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THE NONLOCAL PROBLEM FOR FRACTAL DIFFUSION EQUATION

Ключові слова: рівняння фрактальної дифузії, відхилення змінної, покроковий метод.

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Introduction

In recent decades, fractional diffusion equations are studied very intensively. The fractional reaction-diffusion models are studied due to their usefulness and importance in many areas of science and engineering. The first works in this direction include [1–7]. Any close-to-complete analysis of the multitude of works devoted to the diffusion-wave equation would require a separate special study [8–12].

The result of the analytic theory of heat and mass transfer are represented in [13–15], with a view to the development of computational techniques to determine the fluxes of matter and heat at the interface, including the presence of chemical reactions. Last but not least, the concepts of fractal geometry have entered recently in optics, where they have been successfully used for classification and characterization of rough surfaces and solving numerous related applied problems [16–31].

In [32] we present a formula for classical solutions for time- and space-fractional kinetic equation (also known as fractional diffusion equation) and deviation time variable is given in terms of the Fox's *H*-function, using the step by step method. This equation describes fractal properties of real data arising in applied fields such turbulence, hydrology, ecology, geography, air pollution, economics and finance.

The experimental results of the study of statistical, correlations and fractal parameters, which characterize the real component of the Jones-matrix image of polycrystalline network of flat layers of the main types of human amino acids, are presented in [33]. The use of fractional calculus in mathematical modelling of nonlocal process has been studied by A.M. Nakhushev [34, 35], V.A. Nakhusheva [36], Y.Z. Povstenko [37–39]. It has been noted [35] that the fractional differential and integral calculus in the theory of fractal and systems with memory becomes as important as the classical analysis in physics (mechanics) continua. Thus, fundamental research on non-local problems for pseudo-differential equation is well-timed and relevant.

The Cauchy problem for the fractional diffusion equations is studied in [40–42]. In [40] construction Green-function of Levi-method, and in [41] by method from [43], construction the integral equation.

In this paper we consider a new class of the fractional equation (1) with deviation time variable always supposing that the solution satisfies the nonlocal initial condition (2) using the step by step method. Such equations describe diffusion on inhomogeneous fractals. A fundamental solution of the Cauchy problem is constructed and investigated.

1. Fractional diffusion equation

Introduce into consideration a new class of the following fractional equation with deviation time variable

$$\frac{\partial^{\alpha}u(t,x)}{\partial t^{\alpha}} = a(t,x)\frac{\partial^2 u(t,x)}{\partial x^2} + b(t,x)\frac{\partial u(t,x)}{\partial x} + c(t,x)u(t,x) +$$

$$+ f(t,x,u(t-h,x)),\ \ t > h,\ \ x \in \mathbb{R}. \tag{1}$$

Assume that the function $u(t, x)$ satisfies the nonlocal initial condition

$$u(t,x)|_{0\le t\le h} = u_0(t,x),\ \ x\in\mathbb{R}, \tag{2}$$

where $u = u(t, x)$, $0 \le t < T$, $x \in \mathbb{R}$ is the function to be found and $\alpha \in (0,1]$.

The time derivative of $\alpha \in (0,1]$ order is defined as follows:

$$\frac{\partial^{\alpha}u(t,x)}{\partial t^{\alpha}} = \begin{cases} \dfrac{\partial u(t,x)}{\partial t}, & if\ \alpha = 1, \\ (D_t^{\alpha}u)(t,x), & if\ \alpha \in (0,1), \end{cases} \tag{3}$$

where

$$(D_t^{\alpha}u)(t,x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_h^t (t-\tau)^{-\alpha}u(\tau,x)d\tau - \frac{u(h,x)}{(t-h)^{\alpha}}\right],\ \ h < t < T,$$

is the regulised fractional derivative or fractional derivative in Caputo–Djrbashian sense.

We prove the solvability of the Cauchy problem (1), (2) using the step by step method. The Riemann–Liouville fractional integral of order $\alpha \ge 0$ is defined for $\alpha = 0$ as $J^0 := I$, where I denotes the identity operator, and for $\alpha > 0$ as

$$J^{\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\alpha)^{\alpha-1}f(\tau)d\tau \equiv (g_{\alpha} * f)(t), \tag{4}$$

where

$$g_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

is the Riemann–Liouville kernel and (*) denotes the convolution in time.

Definition. Let $0 < \beta \le 1$. Suppose $u_0 \in C([0,\infty)\times\mathbb{R})$, $f \in C([0,\infty)\times\mathbb{R})$; $C_x^2([0,\infty)\times\mathbb{R})$. Then a function $u \in C_x^2([0,\infty)\times\mathbb{R})$ is a classical solution of the non-local problem (1), (2) if:

1) the coefficients $a(t, x)$, $b(t, x)$, $c(t, x)$, is bounded by number M;

2) a polynomial is ranger if its leading coefficient is $a(t, x) \ge a_0 > 0$,

$|a(t,x) - a(\tau,\xi)| \le \omega(|x-\xi| + (t-\tau)^{\alpha/2})$, $(t,x)\in(0,T)\times\mathrm{R}$, $(\tau,\xi)\in(0,T)\times\mathrm{R}$;

3) $u_0 \in C(0,h)\times\mathrm{R}$, $f \in C_x^{(\omega_1)}((0,T)\times\mathrm{R})$, $\omega(t)$, $\omega_1(t)$ satisfies the Dini conditions [42, p. 11];

4) for every $x\in\mathbb{R}$ the fractional integral $J^{1-\alpha}u$, as defined in (4), is continuously differentiable with respect to $t > 0$, and

5) the function $u(t, x)$ satisfies the pseudodifferential equation of (1) this $\dfrac{\partial^{\alpha}u(t,x)}{\partial t^{\alpha}}$ (3) for every $(t,x)\in(h,\infty)\times\mathrm{R}$ and the initial condition (2) for every $(t,x)\in(0,h)\times\mathrm{R}$.

Let us denote by

$$Au(t, x) := a(t, x)\frac{\partial^2 u(t, x)}{\partial x^2} + b(t, x)\frac{\partial u(t, x)}{\partial x} + c(t, x)u(t, x), \ t > h, \ x \in \mathrm{R}.$$

2. Step method

By the step method we reduce the nonlocal problem for a fractional diffusion equation with deviating argument to the nonlocal problem for an equation with nondeviating argument.

Let $h < t \le 2h$ and $x \in \mathrm{R}$, and $f(t, x, u_0(t-h, x)) \equiv f_0(t, x, h)$. Then $0 < t-h \le h$ and $u(t-h, x) = u_0(t, x)$, $x \in \mathrm{R}$, and problem (1), (2) takes the form

$$\frac{\partial^\alpha u(t, x)}{\partial t^\alpha} = Au(t, x) + f_0(t, x, h), \ h < t \le 2h \ , \tag{5}$$

$$u(t, x)|_{t=h} = u_0(h, x), \ x \in \mathrm{R} \ , \tag{6}$$

To construct a solution of problem (5), (6) we fixed a point (β, y), $h < \beta \le 2h$, $y \in \mathbb{R}$ and consider the Cauchy problem

$$\frac{\partial^\alpha u(t, x)}{\partial t^\alpha} = a(\beta, y)\frac{\partial^2 u(t, x)}{\partial x^2} + f_0(t, x, h), \ h < t \le 2h \ , \tag{7}$$

$$u(t, x)|_{t=h} = u_0(h, x), \ x \in \mathbb{R} \ , \tag{8}$$

The Green's function of this problem according to [40, 42] has the components

$$G_1(t, x; \beta, y) = F^{-1}_{\sigma \to y} E_\alpha(-a(\beta, y)\sigma^2 t^\alpha) ,$$

$$G_2(t, x; \beta, y) = t^{1-\alpha} D_t^\alpha G_1(t, x; \beta, y)$$

and there are estimates of their derivatives that are uniform with respect to (β, y)

$$\left.\begin{aligned} &| D_x^m G_1(t, x; \beta, y) | \le c_m t^{-\alpha\frac{1+m}{2}} \exp\{-c\rho(t, x)\}, \\ &| D_x^m G_2(t, x; \beta, y) | \le c_m t^{-\frac{\alpha}{2}(m+1)+\alpha-1} \exp\{-c\rho(t, x)\}, \end{aligned}\right\} , \tag{9}$$

$$\left.\begin{aligned} &| D_x^m G_1(t, x; \beta_1, y_1) - D_x^m G_1(t, x; \beta_2, y_2) | \le \\ &\le c\omega(| y_1 - y_2 | + | \beta_1 - \beta_2 |^{\frac{\alpha}{2}}) t^{-\alpha\frac{1+m}{2}} \exp\{-c\rho(t, x)\}, \end{aligned}\right\} ,$$

where $\rho(t, x) = (\frac{|x|}{t^{\alpha/2}})^q$, $q = \frac{2}{2-\alpha}$, $m \le 3$, $\omega(h)$ is a function on $[0, \infty)$ as a module of continuity type, ie non-decreasing, non-negative, half-additive, bounded function.

The solution of the problem (5), (6) we will find as a sum

$$u(t, x) = \int_{-\infty}^{\infty} G_1(t, x-\xi; h, \xi)u_0(h, \xi)d\xi + \int_h^t d\tau \int_{-\infty}^{\infty} G_2(t-\tau, x-\xi; \tau, \xi)\mu(\tau, \xi)d\xi , \tag{10}$$

when we suppose that function $\mu(t, x)$ is integrable function and its module of continuity by x satisfies the Dini condition when $t > 0$. Let's apply the operator of equation (5) to (10) for defining function $\mu(t, x)$ and satisfy non-homogeneous equation (5). Considering the properties of differentiation of potentials with kernel G_1 and G_2 we have

$$Lu(t, x) \equiv \left(\frac{\partial^\alpha}{\partial t^\alpha} - a(t, x) \frac{\partial^2}{\partial x^2} - b(t, x) \frac{\partial}{\partial x} - c(t, x) \right) u(t, x) =$$

$$= \int_{-\infty}^{\infty} LG_1(t, x - \xi; h, \xi) u_0(h, \xi) d\xi + \mu(t, x) +$$

$$+ \int_h^t d\tau \int_{-\infty}^{\infty} LG_2(t, \tau, x - \xi; t, \xi) \mu(\tau, \xi) d\xi + f_0(t, x, h), \, t > h, \, x \in \mathrm{R} .$$

We have the integral equation of second order Volterra–Fredholm

$$\mu(t, x) = F(t, x) + \int_h^t d\tau \int_{-\infty}^{\infty} K(t, \tau, x, \xi) \, \mu \, (\tau, \xi) d\xi, \, t > h, \, x \in \mathrm{R}, \tag{11}$$

for defining the function $\mu(t, x)$.

In (11) the functions F and K found via known functions G_1 and G_2. These functions are the solutions of problem (7) and data of original problem as

$$F(t, x) = f_0(t, x, h) - \int_{-\infty}^{\infty} LG_1(t, x - \xi; h, \xi) \, u_0(h, \xi) d\xi = f_0(t, x, h) +$$

$$+ \int_{-\infty}^{\infty} [(a(t, x) - a(h, \xi)) \frac{\partial^2 G_1}{\partial x^2} + b(t, x) \frac{\partial G_1}{\partial \xi} + c(t, x) G_1] u_0(h, \xi) d\xi \equiv f_0 + K_h * u_0,$$

$$K(t, \tau; x, \xi) = (a(t, x) - a(\tau, \xi)) \frac{\partial^2 G_2}{\partial x^2} + b(t, x) \frac{\partial G_2}{\partial x} + c G_2 (t - \tau, x - \xi; \tau, \xi) \equiv -LG_2.$$

Let's estimate the functions $F(t, x)$ and $K(t, \tau; x, \xi)$ using the estimates of (9) and the properties of the module of continuity [43]

$$\omega(t) t^{-1} < 2\omega(\tau) \tau^{-1}, \, 0 < \tau < t.$$

Thus, we have

$$| F(t, x) | \leq | f |_c + c_0 \int_{-\infty}^{\infty} e^{-c\rho(x - \xi, t)} \omega(| x - \xi | + t^{\alpha/2}) \, t^{-\frac{\alpha}{2}(1+2)} d\xi \, | \varphi |_c \leq$$

$$\leq c_0 \omega(t^{\frac{\alpha}{2}}) t^{-\alpha} \int_{-\infty}^{\infty} \exp\{ (\frac{| x - \xi |}{t^{\alpha/2}})^{\frac{2}{2-\alpha}} \} t^{-\frac{\alpha}{2}} d\xi \, | \varphi |_x \leq$$

$$\leq c_0 \omega(t^{\frac{\alpha}{2}}) t^{-\alpha} \int_{-\infty}^{\infty} \exp\{ (\frac{| x - \xi |}{t^{\alpha/2}})^{\frac{2}{2-\alpha}} \} \, t^{-\frac{\alpha}{2}} d\xi \, | \varphi |_x \leq | f |_c + C_0 \omega(t^{\frac{\alpha}{2}}) t^{-\alpha} \, | \varphi |_c , \tag{12}$$

$$| K(t, \tau; x, \xi) | \leq C_0 \frac{\omega(| x - \xi | + (t - \tau)^{\alpha/2})}{(t - \tau)^{\alpha/2 + 3/2 + 1 - \alpha}} e^{-c\rho(t - \tau, \, x - \xi)} \leq$$

$$\leq | f |_c + C_0 \omega(t^{\frac{\alpha}{2}}) t^{-\alpha} \, | \varphi |_c . \tag{13}$$

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Since for module of continuously $\omega(t)=t^{\gamma}$, $\gamma\in(0,1)$ and the kernel $K(t,\tau;x,\xi)$ is quasi-regular, then the resolvent

$$R(t,\tau;x,\xi)=K(t,\tau;x,\xi)+\sum_{i=1}^{\infty}\int_{\tau}^{t}d\beta\int_{-\infty}^{\infty}K(t,\beta;x,\xi)K_i(\beta,\tau;y,\xi)dy$$

defined from uniformly and absolutely convergent Neman series satisies the inequality (13) but with other constants, where

$$K_1(t,\tau;x,\xi)\equiv K(t,\tau;x,\xi),\ \ K(t,h;x,\xi)\equiv K_h(t,x,\xi),$$

$$K_{i+1}(t,\tau;x,\xi)=\int_{\tau}^{t}d\beta\int_{-\infty}^{\infty}K(t,\beta;x,\xi)K_i(\beta,\tau;y,\xi)dy.$$

If for module of continuously $\omega(t)$ the integral $\Phi(t)=\int_0^T\frac{\omega(\tau)}{\tau}d\tau$ converges, then Neman series is estimated by series

$$|R(t,\tau;x,\xi)|\le\sum_{i=1}^{\infty}(C\Phi(T))^I\,\omega((t-\tau)^{\alpha/2})(t-\tau)^{-1-\frac{\alpha}{2}}e^{-c_2\rho(t,\tau,x-\xi)}.$$

This series is converging series for $C\Phi(T)<1$, $t,\tau\in[0,T_0]$, $T_0<T$, ie locally converges.

3. The solution of integral equation

Let's write the integral equation (11) using a resolvent to finding of its solution $\mu(t,x)$ and substitute it into (10). Then we found the components of Green function for problem (5), (6)

$$\mu(t,x)=f_0(t,x,h)+K_h(t,x,\xi)*u_0(h,x)+R(t,\tau;x,\xi)**(f_0(t,x,h)+$$
$$+K_h(t,x,\xi)*u_0(h,x))=F(t,x,h)+R(t,\tau;x,\xi)**f_0(t,x,h)+(K_h(t,x,\xi)+$$
$$+R(t,\tau;x,\xi)**K_h(t,x,\xi)*u_0(h,x)). \qquad (14)$$

If we substitute (14) into (10) we obtain that the solution of the problem (5), (6) we can write in the form

$$u=G_1*u_0+G_2**\left(f_0+R**f_0\right)\ +G_2**\left(K_h+R**K_h\right)*u_0=$$
$$=\left[G_1+G_2**\left(K_h+R*K_h\right)\right]*\ +\left(G_2+G_2**R\right)**f_0 \qquad (15)$$

and the components of Green function in the form

$$Z_1(t,x,\xi)\equiv G_1(t,x;h,\xi)+G_2**(K_h+RK_h), \qquad (16)$$

$$Z_2(t,x;\tau,\xi)\equiv G_2(t-\tau,x-\xi;\tau,\xi)+\int_{\tau}^{t}d\beta\int_{-\infty}^{\infty}G_2(t-\beta,x-y;\beta,y)R(\beta,\tau;y,\xi)dy. \qquad (17)$$

So, we construct the classical solution of the problem (1), (2) on first step $h\le t\le 2h$:

$$u(t,x)=\int_{-\infty}^{\infty}Z_1(t,x;\xi)u_0(h,\xi)d\xi+\int_h^t d\tau\int_{-\infty}^{\infty}Z_2(t,\tau;x,\xi)f_0(\tau,\xi,h)d\xi, \qquad (18)$$

where Z_1 and Z_2 defined via equalities (16) and (17) respectively.

We can prove using the method of mathematical induction that formulas (15)–(18) are correct when $nh \le t \le (n+1)h$ and if we substitute *nh* instead of *h* and define Green function component respectively. So, theorem takes place.

Theorem. The solution of the problem (1), (2) is constructed using the step by step method and it is determined by the formula

$$u(t,x) = \int_{-\infty}^{\infty} Z_1(t,x,\xi)u_0(nh,\xi)d\xi + \int_{nh}^{t} d\tau \int_{-\infty}^{\infty} Z_2(t,\tau;x,\xi) f_0(\tau,\xi,nh)d\xi ,$$

$$nh \le t \le (n+1)h ,\ x \in \mathrm{R} .$$

Conclusion

In this paper we prove the solvability of the nonlocal problem using the step by step method for the first time. We consider a new class of the fractional equation with deviation time variable always supposing that the solution satisfies the nonlocal initial condition using the step by step method. These equations describe diffusion on inhomogeneous fractals. A fundamental solution of the Cauchy problem is constructed and investigated.

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НЕЛОКАЛЬНА ЗАДАЧА ДЛЯ РІВНЯННЯ ФРАКТАЛЬНОЇ ДИФУЗІЇ

Протягом останніх кількох десятиліть інтенсивно розвивається теорія псевдодиференціальних операторів (ПДО) та рівнянь із такими операторами (ПДР). Авторами нового напрямку теорії ПДР, названого параболічні ПДР з негладкими однорідними символами (ППДР), є Ярослав Дрінь і Самуїл Ейдельман. На початку 70-х років минулого століття вони побудували приклад задачі Коші для модифікованого рівняння теплопровідності, що містить замість оператора Лапласа ПДО, що є його квадратним коренем. Такий ПДО має однорідний символ |σ|, негладкий у початку координат. Фундаментальний розв'язок задачі Коші (ФРЗК) для такого рівняння є точною степеневою функцією. Для рівняння теплопровідності ФРЗК є точною експонентною функцією. Оператор Лапласа можна тлумачити як ПДО з однорідним гладким символом |σ|^2, σ ∈ Rn. Узагальненням рівняння теплопровідності є ППДР, що містять ПДО з однорідними негладкими символами. Вони мають важливе застосування в теорії випадкових процесів, зокрема, при побудові розривних марківських процесів з твірними інтегро-диференціальними операторами, які відносяться до ПДО, у сучасній теорії фракталів, яка останнім часом бурхливо розвивається. Якщо символ ПДО не залежить від просторових координат, то задача Коші для ППДР коректно розв'язна у просторі узагальнених функцій типу розподілів. Розв'язок при цьому записується як згортка ФРЗК із початковою узагальненою функцією. Ці результати належать низці вітчизняних та зарубіжних математиків, зокрема С. Ейдельману та Я. Дріню (які першими визначили ППДО з негладкими символами та розпочали дослідження задачі Коші для відповідних ППДР), М. Федорюку, О. Кочубею, В. Городецькому, Літовченку та ін. Для певних нових класів ППДР доведено коректну розв'язність задачі Коші у просторі гельдерових функцій, побудовано класичні ФРЗК, отримано точні оцінки їх похідних степеневого характеру [1–4]. Принципово важливим є запропоноване А. Кочубеєм тлумачення ПДО через гіперсингулярні інтеграли (ГСІ). При цьому за відомим символом ПДО будується символ ГСІ і навпаки [6]. Теорія ГСІ, що суттєво розширює клас ПДО, розроблена С. Самком [7]. Це поняття розповсюджено на матричні ГСІ [5]. Узагальненням задачі Коші є нелокальні багатоточкові за часовою змінною задачі та задача з відхиленням аргументу. Тут доведено розв'язність нелокальної задачі з використанням методу кроків. Розглядаємо еволюційне нелінійне рівняння з регуляризованою фрактальною похідною дробового порядку α ∈ (0, 1] за часовою змінною та еліптичний оператор зі змін-

ними коефіцієнтами просторової змінної. Це рівняння описує фрактальні властивості реальних даних, що виникають у таких прикладних областях, як турбулентність, гідрологія, екологія, геофізика, забруднення середовища, економіка та фінанси.

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THE NONLOCAL PROBLEM FOR FRACTAL DIFFUSION EQUATION

Over the past few decades, the theory of pseudodifferential operators (PDO) and equations with such operators (PDE) has been intensively developed. The authors of a new direction in the theory of PDE, which they called parabolic PDE with non-smooth homogeneous symbols (PPDE), are Yaroslav Drin and Samuil Eidelman. In the early 1970s, they constructed an example of the Cauchy problem for a modified heat equation containing, instead of the Laplace operator, PDO, which is its square root. Such a PDO has a homogeneous symbol |σ|, which is not smooth at the origin. The fundamental solution of the Cauchy problem (FSCP) for such an equation is an exact power function. For the heat equation, FSCP is an exact exponential function. The Laplace operator can be interpreted as a PDO with a smooth homogeneous symbol |σ|^2, σ ∈ Rn. A generalization of the heat equation is PPDE containing PDO with homogeneous non-smooth symbols. They have an important application in the theory of random processes, in particular, in the construction of discontinuous Markov processes with generators of integro-differential operators, which are related to PDO; in the modern theory of fractals, which has recently been rapidly developing. If the PDO symbol does not depend on spatial coordinates, then the Cauchy problem for PPDE is correctly solvable in the space of distribution-type generalized functions. In this case, the solution is written as a convolution of the FSCP with an initial generalized function. These results belong to a number of domestic and foreign mathematicians, in particular S. Eidelman and Y. Drin (who were the first to define PPDO with non-smooth symbols and began the study of the Cauchy problem for the corresponding PPDE), M. Fedoruk, A. Kochubey, V. Gorodetsky, V . Litovchenko and others. For certain new classes of PPDE, the correct solvability of the Cauchy problem in the space of Hölder functions has been proved, classical FSCP have been constructed, and exact estimates of their power-law derivatives have been obtained [1–4]. Of fundamental importance is the interpretation of PDO proposed by A. Kochubey in terms of hypersingular integrals (HSI). At the same time, the HSI symbol is constructed from the known PDO symbol and vice versa [6]. The theory of HSI, which significantly extend the class of PDO, was developed by S. Samko [7]. We extends this concept to matrix HSI [5]. Generalizations of the Cauchy problem are non-local multipoint problems with respect to the time variable and the problem with argument deviation. Here we prove the solvability of a nonlocal problem using the method of steps. We consider an evolutionary nonlinear equation with a regularized fractal fractional derivative α ∈ (0, 1] with respect to the time variable and a general elliptic operator with variable coefficients with respect to the second-order spatial variable. Such equations describe fractal properties in real processes characterized by turbulence, in hydrology, ecology, geophysics, environment pollution, economics and finance.

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