that $u(x,t) \geqslant 0$ is a weak solution to the problem (1), (2) if for an arbitrary function $\psi \in C^\infty(\Omega_T)$, vanishing in a neighborhood of $(x_0, 0)$, we have an inclusion $u\psi \in V_p(\Omega_T)$ and the integral identity

$$\int\limits_{\Omega}(u\varphi\psi)(\cdot,\tau)dx - \int\limits_0^\tau\int\limits_\Omega u\frac{\partial(\varphi\psi)}{\partial t}dxdt +$$

$$+\sum_{i=1}^{n}\int\limits_0^\tau\int\limits_\Omega a_i\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial(\varphi\psi)}{\partial x_i}dxdt - \int\limits_0^\tau\int\limits_\Omega b\left(x,t,u,\frac{\partial u}{\partial x}\right)\varphi\psi\, dxdt = 0 \tag{9}$$

holds for every $\varphi \in \overset{o}{V}_p(\Omega_T)$ and for all $\tau \in (0,T)$.

The properties of the above-mentioned Sobolev spaces can be viewed in [1–4].

Many authors researched problems of singularities for solutions of parabolic equations with Laplace of p-Laplace operators in principle part. Review of these results can be found in monograph [5]. We are concerned here to find pointwise conditions on solutions of (1), (2) to guarantee that the singularity at $(x_0, 0)$ is removable. And in the following sections we prove that the singularity at the point $(x_0, 0)$ is removable (that is the integral identity (9) holds for $\psi \equiv 1$) if

$$\lim_{r\to 0} M(r) r^n = 0. \tag{10}$$

The equation

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 0 \tag{11}$$

is the simplest model of equations from class (1). For nonnegative solutions of problems (1), (2) with $p_1 = p_2 = \ldots = p_n = p = 2$ the result analogous to (10) follows from the paper of D.G. Aronson and J. Serrin [6]. In isotropic case ($2 < p_1 = p_2 = \ldots = p_n = p$) condition for removable singularity was obtained by F. Nicolosi, I.I. Skrypnik, I.V. Skrypnik [7]. For anisotropic evolution p-Laplacian equation (11) local boundedness of solutions has been obtained in [2] and the question of removability of isolated singularity with $p_i > 2$ was studied in [8]. The important difference our research lies in the fact that part of $p_i < 2$ and another part of $p_i > 2$.

Let's formulate the main results:

Theorem 1. *Assume that conditions* (3), (4) *are fulfilled. Let u be a weak solution of the problem* (1), (2). *Then the singularity of solution u at the point* $(x_0, 0)$ *is removable if condition* (10) *is satisfied.*

Using the standard approach, the proof of Theorem 1 is obtained from the following two theorems.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then there exists a positive constants K_1, β depending only on $\nu_1, \nu_2, n, p_1, ..., p_n, p, R_0$ such that the following inequality is true*

$$M(\rho) \leqslant K_1 \rho^{-n+\beta},\ 0 < \rho < R_0. \tag{12}$$

Theorem 3. *Let the conditions of Theorem 1 be fulfilled. Then there exists a positive constant K_2 depending only on $\nu_1, \nu_2, n, p_1, ..., p_n, p, R_0$ such that the following inequality is valid*

$$|u(x,t)| \leqslant K_2,\ \ \forall (x,t) \in D\left(\frac{R_0}{2}\right). \tag{13}$$

The rest of the paper contains the proof of the above theorems.

2. Auxiliary integral estimates of solutions.

We accomplish proof of the main result in the several steps. First, we obtain some integral estimates of the solution under consideration. Without loss of generality it can be assumed that the function $M(r)$ defined by (8) satisfies

$$\lim_{r\to 0} M(r)=\infty \tag{14}$$

and fixe sufficiently small the number $R_0 : M(R_0)\geq 1$.

For every $\rho : 2\rho \leq R_0$ we define $u_{2\rho}(x,t)=(u(x,t)-M(2\rho))_+$, $E_{2\rho}=\{(x,t)\in D(2\rho) : u(x,t)>M(2\rho)\}$.

We further denote by γ constant, depending only on the known parameters ν_1,ν_2,n, $p_1,...,p_n,p,R_0$, which may vary from line to line.

Let $\eta_r\in C^\infty(\Omega_T)$ be the cut-off function such that

(i) $0\leq \eta_r(x,t)\leq 1$ in Ω_T,

(ii) $\eta_r\equiv 0$ in $D(r)$, $\eta_r\equiv 1$ outside $D(2r)$,

(iii)$\left|\frac{\partial\eta_r}{\partial t}\right|\leqslant \gamma r^{-k}$, $\left|\frac{\partial\eta_r}{\partial x_i}\right|\leqslant \gamma r^{-k_i}$, where $k,k_i, i=\overline{1,n}$ are defined by (5), (7).

Introduce the following notations:

$$\Phi_{\rho,2\rho}(u(x,t))=\min\{u_{2\rho}, M(\rho)-M(2\rho)\},$$

$$E(\rho,2\rho)=\{(x,t)\in\Omega_T : 0<u_{2\rho}(x,t)<M(\rho)-M(2\rho)\},$$

$$\varepsilon(r)=M(r)r^n+r^2(M(r)r^n)^{p-1}+\sum_{i=1}^n (M(r)r^n)^{p_i-1}.$$

Lemma 1. *Let the conditions of Theorem 1 be fulfilled. Then the following inequality is valid for every* $r : 0<r<\rho<R_0$

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_\Omega u^2(x,t)\eta_r^2(x,t)dx+\sum_{i=1}^n\iint\limits_{E_{2\rho}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2 dxdt\leqslant \gamma M(r)\varepsilon(r). \tag{15}$$

Proof. Testing integral identity (9) by

$$\varphi(x,t)=u_{2\rho}(x,t)\eta_r(x,t),\quad \psi(x,t)=\eta_r(x,t).$$

Applying structural conditions (3), we obtain:

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_\Omega u_{2\rho}^2(x,t)\eta_r^2(x,t)dx+\iint\limits_{E_{2\rho}}\sum_{i=1}^n\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2 dxdt\leqslant$$

$$\leqslant \gamma\sum_{i=1}^n\iint\limits_{E_{2\rho}} u_{2\rho}\left(\sum_{j=1}^n\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}\left|\frac{\partial\eta_r}{\partial x_i}\right|\eta_r\, dxdt+$$

$$+\gamma\sum_{i=1}^n\iint\limits_{E_{2\rho}} u_{2\rho}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)^{1-\frac{1}{p}}\eta_r^2\, dxdt+\gamma\iint\limits_{E_{2\rho}} u_{2\rho}^2\left|\frac{\partial\eta_r}{\partial t}\right|\eta_r dxdt.$$

Now we use Hölder's and Young's inequalities:

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_{\Omega} u_{2\rho}^2(x,t)\eta_r^2(x,t)dx+\sum_{i=1}^n\iint\limits_{E_{2\rho}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2 dxdt\leqslant$$

$$\leqslant\gamma\sum_{i=1}^n\iint\limits_{E_{2\rho}}u_{2\rho}^{p_i}\left|\frac{\partial\eta_r}{\partial x_i}\right|^{p_i}dxdt+\gamma\iint\limits_{E_{2\rho}}u_{2\rho}^p\eta_r^2dxdt+\gamma\iint\limits_{E_{2\rho}}u_{2\rho}^2\left|\frac{\partial\eta_r}{\partial t}\right|\eta_r dxdt. \qquad (16)$$

Using the definitions of $M(r)$ and $\eta_r(x,t)$, we get:

$$\sum_{i=1}^n\iint\limits_{E_{2\rho}}u_{2\rho}^{p_i}\left|\frac{\partial\eta_r}{\partial x_i}\right|^{p_i}dxdt+\iint\limits_{E_{2\rho}}u_{2\rho}^2\left|\frac{\partial\eta_r}{\partial t}\right|\eta_r dxdt$$

$$\leqslant\gamma\left(\sum_{i=1}^n M^{p_i}(r)r^{-p_ik_i}+M^2(r)r^{-k}\right)|D(2r)\setminus D(r)|\leqslant$$

$$\leqslant\gamma M(r)\left(\sum_{i=1}^n(M(r)r^n)^{p_i-1}+M(r)r^n\right)=\gamma M(r)\varepsilon(r). \qquad (17)$$

Estimating the second term in the right-hand side of (16), we derive

$$\iint\limits_{E_{2\rho}}u_{2\rho}^p\eta_r^2dxdt\leqslant\gamma\iint\limits_{E_{2\rho}\setminus D(r)}u_{2\rho}^p\eta_r^2dxdt\leqslant\gamma M(r)(M(r)r^n)^{p-1}=\gamma M(r)\varepsilon(r). \qquad (18)$$

Combining estimates (16)–(18), we obtain the required estimate (15). □

Integral estimates (15) is used in the proof of the following lemma.

Lemma 2. *Let the conditions of Theorem 1 be fulfilled. Then the following estimate is true*

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_{\Omega}\Phi_{\rho,2\rho}^2(u(x,t))\eta_r^2(x,t)dx+\sum_{i=1}^n\iint\limits_{E(\rho,2\rho)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2dxdt\leqslant$$

$$\leqslant\gamma(M(\rho)-M(2\rho))\sum_{i=1}^n\iint\limits_{E_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2dxdt+$$

$$+\gamma\varepsilon^2(r)(M(\rho)-M(2\rho))+\gamma M(r)\varepsilon(r)+\gamma(M(\rho)-M(2\rho))^{2-\delta_1}, \qquad (19)$$

where $\delta_1\in\left(0,\frac{k}{n}\right), 0<r<\rho<R.$

Proof. Testing (9) by

$$\varphi(x,t)=\Phi_{\rho,2\rho}(u(x,t))\eta_r(x,t),\qquad \psi(x,t)=\eta_r(x,t).$$

Using structural inequalities (3), we have

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_{\Omega}\Phi^2_{\rho,2\rho}(u(x,t))\eta_r^2(x,t)dx+\sum_{i=1}^{n}\iint\limits_{E_{(\rho,2\rho)}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2dxdt\leqslant$$

$$\leqslant\gamma(M(\rho)-M(2\rho))\sum_{i=1}^{n}\iint\limits_{E_\rho}\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}\left|\frac{\partial\eta_r}{\partial x_i}\right|\eta_r dxdt+$$

$$+\gamma\sum_{i=1}^{n}\iint\limits_{E_\rho}\Phi_{\rho,2\rho}(u(x,t))\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2dxdt+\gamma\iint\limits_{E_\rho}\Phi^2_{\rho,2\rho}(u(x,t))\left|\frac{\partial\eta_r}{\partial t}\right|\eta_r dxdt. \quad (20)$$

Applying Hölder's inequality and Lemma 1 to the first term on the right-hand side of estimate (20), we derive

$$\sum_{i=1}^{n}\iint\limits_{E_\rho}\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}\left|\frac{\partial\eta_r}{\partial x_i}\right|\eta_r^2dxdt\leqslant$$

$$\leqslant\sum_{i=1}^{n}\left(\iint\limits_{E_\rho}\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\eta_r^2dxdt\right)^{\frac{p_i-1}{p_i}}\left(\iint\limits_{E_\rho}\left|\frac{\partial\eta_r}{\partial x_i}\right|^{p_i}\eta_r^2dxdt\right)^{\frac{1}{p_i}}\leqslant$$

$$\leqslant\gamma\sum_{i=1}^{n}r^{\frac{n+k}{p_i}-k_i}\left(\sum_{j=1}^{n}\iint\limits_{E_\rho}\left|\frac{\partial u}{\partial x_j}\right|^{p_i}\eta_r^2dxdt\right)^{1-\frac{1}{p_i}}\leqslant$$

$$\leqslant\gamma\sum_{i=1}^{n}r^{\frac{n(p_i-1)}{p_i}}\left[M(r)\varepsilon(r)\right]^{1-\frac{1}{p_i}}\leqslant\gamma\varepsilon^2(r). \quad (21)$$

Estimating the third term on the right-hand side of (20), we get

$$\iint\limits_{E_\rho}\Phi^2_{\rho,2\rho}(u(x,t))\left|\frac{\partial\eta_r}{\partial t}\right|\eta_r dxdt\leqslant\gamma M(r)\varepsilon(r). \quad (22)$$

The second term on the right-hand side of (20) can be represented in the following way:

$$\gamma\sum_{i=1}^{n}\iint\limits_{E_\rho}\Phi_{\rho,2\rho}(u(x,t))\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2dxdt\leqslant\sum_{i=1}^{n}\iint\limits_{E_\rho}\Phi_{\rho,2\rho}(u(x,t))\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2dxdt+$$

$$+\sum_{i=1}^{n}\iint\limits_{E_{(\rho,2\rho)}}\Phi_{\rho,2\rho}(u(x,t))\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2dxdt. \tag{23}$$

Estimating the second integral on the right-hand of the last inequality, we have

$$\sum_{i=1}^{n}\iint\limits_{E_{(\rho,2\rho)}}\Phi_{\rho,2\rho}(u(x,t))\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2dxdt\leqslant$$

$$\leqslant\sum_{i=1}^{n}\iint\limits_{E_{(\rho,2\rho)}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2dxdt+\iint\limits_{E_{(\rho,2\rho)}}\Phi^p_{\rho,2\rho}(u(x,t))\eta_r^2dxdt\leqslant$$

$$\leqslant\gamma\sum_{i=1}^{n}\iint\limits_{E_{(\rho,2\rho)}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2dxdt+\gamma(M(\rho)-M(2\rho))^{2-\delta_1}. \tag{24}$$

Combining estimates (20)–(24), we obtain (19). This completes the proof of the Lemma 2. □

Lemma 3. *Let the conditions of Theorem 1 be fulfilled. Then the following estimate*

$$\sum_{i=1}^{n}\iint\limits_{E_\rho}u_{2\rho}^{-q}\left|\frac{\partial u}{\partial x_i}\right|^2\eta_r^2dxdt\leq$$

$$\leq\gamma(M(\rho)-M(2\rho))^{2(1-q)}+\gamma(M(\rho)-M(2\rho))^{1-q}\varepsilon(r) \tag{25}$$

holds with $1<q<1+\min\left[\frac{2}{n},\frac{1}{2}\right]$.

Proof. Testing identity (9) by

$$\varphi=\left([M(\rho)-M(2\rho)]^{1-q}-[max(u_{2\rho},M(\rho)-M(2\rho))]^{1-q}\right)_+\eta_r,\qquad \psi=\eta_r.$$

Using conditions (3) and Young's inequality, we obtain

$$\sum_{i=1}^{n}\iint\limits_{E_\rho}\sum_{i=1}^{n}u_{2\rho}^{-q}\left|\frac{\partial u}{\partial x_i}\right|^2\eta_r^2dxdt\leqslant$$

$$\leqslant\gamma(M(\rho)-M(2\rho))^{1-q}\sum_{i=1}^{n}\iint\limits_{E_\rho}\left|\frac{\partial\eta_r}{\partial x_i}\right|\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}\eta_rdxdt+$$

$$+\gamma(M(\rho)-M(2\rho))^{1-q}\sum_{i=1}^{n}\iint\limits_{E_\rho}\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_i}\right)^{1-\frac{1}{p}}\eta_r^2dxdt+$$

$$+\gamma\left(M(\rho)-M(2\rho)\right)^{2(1-q)}\iint\limits_{E_\rho}\eta_r\left|\frac{\partial\eta_r}{\partial t}\right|dxdt. \tag{26}$$

Estimating the terms on the right-hand side of (26), we obtain

$$\sum_{i=1}^{n}\iint\limits_{E_\rho}\left|\frac{\partial\eta_r}{\partial x_i}\right|\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}\eta_r dxdt\leqslant$$

$$\leqslant\sum_{i=1}^{n}\left(\iint\limits_{E_\rho}\left|\frac{\partial\eta_r}{\partial x_i}\right|^{p_i}\eta_r^2 dxdt\right)^{\frac{1}{p_i}}\left(\sum_{j=1}^{n}\iint\limits_{E_\rho}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\eta_r^2 dxdt\right)^{1-\frac{1}{p_i}}\leqslant$$

$$\leqslant\gamma\sum_{i=1}^{n}r^{\frac{n(p_i-1)}{p_i}}\left(\gamma M(r)\varepsilon(r)\right)^{\frac{p_i-1}{p_i}}\leq\gamma\varepsilon(r) \tag{27}$$

$$\left(M(\rho)-M(2\rho)\right)^{1-q}\sum_{i=1}^{n}\iint\limits_{E_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2 dxdt\leqslant$$

$$\leqslant\left(M(\rho)-M(2\rho)\right)^{1-q}\iint\limits_{E_\rho}u_{2\rho}^{q(p-1)}\eta_r^2 dxdt+\sum_{i=1}^{n}\iint\limits_{E_\rho}u_{2\rho}^{-q}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2 dxdt. \tag{28}$$

Combining estimates (26)–(28), we derive required inequality (25). □

Combining lemmas 2 and 3, we obtain:

Lemma 4. *Let the conditions of Theorem 1 be fulfilled. Then the following estimate is true*

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_{\Omega}\Phi_{\rho,2\rho}^2(u(x,t))\eta_r^2(x,t)dx+\sum_{i=1}^{n}\iint\limits_{E(\rho,2\rho)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2 dxdt\leqslant$$

$$+\gamma\varepsilon^2(r)(M(\rho)-M(2\rho))+\gamma M(r)\varepsilon(r)+\gamma(M(\rho)-M(2\rho))^{2-\delta_1}+$$

$$\gamma(M(\rho)-M(2\rho))^{2-\delta_2}+\gamma(M(\rho)-M(2\rho))^{2-q}\varepsilon(r), \tag{29}$$

where $0<\delta_1,\delta_2<1$.

Proof. By the Young's inequality, we have

$$\sum_{i=1}^{n}\iint\limits_{E_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i\left(1-\frac{1}{p}\right)}\eta_r^2 dxdt\leqslant$$

$$\leqslant\gamma\sum_{i=1}^{n}\iint\limits_{E_\rho}u_{2\rho}^{-q}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\eta_r^2 dxdt+\gamma\iint\limits_{E_\rho}u_{2\rho}^{q(p-1)}\eta_r^2 dxdt. \tag{30}$$

We estimate the last integral on the right-hand side of (30) similarly to (18)

$$\iint\limits_{E_\rho} u_{2\rho}^{q(p-1)}\eta_r^2 dxdt \leqslant \gamma. \tag{31}$$

Combining inequalities (19), (25), (30), (31), we obtain estimate (29). □

Taking into account condition (10), we can pass to the limit as $r \to 0$ in (29) and obtain the following statement.

REMARK 1. Let the conditions of Theorem 1 be fulfilled. Then the following estimate is valid

$$\operatorname*{ess\,sup}_{0<t<T} \int\limits_{\Omega} \Phi_{\rho,2\rho}^2(u(x,t))dx + \sum_{i=1}^{n} \iint\limits_{E_{\rho,2\rho}} \sum_{i=1}^{n} u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2 dxdt \leqslant$$

$$\leqslant \gamma(M(\rho)-M(2\rho))^{2-\delta_1} + \gamma(M(\rho)-M(2\rho))^{2-\delta_2}, \tag{32}$$

where $\delta_1, \delta_2 \in (0,1)$.

3. The pointwise estimate of solutions.

In this section we prove Theorem 2. First, we obtain the additional integral estimate for the solution of the problem (1), (2).

Let $(\widetilde{x},\widetilde{t})$ be an arbitrary point in $D(R_0)\setminus D(\rho)$. For any $\rho : 0<r<\rho<R$ and any positive h we define the sequences of numbers $\rho_j := \frac{\rho}{2}(1+2^{-j}),\quad h_j := h(1-2^{-j}),\ j=\overline{1,n}$ and the families of sets:

$$Q(\rho_j) := \left\{(x,t) : \sum_{i=1}^{n}\left(\frac{|\,x_i-\widetilde{x}_i\,|}{\rho_j^{k_i}}\right)^{p_i} + \frac{|\,t-\widetilde{t}\,|}{\rho_j^{k}} < 1\right\},$$

$$A_j := \{(x,t)\in Q(\rho_j) : u_{2\rho} > h_j\}.$$

We denote by $\zeta_j \in C_0^\infty(Q(\frac{\rho_j+\rho_{j+1}}{2}))$ the cutt-off function such that: (i) $\zeta_j(x,t)\equiv 1$ outside $Q(\rho_j)$, $\zeta_j(x,t)\equiv 0$ for $(x,t)\in Q(\rho_{j+1})$; (ii) $\left|\frac{\partial\zeta_j}{\partial t}\right| \leqslant \gamma 2^{jk}\rho^{-k}, \left|\frac{\partial\zeta_j}{\partial x_i}\right| \leqslant \gamma 2^{jk_i}\rho^{-k_i}, i=\overline{1,n}$.

Lemma 5. *Let the conditions of Theorem 1 be fulfilled. Then the following inequality is valid for every* $r : 0<r<\rho<R_0$

$$\operatorname*{ess\,sup}_{0<t<T} \int\limits_{A_j(t)} (u_{2\rho}-h_j)_+^2\zeta_j^2 dx + \sum_{i=1}^{n}\iint\limits_{A_j}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\zeta_j^2 dxdt \leqslant$$

$$\leqslant \gamma 2^{j\gamma}\rho^{-n(p+1)}\,|A_j|\,. \tag{33}$$

Proof. Testing identity by

$$\varphi = (u_{2\rho}-h_j)_+\zeta_j, \qquad \psi = \zeta_j.$$

From the conditions (3), Young's inequality and properties of the cut-off functions ζ_j, we obtain

$$\operatorname*{ess\,sup}_{0<t<T} \int\limits_{A_j(t)} (u_{2\rho}-h_j)_+^2 \zeta_j^2 dx + \sum_{i=1}^{n} \iint\limits_{A_j} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \zeta_j^2 dxdt \leqslant$$

$$\leqslant \gamma \sum_{i=1}^{n} \iint\limits_{A_j} \left|\frac{\partial \zeta_j}{\partial x_i}\right|^{p_i} (u_{2\rho}-h_j)_+^{p_i} \zeta_j dxdt +$$

$$+\gamma \iint\limits_{A_j} (u_{2\rho}-h_j)^2 \left|\frac{\partial \zeta_j}{\partial t}\right| \zeta_j dxdt + \gamma \iint\limits_{A_j} (u_{2\rho}-h_j)_+^p \zeta_j^2 dxdt. \tag{34}$$

Using the definition of $M(\rho)$, we have

$$\sum_{i=1}^{n} \iint\limits_{A_j} \left|\frac{\partial \zeta_j}{\partial x_i}\right|^{p_i} (u_{2\rho}-h_j)_+^{p_i} \zeta_j dxdt \leqslant \gamma \sum_{i=1}^{n} 2^{2jk_i} \rho^{-p_i k_i} M\left(\frac{\rho}{2}\right)^{p_i} |A_j| =$$

$$= \gamma \sum_{i=1}^{n} 2^{2jk_i} \rho^{-p_i k_i} \left(M\left(\frac{\rho}{2}\right)\left(\frac{\rho}{2}\right)^{n}\right)^{p_i} |A_j| \left(\frac{\rho}{2}\right)^{-np_i} \leqslant \gamma 2^{j\gamma} \rho^{-p(n+1)} |A_j|, \tag{35}$$

$$\iint\limits_{A_j} (u_{2\rho}-h_j)_+^2 \left|\frac{\partial \zeta_j}{\partial t}\right| \zeta_j dxdt \leq \gamma 2^{jk} \rho^{-k} M_u^2 \left(\frac{\rho}{2}\right) |A_j| =$$

$$= \gamma 2^{jk} \rho^{-k} \left(M_u\left(\frac{\rho}{2}\right)\left(\frac{\rho}{2}\right)^{n}\right)^{2} |A_j| \left(\frac{\rho}{2}\right)^{-2n} \leqslant \gamma 2^{jk} \rho^{-p(n+1)} |A_j|, \tag{36}$$

$$\iint\limits_{A_j} (u_{2\rho}-h_j)_+^p \zeta_j^2 dxdt \leqslant \gamma M^p \left(\frac{\rho}{2}\right) |A_j| =$$

$$= \gamma \left(M\left(\frac{\rho}{2}\right)\left(\frac{\rho}{2}\right)^{n}\right)^{p} |A_j| \left(\frac{\rho}{2}\right)^{-np} \leqslant \gamma \rho^{-np} |A_j|. \tag{37}$$

Combining inequalities (34)–(37), we derive the required integral estimate. □

Now we are ready to prove Theorem 2. Applying Hölder's inequality, Lemma 7 from the appendix with $\alpha_i = 0,\ i = 1, ..., n$, we derive:

$$Y_{j+1} = \iint\limits_{A_{j+1}} (u_{2\rho}-h_{j+1})_+^p dxdt \leqslant$$

$$\leqslant |A_{j+1}|^{\frac{p}{n+p}} \operatorname*{ess\,sup}_{0<t<T} \left(\int\limits_{A_{j+1}(t)} (u_{2\rho}-h_{j+1})_+^p \zeta_j^p dx \right)^{\frac{p}{n+p}} \times$$

$$\times\left(\sum_{i=1}^{n}\left(\iint\limits_{A_j}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\zeta_j^{p_i}dxdt+\left|\frac{\partial \zeta_j}{\partial x_i}\right|^{p_i}(u_{2\rho}-h_{j+1})_+^{p_i}dxdt\right)\right)^{\frac{n}{n+p}}. \tag{38}$$

Using Lemma 5 and the properties of the functions ζ_j, estimate the second multiplier in the last inequality

$$\sum_{i=1}^{n}\left(\iint\limits_{A_j}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\zeta_j^{p_i}dxdt+\left|\frac{\partial \zeta_j}{\partial x_i}\right|^{p_i}(u_{2\rho}-h_{j+1})_+^{p_i}dxdt\right)\leqslant$$

$$\leqslant \gamma 2^{j\gamma}\rho^{-n(p+1)}\left|A_j\right|+2^{j\gamma}\left|A_j\right|\sum_{i=1}^{n}2^{i\gamma}\rho^{-k_ip_i}\rho^{-np_i}\leq\gamma 2^{j\gamma}\rho^{-n(p+1)}\left|A_j\right|. \tag{39}$$

Depending on the value of the exponent p, we estimate the first multiplier as follows

$$p<2:\qquad \operatorname*{ess\,sup}_{0<t<T}\left(\int\limits_{A_{j+1}(t)}(u_{2\rho}-h_{j+1})_+^{p}\zeta_j^{p}dx\right)^{\frac{p}{n+p}}\leqslant$$

$$\leqslant \operatorname*{ess\,sup}_{0<t<T}\left(\left(\int\limits_{A_{j+1}(t)}(u_{2\rho}-h_{j+1})_+^{2}\zeta_j^{2}dx\right)^{\frac{p}{2}}\left|A_{j+1}(t)\right|^{\frac{2-p}{2}}\right)^{\frac{p}{n+p}}\leqslant$$

$$\leqslant\gamma 2^{j\gamma}\rho^{-\frac{np^2(p+1)}{2(n+p)}}\left|A_j\right|^{\frac{p}{n+p}} \tag{40}$$

$$p>2:\qquad \operatorname*{ess\,sup}_{0<t<T}\left(\int\limits_{A_{j+1}(t)}(u_{2\rho}-h_{j+1})_+^{p-2}(u_{2\rho}-h_{j+1})_+^{2}\zeta_j^{p}dx\right)^{\frac{p}{n+p}}\leqslant$$

$$\leqslant (M(\rho))^{\frac{p(p-2)}{n+p}}\operatorname*{ess\,sup}_{0<t<T}\left(\int\limits_{A_{j+1}(t)}(u_{2\rho}-h_{j+1})_+^{2}\zeta_j^{2}dx\right)^{\frac{p}{n+p}}\leqslant$$

$$\leqslant\gamma 2^{j\gamma}\rho^{-\frac{np(2p-1)}{n+p}}\left|A_j\right|^{\frac{p}{n+p}}. \tag{41}$$

Applying (39) to estimate (38) and choosing $m=\min\left\{\frac{np(2p-1)}{n+p},\frac{np^2(p+1)}{2(n+p)}\right\}$ from (40), (41), we get

$$Y_{j+1}\leq\gamma 2^{j\gamma}\rho^{-m}\left|A_{j+1}\right|^{1+\frac{p}{n+p}}\leq 2^{j\gamma}\rho^{-m}h^{-p\left(1+\frac{p}{n+p}\right)}\left(\iint\limits_{A_j}(u_{2\rho}-h_j)_+^2\,dxdt\right)^{1+\frac{p}{n+p}}.$$

Choosing h from the condition

$$\iint\limits_{Q(\rho)} u_{2\rho}^{p} dxdt \le \left[2^{j\gamma}\rho^{-m} h^{-p\left(1+\frac{p}{n+p}\right)}\right]^{-\frac{n+p}{p}} \Longrightarrow h^{n+2p} \ge 2^{j\gamma}\rho^{-\frac{m(n+p)}{p}} \iint\limits_{Q(\rho)} u_{2\rho}^{p} dxdt.$$

Using the Lemma 6 from the appendix, we obtain

$$\left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right)^{n+2p} \le \gamma \rho^{-\frac{m(n+p)}{p}} \iint\limits_{Q(\rho)} u_{2\rho}^{p} dxdt$$

or

$$M\left(\frac{\rho}{2}\right) - M(2\rho) \leqslant \gamma \rho^{-\frac{m(n+p)}{p(n+2p)}} \left(\iint\limits_{Q(\rho)} u_{2\rho}^{p} dxdt\right)^{\frac{1}{n+2p}}. \tag{42}$$

To estimate the integral on the right-hand side of (42) we apply Lemma 7 with $\alpha_i = 0, i = 1, ..., n$ and Remark 1

$$\iint\limits_{Q(\rho)} u_{R}^{p} dxdt \leqslant \rho^{p} \sum_{i=1}^{n} \iint\limits_{Q(\rho)} \left|\frac{\partial u}{\partial x_i}\right|^{2} dxdt \le \gamma \rho^{p} (M(\rho) - M(R))^{2-\delta}. \tag{43}$$

Eventually, from (42), (43), we derive

$$M\left(\frac{\rho}{2}\right) - M(2\rho) \le \gamma \rho^{-n+\beta}, \tag{44}$$

where

$$\beta = \frac{np(n+2p) - m(n+p) + p^2 - np(1-\delta)}{p(n+2p)} > 0, \ \ \delta = min\{\delta_1, \, \delta_2\}.$$

Iterating inequality (44), we obtain estimate (12). This proves Theorem 2. □

4. Proof of Theorem 3.

Let $\xi_j \in C_0^{\infty}(D(\overline{\rho}_j))$, $0 \le \xi_j \le 1$, $\xi_j = 1$ in $D(\rho_{j+1})$ and $\left|\frac{\partial \xi_j}{\partial t}\right| \le \gamma 2^{j\gamma} R_0^{-k}$, $\left|\frac{\partial \xi_j}{\partial x_i}\right| \le \gamma 2^{j\gamma} R_0^{-k_i}, i = \overline{1,n}$, where k, k_i are defined in (5), (7) and $0 < \lambda < min\{\frac{\beta}{n-\beta}, \frac{\beta n}{(n-\beta)(n+1)}\}$ with β from (12).

Testing the integral identity (9) by

$$\varphi = (u-h)_{+}^{\lambda} \, \xi_j^2 \eta_r, \quad \psi = \eta_r,$$

where η_r was defined in Section 2, for every $h > 0$. Using conditions (3) and the Young inequality, we obtain

$$\operatorname*{ess\,sup}_{0<t<T} \int\limits_{\Omega} (u-h)_{+}^{\lambda+1} \, \xi_j^2 \eta_r^2 dx + \sum_{i=1}^{n} \iint\limits_{\Omega_T} (u-h)_{+}^{\lambda-1} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \xi_j^2 \eta_r^2 dxdt \le \gamma \sum_{i=1}^{5} J_i, \tag{45}$$

where

$$J_1 = \sum_{i=1}^{n} \iint\limits_{\Omega_T} (u-h)_+^{\lambda+p_i-1} \left|\frac{\partial \eta_r}{\partial x_i}\right|^{p_i} \xi_j^2 \eta_r dxdt,$$

$$J_2 = \iint\limits_{\Omega_T} (u-h)_+^{\lambda+1} \left|\frac{\partial \eta_r}{\partial t}\right| \xi_j^2 \eta_r dxdt,$$

$$J_3 = \sum_{i=1}^{n} \iint\limits_{\Omega_T} (u-h)_+^{\lambda+p_i-1} \left|\frac{\partial \xi_j}{\partial x_i}\right|^{p_i} \xi_j \eta_r^2 dxdt,$$

$$J_4 = \iint\limits_{\Omega_T} (u-h)_+^{\lambda+1} \eta_r^2 \xi_j \left|\frac{\partial \xi_j}{\partial t}\right| dxdt,$$

$$J_5 = \iint\limits_{\Omega_T} (u-h)^{\lambda+p-1} \xi_j^2 \eta_r^2 dxdt.$$

Denote by $A(k)$ the following set

$$A(h) := \{(x,t) \in D(R_0) : u(x,t) > h\}.$$

Condition (12) imply that

$$J_1 \leq \gamma r^{(\lambda+p_i-1)(\beta-n)} r^{-p_i k_i} r^{n+k} = \gamma r^{\lambda(\beta-n)+\beta(p_i-1)},$$

$$J_2 \leq \gamma r^{(\lambda+1)(\beta-n)} r^{-k} r^{n+k} = \gamma r^{\lambda(\beta-n)+\beta},$$

$$J_3 \leq \gamma 2^{j\gamma} R_0^{-n(p-2)-p} \iint\limits_{A(h)} (u-h)^{\lambda+p-1} dxdt, \tag{46}$$

$$J_4 \leqslant \gamma 2^{j\gamma} R_0^{-n(p-2)-p} \iint\limits_{A(h)} (u-h)^{\lambda+p-1} dxdt,$$

$$J_5 \leq \gamma \iint\limits_{A(h)} (u-h)^{\lambda+p-1} \eta_r^2 \xi_j^2 dxdt.$$

We combine estimates (45), (46) and pass to the limit in the resulting inequality as $r \to 0$. Eventually, we obtain the boundedness of $u(x,t)$ in $D\left(\frac{R_0}{2}\right)$ analogously to the argumentations from Section 3. This proves Theorem 3. □

5. Appendix.

Lemma 6. [3]

Let $\{Y_j\}$, $j = 0, 1 \ldots$ *be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{j+1} \leqslant C b^j Y_j^{1+\alpha},$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If

$$Y_0 \leqslant C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then Y_j converges to zero as $j \to \infty$.

Lemma 7. [9] *Let $\Omega \subset R_n, n \geq 3$ be a bounded domain. Let ν be an arbitrary function from the Sobolev space $W_0^{1,1}$ such that*

$$\int\limits_{\Omega} \int\limits_{E_R} |\nu|^{\alpha_i} \left| \frac{\partial \nu}{\partial x_i} \right|^{p_i} dx < \infty,$$

where $p_i \geqslant 1, 1 + \frac{\alpha_i}{p_i} > 0, i = 1, ..., n, \sum\limits_{i=1}^{n} \frac{1}{p_i} > 1$. Then

$$\nu \in L_{q_*}(\Omega), q_* = \frac{np}{n-p} \left(1 + \frac{1}{n} \sum_{k=1}^{n} \frac{\alpha_k}{p_k} \right),$$

where $\frac{1}{p} = \frac{1}{n} \sum\limits_{i=1}^{n} \frac{1}{p_i}$ and there exists a positive constant K depending only on $n, \alpha_i, p_i, i = 1, ..., n$, such that the following inequality is valid

$$\parallel \nu \parallel_{L_{q_*}(\Omega)} \leqslant K \prod_{i=1}^{n} \left(\int\limits_{\Omega} |\nu|^{\alpha_i} \left| \frac{\partial \nu}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i n \left(1 + \frac{1}{n} \sum\limits_{k=1}^{n} \frac{\alpha_k}{p_k} \right)}}.$$

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Усувні ізольовані особливості розв'язків анізотропного еволюційного рівняння p-Лапласа.

У роботі досліджено широкий клас квазілінійних параболічних рівнянь другого порядку в дивергентній формі, одним з найпростіших модельних представників таких рівнянь є анізотропне еволюційне рівняння p-Лапласу. Відомо, що загальна якісна теорія для анізотропних еліптичних і параболічних рівнянь не побудована. Крім того, точний вигляд фундаментального розв'язку для таких рівнянь невідомий, тому природно постають питання про дослідження локальної поведінки розв'язків вищевказаних рівнянь в околі сингулярної точки та про отримання точної умови усунення особливості. Хоча квазілінійні рівняння з анізотропією мало вивчені, однак для їх дослідження можна застосувати деякі з нароблених методів, а саме для вирішення поставлених задач було використано такі методи: метод локальних енергетичних оцінок, ітераційна техніка Де Джорджі, метод поточкових оцінок. Нам вдалося для слабких обмежених невід'ємних розв'язків анізотропного еволюційного рівняння p-Лапласу встановити нові інтегральні оцінки, поточкові оцінки та отримати достатню умову усунення ізольваної особливості розв'язків у вигляді асимптотичної оцінки. Ще одна складність дослідження виникає з різними значеннями показників анізотропії, а саме вона полягає в тому, що розглядається випадок, коли частина показників анізотропії $p_i < 2$ (так званий "сингулярний випадок"), а інша частина $p_i > 2$ (так званий "вироджений випадок").

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