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MULTIPARAMETRIC IDENTIFICATION OF SEVERAL THERMOPHYSICAL CHARACTERISTICS BY SOLVING THE INTERNAL INVERSE HEAT CONDUCTION PROBLEM

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Approaches to the identification of thermophysical characteristics, using methods for solving inverse heat conduction problems and A. N. Tikhonov's regularization method, are developed. According to the results of the experiment, temperature-dependent coefficients of heat conductivity, heat capacity, and internal heat sources are determined. In this case, the thermophysical characteristics are approximated by Schoenberg's cubic splines, as a result of which their identification reduces to determining unknown coefficients in the approximated dependencies. Therefore, the temperature in the body will depend on these coefficients, and it can be represented using two members of the Taylor series as a linear combination of its partial derivatives with respect to the unknown coefficients, multiplied by the increments of these coefficients. Substituting this expression into the Tikhonov functional and using the minimum property of the quadratic functional, we can reduce the solution of the problem to the solution of a system of linear equations with respect to the increments of unknown coefficients. By choosing a certain regularization parameter and some functions as an initial approximation, we can implement an iterative process in which the vector of unknown coefficients for the current iteration will be equal to the sum of the vector of the coefficients obtained in the previous iteration and the coefficient increment vector as a result of solving a system of linear equations. Such an iterative process of identifying the thermophysical characteristics for each regularization parameter makes it possible to determine the mean-square discrepancy between the resulting temperature and the temperature measured as a result of the experiment. It remains to choose the regularization parameter so that this discrepancy is within the root-mean-square measurement error. Such a search, for example, is identical to algorithms for searching roots of nonlinear equations. When checking the efficiency of using the proposed method, a number of test problems were solved for bodies with known thermophysical characteristics. An analysis of the influence of random measurement errors on the error of the identifiable thermophysical characteristics of the body being studied was carried out.

Keywords: *inverse heat conduction problem, Tikhonov's regularization method, stabilization functional, regularization parameter, identification, approximation, Schoenberg's cubic splines.*

Introduction

Solution of inverse heat conduction problems (HCP) of identifying the parameters of mathematical models is of particular importance as an important step in ensuring the adequacy of these models in the presence of experimental information about the thermal process being studied. This article discusses the nonlinear internal inverse HCP of identifying thermophysical characteristics. These can be temperature-dependent coefficient of heat conductivity, heat capacity, internal heat sources, etc. The authors of [1–6] propose classifications of inverse HCPs and consider methods for their solution. In [4, 5], the problems of identifying the coefficient of heat conductivity and heat capacity are called coefficient inverse HCPs. We follow the classification given in [2, 6] and assign all the problems of identifying the thermophysical characteristics inside the body being studied to a class of internal problems by analogy with the external inverse HCPs [2, 4, 5, 6] of identifying heat flows and other thermophysical characteristics on the surface of the body. The solution for the class of internal inverse HCPs of simultaneously identifying the internal thermal characteristics, such as the coefficient of heat conductivity and heat capacity, is considered the most time-consuming, since it is very difficult and in some cases almost impossible to take into account the influence of changes in these characteristics on the thermal process of the body being studied. One of the first works devoted to the solution of such problems was work [7], which considered the simultaneous identification of the heat conductivity coefficient and specific volumetric heat capacity of artificial diamonds by solving a multi-parameter inverse HCP, using an iterative filter. Its authors conclude that while searching for different thermophysical characteristics, an inaccuracy in identifying one of them causes a corresponding inaccuracy in identifying another characteristic, i.e. for their correct simultaneous identification, the a-priori information about one of the identifiable characteristics must be known. In [8], a method for identifying one of the thermal characteristics is proposed, which reduces to constructing an iterative process of determining the coefficient vector in the approximative dependence of the identifiable function on temperature. In this work, to obtain the solution, the author uses M. M. Lavrentyev's α-regularization method [9, 10], which is less flexible than A. N. Tikhonov's regularization method [5], since when the former is used, it is more difficult to take into account the a-priori information about the desired thermophysical characteristic. Using the iterative process proposed in [8] and A. N. Tikhonov's regularization method [5], we propose the following approach to the simultaneous identification of several desired characteristics.

Problem Formulation

In this paper, we consider a nonstationary internal inverse HCP, which can be formalized as follows:

$$A[\varphi_1(T),\dots,\varphi_i(T),\dots,\varphi_n(T)] = T^{ex}\ ,\ i=\overline{1,n}\ ,$$

where $\varphi_i(T), i=\overline{1,n}$ are the desired temperature-dependent thermophysical characteristics; T^{ex} is the process state variable, which in most cases is known from an experiment (initial data); A is the operator that connects the desired dependencies $\varphi_i(T), i=\overline{1,n}$ with the initial data T^{ex}; n is the number of the required dependencies.

This problem, like any inverse HCP, due to the violation of causality, is incorrect according to Hadamard, which leads to the instability of the solution being obtained. To solve such problems, they are either reduced to conditionally correct, or left incorrect, with one of the regularization methods being used [1–10]. Here, A. N. Tikhonov's regularization method is used [5].

Consider the nonstationary thermal process in the body with boundary conditions of the second and third kinds, which is described as follows [2, 11]:

$$C(T)\frac{\partial T}{\partial \tau} = div(\lambda(T)\cdot grad(T)) + S(T),\ M\in D\ , \tag{1}$$

$$-\lambda(T)\frac{\partial T}{\partial \nu}\bigg|_{M\in\Gamma_1} = q\ , \tag{2}$$

$$-\lambda(T)\frac{\partial T}{\partial \nu}\bigg|_{M\in\Gamma_2} = \alpha(T - T_S)\ , \tag{3}$$

$$T(M,\tau)\big|_{\tau=0} = T_0\ , \tag{4}$$

$$\text{at } T(M_k,\tau_i) = T_{ik}^{ex},\quad i=\overline{1,n_\tau}\ ,\quad k=\overline{1,m}\ , \tag{5}$$

where T=$T(M, \tau)$ is the body temperature; D is the area of space occupied by the body; Γ_1 and Γ_2 are parts of the border area D; M is a point in the region D; τ is a time coordinate; $\lambda(T), C(T), S(T)$ are the desired dependencies of the coefficient of heat conductivity, heat capacity and internal heat source; α is the heat-transfer coefficient of the surface Γ_2; T_S is the required ambient temperature of the surface Γ_2; q is the required heat flows at the boundary Γ_1; ν is the external normal to the boundary of the body; T_0 is the initial body temperature; m is the number of measurements in the time coordinate; n_τ is the number of measurement points in the body; M_k are the individual points of the area D in which the temperature T_{ik}^{ex} is measured. The measurement error is a random variable distributed according to the normal law with zero mathematical expectation and σ^2 dispersion.

Based on the thermophysical experiment T_{ik}^{ex}, the dependencies of the thermophysical characteristics of the body being studied, $\lambda(T), C(T), S(T)$, are determined over the entire temperature range, with account taken of the available a priori information about these dependencies.

Below, we consider a methodological approach to solving the problem.

Regularization Algorithm for Solving the Inverse HCP

To solve the inverse HCP, we use A. N. Tikhonov's regularization method, which reduces to minimizing the following functional:

$$J = \int_0^{\tau_0}\int_D \left[T(M,\tau) - T^{ex}(M,\tau)\right]^2 dDd\tau + \beta\cdot\Omega[\varphi_1(T),\dots,\varphi_n(T)] + \beta\cdot\Delta[\varphi_1(T),\dots,\varphi_n(T)], \quad (6)$$

where $T(M, \tau)$ is the temperature obtained as a result of solving the direct HCP; $T^{ex}(M,\tau)$ is the experimentally obtained temperature; τ_0 is the completion moment of thermal process analysis; β is the regularization parameter; $\Omega[\varphi_1(T),\dots,\varphi_n(T)]$ is a stabilization functional; $\Delta[\varphi_1(T),\dots,\varphi_n(T)]$ is a quadratic functional characterizing the discrepancy between the values of the desired thermophysical characteristics and the values of these characteristics, a-priori specified at certain temperatures in the range being studied.

If the desired functions $\varphi_1(T),\dots,\varphi_n(T)$ are represented as

$$\varphi_i(T) = \sum_{k=1}^{n_i} \alpha_{ks} B_3^{ki}(T), \; i = \overline{1,n}, \quad (7)$$

where $(\alpha_{1i},\dots,\alpha_{n_i i}) = \overrightarrow{\Lambda_i}$ are the vectors of the desired parameters, and $B_3^{ki}(T)$ are Schoenberg's cubic splines, then the identification of the desired functions reduces to the determination of the unknown vectors $\overrightarrow{\Lambda_i}$, $i = \overline{1,n}$.

We minimize functional (6) by the iterative method [8]. Since the temperature $T(M, \tau)$ depends on the vectors $\overrightarrow{\Lambda_i}$, $i = \overline{1,n}$, it can be represented at the $(p+1)$ th iteration by using Taylor's series, as

$$T^{p+1}(M,\tau,\varphi_1^{p+1}(T),\dots,\varphi_n^{p+1}(T)) \approx T^p(M,\tau,\varphi_1^p(T),\dots,\varphi_n^p(T)) + \sum_{i=1}^{n}\sum_{k=1}^{n_i} \frac{\partial T^p}{\partial \alpha_{ki}} \Delta\alpha_{ki}^{p+1}, \quad (8)$$

where $(\Delta\alpha_{1i}^{p+1},\dots,\Delta\alpha_{n_i i}^{p+1}) = \overrightarrow{\Delta\Lambda_i^{p+1}}$, $i = \overline{1,n}$ are the incremental vectors $\overrightarrow{\Delta\Lambda_i^{p+1}} = \overrightarrow{\Lambda_i^{p+1}} - \overrightarrow{\Lambda_i^p}$.

At the $(p+1)$ th iteration, we write the stabilization functional as

$$\Omega[\varphi_1^{p+1}(T),\dots,\varphi_n^{p+1}(T)] = \sum_{i=1}^{n}\int_{T_{\min}}^{T_{\max}} \left(w_{0i}\left(\varphi_i^{p+1}\right)^2 + w_{1i}\left(\frac{\partial \varphi_i^{p+1}}{\partial T}\right)^2 + w_{2i}\left(\frac{\partial^2 \varphi_i^{p+1}}{\partial T^2}\right)^2 \right) dT, \quad (9)$$

where $T_{\min}$, $T_{\max}$ are the minimum and maximum temperatures in problem (1) – (5), and w_{0i}, w_{1i}, w_{2i} are the weighting factors, which are selected depending on the a priori information about the desired thermophysical characteristics.

The quadratic functional $\Delta[\varphi_1(T),\dots,\varphi_n(T)]$ that characterizes the discrepancy between the values of the desired thermophysical characteristics and the values of the same characteristics, a-priori specified at certain temperatures in the range being studied, can be constructed as follows.

Let $T_{i1},\dots,T_{is_i}$ be some temperatures from the interval $[T_{\min}, T_{\max}]$ for the i-th desired function, and $f_{i1}, f_{i2},\dots,f_{is_i}$, respectively, a priori values of this function. Then the quadratic functional $\Delta[\varphi_1(T),\dots,\varphi_n(T)]$ takes the form

$$\Delta[\varphi_1(T), \dots, \varphi_n(T)] = \sum_{i=1}^{n} \sum_{j=1}^{s_i} W_{ij} \left(\varphi_i\left(T_{ij}\right) - f_{ij}\right)^2 , \tag{10}$$

where W_{ij} are the weighting factors that are selected based on how accurately the a priori values $f_{i1}, f_{i2}, \dots, f_{is_i}$ are specified.

We substitute expressions (7), (8), (9), (10) into functional (6). Replacing $T(M,\tau)$ by the approximate temperature value at the points of thermometry and using the necessary condition for the minimum of functional (6), we obtain a system of linear equations with respect to $\Delta\alpha_{ki}^{p+1}$, $i = \overline{1,n}$, $k = \overline{1,n_s}$ in the i-th iteration. The system of linear equations includes the regularization parameter β, which is determined, as we did in [12, 13], based on the condition

$$\left(1 - \sqrt{\frac{2}{N}}\right)\sigma \le \delta \le \left(1 + \sqrt{\frac{2}{N}}\right)\sigma , \tag{11}$$

which is proposed in [1]. Here, N is the total number of thermometric measurements; δ is the root-mean-square deviation of the obtained temperature from the temperature measured at the points of thermometry.

We believe that the regularization parameter is chosen correctly if the two-sided inequality (11) is satisfied for the obtained solution according to the iterative scheme proposed above.

Numerical Experiment

Consider the process of heating a body (an infinite plate), with a temperature-dependent internal heat source and convective heat flux, taking into account the fact that in the case of the dependence of the heat capacity and heat conductivity of the body on temperature, the boundary value problem (1–4) can be represented as follows:

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + S(T), \quad x \in [0, l], \tag{12}$$

$$\lambda(T)\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 , \tag{13}$$

$$\left(-\lambda(T)\frac{\partial T}{\partial x} + \alpha T\right)\bigg|_{x=l} = \alpha T_s , \tag{14}$$

$$T(M,\tau)\big|_{\tau=0} = T_0 , \tag{15}$$

where l is the thickness of the plate; $C(T)$ is the heat capacity of the plate; $\lambda(T)$ is the coefficient of the heat conductivity of the plate; $S(T)$ is an internal heat source; α is the heat transfer coefficient; T_S is the convective flow temperature; T_0 is the initial temperature of the plate.

To conduct the numerical experiment of the boundary value problem (12–15), the dimensionless dependencies $C(T), \lambda(T), S(T)$ are taken, which are approximated quite accurately by Schoenberg's cubic splines with a small number of the required parameters

$$C(T) = 3.4 - 3.2T - 0.8T^2 , \tag{16}$$

$$\lambda(T) = 0.1 + 1.8T - 0.9T^2 , \tag{17}$$

$$S(T) = 10.0 - 15.0T + 5.0T^2 . \tag{18}$$

The points of thermometry are evenly distributed across the plate thickness. A random error, distributed according to the normal law at σ=0.05, is superimposed on the obtained numerical solution at the points of thermometry. Such a random error is large enough in comparison with the measurement errors of modern devices, which makes it possible to illustrate the effectiveness of the proposed method.

Figs. 1–3 show the dependencies of heat capacity $C(T)$, heat conductivity coefficient $\lambda(T)$, and internal heat source $S(T)$, obtained using the method described above, and the dependencies of these thermophysical characteristics (16)–(18) for the following dimensionless data: l=1, n_τ=400, m=21, $\Delta\tau$=0.001, α=2.0, T_s=2.5, T_0=1.0, $T_{\min}$=1.0, $T_{\max}$=2.0, $S(T_{\min})$=0, $S(T_{\max})$=0, $W_{T_{\min}S}=10^5$, $W_{T_{\max}S}=10^5$, $W_{T_{\min}C}=10^5$. For all the required dependencies, as noted above, a small number of the required parameters n_C=5, n_λ=5, n_S=5 were chosen. The weighting factors in functional (9) were chosen according to the recommendations of [1]. In this problem, for the required dependencies of heat conductivity and heat capacity, the second-order regularization was used, and for the desired dependence of the internal source, the zero-order regularization.

Fig. 4 shows the dependencies $T(M_0, \tau)$, obtained as a result of solving the direct and inverse HCPs, as well as the noisy temperature at the point $M_0=\{0\}$.

The selection of the regularization parameter β began with β=1.0, and as the required functions, the initial dependencies $\lambda(T)$=0.6, $C(T)$=1.0 and $S(T)$=0 were selected. The iterative selection process β after the fifth iteration ended at $\beta\approx$0.003906 when the root-mean-square error reached $\delta\approx$0.05011.

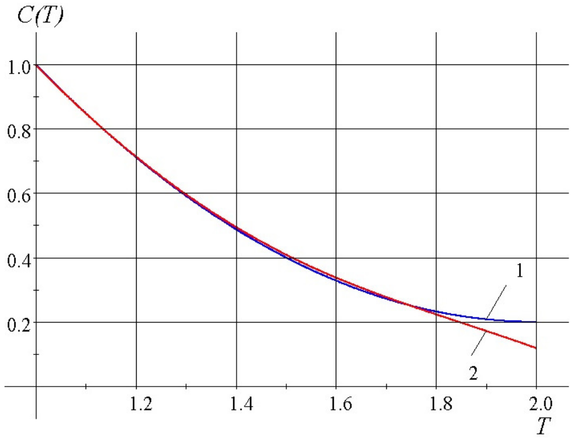

Fig. 1. Dependencies C(T):
1 – in the form of (16);
2 – obtained by the iterative method

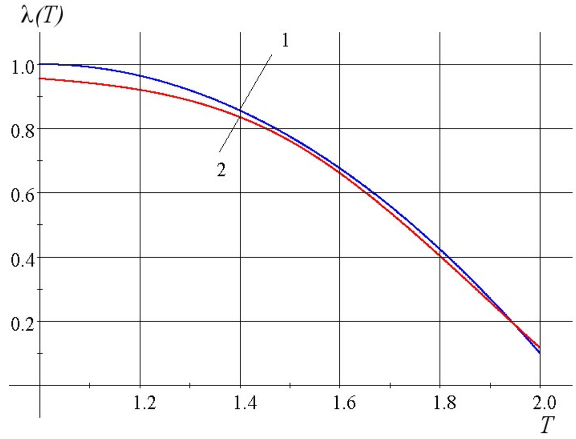

Fig. 2. Dependencies λ(T):
1 – in the form of (17);
2 – obtained by the iterative method

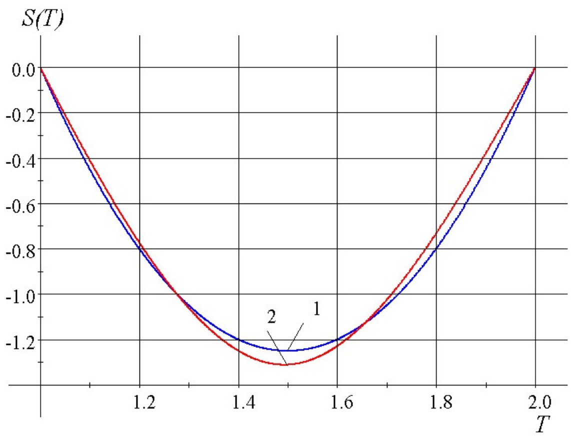

Fig. 3. Dependencies S(T):
1 – in the form of (18);
2 – obtained by the iterative method

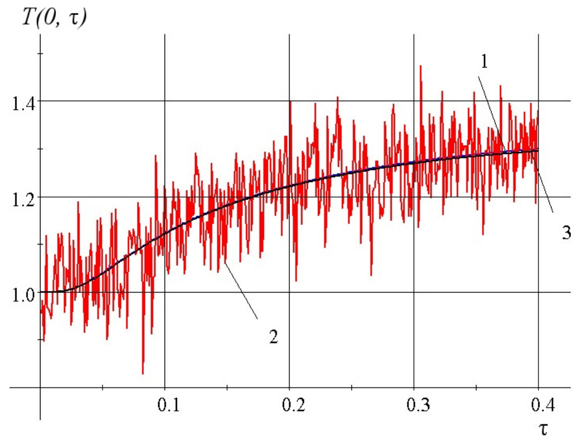

Fig. 4 Dependencies T(0, τ):
1 – obtained by solving the direct HCP,
using characteristics in the form (16–18);
2 – the noisy solution to the direct problem;
3 – obtained using the identified characteristics
by solving the inverse HCP

Conclusions

The above solution to the internal inverse HCP of identifying temperature-dependent heat capacity, heat conductivity coefficient, and internal heat source indicates that the presented method for identifying thermophysical characteristics can be successfully used if there is an a-priori information about the desired functions. If this is absent, then the proposed method can also be applied, but the measurement errors should be comparable with the errors in solving direct problems.

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Багатопараметрична ідентифікація теплофізичних характеристик шляхом розв’язання внутрішньої оберненої задачі теплопровідності

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Розроблено підходи до ідентифікації теплофізичних характеристик з використанням методів розв'язання обернених задач теплопровідності і методу регуляризації А. М. Тихонова. За результатами проведеного експерименту визначаються залежні від температури коефіцієнт теплопровідності, теплоємність, внутріш-

ні джерела теплоти. При цьому теплофізичні характеристики апроксимуються кубічними сплайнами Шьонберга, внаслідок чого їх ідентифікація зводиться до визначення невідомих коефіцієнтів в апроксимаційних залежностях. Отже, температура в тілі буде залежати від цих коефіцієнтів і її можна буде зобразити, використовуючи два члени ряду Тейлора як лінійну комбінацію її частинних похідних з невідомих коефіцієнтів, помножених на приріст цих коефіцієнтів. Підставляючи цей вираз в функціонал Тихонова і використовуючи властивість мінімуму квадратичного функціонала, можна звести розв'язок задачі до розв'язання системи лінійних рівнянь щодо збільшень невідомих коефіцієнтів. Вибравши для початкового наближення певний параметр регуляризації і деякі функції, можна реалізувати ітераційний процес, в якому вектор невідомих коефіцієнтів для поточної ітерації буде дорівнювати сумі вектора коефіцієнтів з попередньої ітерації і вектора приростів цих коефіцієнтів внаслідок розв'язання системи лінійних рівнянь. Такий ітераційний процес з ідентифікації теплофізичних характеристик для кожного параметра регуляризації дає можливість визначити середньоквадратичний відхил між одержуваною температурою і температурою, яку виміряли внаслідок проведеного експерименту. Залишається підібрати параметр регуляризації таким чином, щоб цей відхил був в межах середньоквадратичної похибки вимірювань. Такий пошук, наприклад, ідентичний алгоритмам пошуку кореня нелінійного рівняння. Під час перевірки ефективності використання запропонованого методу було розв'язано низку тестових задач для тіл з відомими теплофізичними характеристиками. Проведено аналіз впливу випадкових похибок вимірювань на похибку ідентифікованих теплофізичних характеристик досліджуваного тіла.

Ключові слова: *обернена задача теплопровідності, метод регуляризації А. М. Тихонова, стабілізуючий функціонал, параметр регуляризації, ідентифікація, апроксимація, кубічні сплайни Шьонберга.*

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