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CHRISTIAN SELINGER

ZETA FUNCTION REGULARIZED LAPLACIAN ON THE SMOOTH WASSERSTEIN SPACE ABOVE THE UNIT CIRCLE

Via elements of second order differential geometry on smooth Wasserstein spaces of probability measures we give an explicit formula for a Laplacian in the case that the Wasserstein space is based on the unit circle. The Laplacian on this infinite dimensional manifold is calculated as trace of the Hessian in the sense of Zeta function regularization. Its square field operator is the square norm of the Wasserstein gradient.

1. Differential geometry of smooth probability densities

Definition 1.1. Let $(M, \langle ., . \rangle_x)$ denote a complete simply connected Riemannian manifold without boundary and $\mathbb{T}^1$ denote $\mathbb{R}$ mod $\mathbb{Z}$ equipped with the flat metric.

- $P(M) := P = \{\mu$ Borel probability measure on M and $\int d_M(x,y)^2 \mu(dx) < \infty\}$
- $P_{ac}(M) := P_{ac} = \{\mu \in P : \mu \ll \mathrm{vol}_M\}$
- $P^\infty(M) := P^\infty = \{\mu \in P_{ac} : m(x) := \frac{d\mu}{d\ \mathrm{vol}_M}(x) > 0;$ for a.e. $x \in M\}$
- $C_c^\infty(M)$ is the space of smooth compactly supported funtions on M.
- Given $\mu, \nu \in P$: $\Pi(\mu,\nu) = \Pi := \{\pi \in P(M \times M) : \pi(A \times M) = \mu(A); \pi(M \times B) = \nu(B)$ for all Borel sets $A, B.\}$.
  $d_W(\mu,\nu)^2 := \inf_{\pi \in \Pi} \int d_M(x,y)^2 \pi(dx,dy)$ is called quadratic Wasserstein distance.
- The metric space (P, d_W) is called Wasserstein space. Convergence in Wasserstein distance is equivalent to weak convergence plus convergence of second moments.
  By Prokhorov's theorem the Wasserstein space is Polish if the underlying space is so, which is the case for M. The subspace $(P^\infty, d_W) \subset (P, d_W)$ is not complete, e.g. convolution of a positive density with rescaled Gaussians converges in Wasserstein distance to a Dirac measure.

Theorem 1.1 (Brenier-McCann [2][5])**.** *Given $\mu, \nu \in P_{ac}(M)$, then the optimal transport plan π realizing the Wasserstein distance between μ and ν is given by*

$$\pi = (id, \exp(\nabla\varphi))\#\mu,$$

i.e.

$$\pi(dx,dy) = \delta_{\{y = \exp_x(\nabla\varphi(x))\}}\mu(dx),$$

where φ is a μ-a.s. unique (up to constants) function on M which satisfies $(\varphi^c)^c = \varphi$ for $\varphi^c(y) := \min\{x \in M : d^2(x,y)/2 - \varphi(x)\}$

Theorem 1.2 (Benamou-Brenier [1])**.** *Given $\mu, \nu \in P_{ac}(\mathbb{R}^d)$. We denote by $C_{\mu,\nu}$ the set of all curves $c : [0,1] \to P_{ac}(\mathbb{R}^d)$ satisfying $c(0) = \mu$ and $c(1) = \nu$ such that there exists a time-dependent $L^2(c(t))$-integrable compactly supported vector field v_t for which the continuity equation*

$$\dot{c}(t) = -\nabla.(c(t)v_t)$$

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holds, i.e. for all $\varphi \in C_0^\infty([0,1]\times M)$ *it holds that*

$$\int_{[0,1]\times M} c(t)\dot{\varphi} vol_M dt = -\int_{[0,1]\times M} c(t) v_t.\nabla\varphi vol_M dt.$$

Then

$$d_W(\mu,\nu)^2 = \inf_{C_{\mu,\nu}} \int_{[0,1]\times M} |v_t|^2 c(t) vol_M dt.$$

Definition 1.2 (Tangent space)**.** We define the tangent space at a measure $\mu \in P^\infty(M)$ by:

$$T_\mu P^\infty(M) := \overline{\{\nabla\varphi \quad ; \varphi \in C_c^\infty(M)\}}^{L^2(\mu)}$$

and denote the inner product on $T_\mu P^\infty(M)$ by $\langle \nabla\varphi, \nabla\varphi'\rangle_\mu := \int_M \langle\nabla\varphi,\nabla\varphi'\rangle_x \mu(dx)$ for any $\mu \in P^\infty(M)$.

Remark 1.1. $P^\infty(M)$ is an infinite dimensional manifold in the sense of convenient calculus [3], i.e. the modelling space is not a Banach but a locally convex one, for instance if M is compact then $P^\infty(M) \subset C^\infty(M)$, where the latter is a locally convex vector space which is complete (in the sense that each Mackey-Cauchy sequence this space is also converging in this space). As topological space $P^\infty(M)$ is endowed with the smooth topology, which is the final topology with respect to smooth curves into it. The tangent space $T_\mu P^\infty$ corresponds to notion of kinematic tangent space in [3] p.284. See also [4].

Vector fields are defined as smooth (in the sense of c^∞-topology, see [3]) sections of the tangent bundle and Otto's formalism suggests to write $V \in \Gamma(P^\infty \leftarrow TP^\infty)$ as a distribution:

$$V(\mu) = -\mathrm{div}(\mu\,\nabla v),$$

i.e. for all $\varphi \in C_c^\infty(M)$, a function $v \in C^\infty(M)$ and $m_0 = \frac{d\mu}{d\mathrm{vol}}$:

$$(V(\mu)|\varphi) = \int_M \langle \nabla v,\, \nabla\varphi\rangle_x m_0(x) vol(dx).$$

We emphasize that the smooth function $v : M \ni x \mapsto \bar{v}(m_0(x)) \in \mathbb{R}$ for $\bar{v} : \mathbb{R} \to \mathbb{R}$ is a possible choice, i.e. take $\bar{v}(x) = \log(x) + 1$, then $\nabla\, v(x) = \nabla\,(\log(m_0(x)) + 1) = \frac{\nabla m_0(x)}{m_0(x)}$ gives the vector field associated to the entropy via Wasserstein gradient [6]:

$$\nabla^{\mathrm{P}^\infty} \int_M m_0 \log(m_0)\, \mathrm{vol}(dx) = -\mathrm{div}(m_0\,\nabla\, v)$$

Tangent map. For a smooth mapping $F : P^\infty(M) \to \mathbb{R}$, an interval $I = (-a,a)$ and any smooth curve $c : I \to P^\infty(M)$ such that $c(0) = \mu$ and $\dot{c}(0) = -\mathrm{div}(\mu\,\nabla\, u)$ the tangent map

$$\begin{aligned} \mathrm{T(F)}: \quad & TP \to \mathbb{R}\times\mathbb{R} \\ & (\mu,\dot{c}(0)) \mapsto (F(\mu), T(F)(\mu).\dot{c}(0)) := (F(\mu), \tfrac{d}{dt}|_0 (F\circ c)(t)) \end{aligned}$$

Proposition 1.1 (smooth Lie bracket [7])**.** *Given* $U, V \in \Gamma(TP^\infty \leftarrow P^\infty)$*. Since vector fields are not complete we have to construct their respective flows explicitely: For* $0 < a \ll \epsilon$ *we define* $Fl_t^U, Fl_t^V : (-a,a)\times P^\infty \to P^\infty$ *by*

$$\begin{aligned} \frac{\partial}{\partial t} Fl_t^U(\mu) &= U(Fl_t^U(\mu)) \\ &= -div(Fl_t^U(\mu)\nabla\bar{u}(Fl_t^U(\mu))) \end{aligned}$$

resp.

$$\begin{aligned} \frac{\partial}{\partial t} Fl_t^V(\mu) &= V(Fl_t^V(\mu)) \\ &= -div(Fl_t^U(\mu)\nabla\bar{v}(Fl_t^V(\mu))) \end{aligned}$$

Then the Lie bracket reads as follows:

$$\begin{aligned}[U,V](\mu) &= div(V(\mu)\,\nabla u) - div(U(\mu)\,\nabla v) + \\ &\quad + div(\mu\,\nabla T(u)(\mu).V(\mu)) - div(\mu\,\nabla T(v)(\mu).U(\mu))\end{aligned}$$

Here $T(u)(\mu)$ is the tangent map of u at μ, since u is a real-valued function on P it is the differential of u at μ.

Proof. As a prerequisite we calculate

$$\frac{\partial}{\partial t}T\left(\mathrm{Fl}^U_{-t}\right)(\mu),$$

i.e. the expression we differentiate is the tangent map of Fl^U_{-t} at μ. By the product rule applied to the flow equation:

$$\begin{aligned}\frac{\partial}{\partial t}T(\mathrm{Fl}^U_{-t})(\mu) &= T\left(\frac{\partial}{\partial t}\mathrm{Fl}^U_{-t}\right)(\mu) \\ &= -T\left(-\mathrm{div}(\mathrm{Fl}^U_{-t}\nabla\left\{u\circ\mathrm{Fl}^U_{-t}\right\})\right)(\mu) \\ &= \mathrm{div}\left[T(\mathrm{Fl}^U_{-t})(\mu)\nabla\left\{u\circ\mathrm{Fl}^U_{-t}\right\}(\mu)\right] + \\ &\quad + \mathrm{div}\left[(\mathrm{Fl}^U_{-t})(\mu)\nabla\left\{T(u\circ\mathrm{Fl}^U_{-t})\right\}(\mu)\right] \\ &= \mathrm{div}\left[T(\mathrm{Fl}^U_{-t})(\mu)\nabla\left\{u\circ\mathrm{Fl}^U_{-t}\right\}(\mu)\right] + \\ &\quad + \mathrm{div}\left[(\mathrm{Fl}^U_{-t})(\mu)\nabla\left\{T(u)(\mathrm{Fl}^U_{-t}(\mu))T(\mathrm{Fl}^U_{-t})(\mu)\right\}\right]\end{aligned}$$

By definition

$$\begin{aligned}[U,V](\mu) &= \frac{\partial}{\partial t}|_0(\mathrm{Fl}^U_t)^*V(\mu) \\ &= \frac{\partial}{\partial t}|_0(T(\mathrm{Fl}^U_{-t})\circ V\circ\mathrm{Fl}^U_t)(\mu) \\ &= \left(\frac{\partial}{\partial t}|_0 T(\mathrm{Fl}^U_{-t})(V\circ\mathrm{Fl}^U_t|_0)\right)(\mu) + T(\mathrm{Fl}^U_{-t})|_0\circ\left(\frac{\partial}{\partial t}|_0 V\circ\mathrm{Fl}^U_t\right)(\mu) \\ &= \frac{\partial}{\partial t}|_0 T(\mathrm{Fl}^U_{-t})(V(\mu)) + \frac{\partial}{\partial t}|_0 V(\mathrm{Fl}^U_t(\mu)) \\ &= \mathrm{div}\,[V(\mu)\nabla\,u(\mu)] + \mathrm{div}\,[\mu\,\nabla\,\{T(u)(\mu).V(\mu)\}] + \\ &\quad + \frac{\partial}{\partial t}|_0\left(-\mathrm{div}\left[\mathrm{Fl}^U_t(\mu)\nabla\,v(\mathrm{Fl}^U_t(\mu))\right]\right) \\ &= \mathrm{div}\,[V(\mu)\nabla\,u(\mu)] + \mathrm{div}\,[\mu\,\nabla\,\{T(u)(\mu).V(\mu)\}] + \\ &\quad - \mathrm{div}\,[U(\mu)\nabla\,v(\mu)] - \mathrm{div}\,[\mu\,\nabla\,\{T(v)(\mu).U(\mu)\}]\end{aligned}$$

□

In view of the Lie bracket we define the covariant derivative (compare to [4]):

Proposition 1.2 (smooth Levi-Cività connection)**.**

$$\begin{aligned}\widetilde{\nabla}_U V(\mu) &:= -div\,[U(\mu)\nabla\,v(\mu)] - div\,[\mu\,\nabla(\,T(v)(\mu).U(\mu))] \\ &= div\,[div(\mu\,\nabla\,u(\mu))\nabla\,v(\mu)] - div\,[\mu\,\nabla(\,T(v)(\mu).U(\mu))]\,,\end{aligned}$$

i.e. for all $\varphi\in C^\infty_c(M)$:

$$(\widetilde{\nabla}_U V(\mu)|\varphi) := \int_M\langle\nabla\langle\nabla\varphi,\nabla v\rangle_x,\nabla u\rangle_x\mu(dx) + \int_M\langle\nabla(T(v)(\mu).U(\mu)),\nabla\varphi\rangle_x\mu(dx)$$

Note that if v does not depend on the density μ, then $\widetilde{\nabla}_U V(\mu) \in T_\mu P^\infty$, for details see Lemma 4.14 in [4].

Proof. We have to show that

$$U\langle V, W\rangle_\mu = \langle \widetilde{\nabla}_U V, W\rangle_\mu + \langle V, \widetilde{\nabla}_U W\rangle_\mu,$$

i.e. for $I(\mu) = \langle V(\mu), W(\mu)\rangle_\mu$

$$\begin{aligned}
U\langle V, W\rangle_\mu = & \\
& T(I).U \\
= & \frac{d}{dt}|_0 \int_M \{\langle \nabla v((\mathrm{id} + t\nabla u(\mu))\#\mu), \nabla w((\mathrm{id} + t\nabla u(\mu))\#\mu)\rangle_x \times \\
& \quad \times (\mathrm{id} + t\nabla u(\mu))\#\mu)\} \\
= & \int_M \left\langle \frac{d}{dt}|_0 \nabla v((\mathrm{id} + t\nabla u(\mu))\#\mu), \nabla w(\mu) \right\rangle_x \mu(dx) \\
& \quad + \int_M \left\langle \nabla v(\mu), \frac{d}{dt}|_0 \nabla w((\mathrm{id} + t\nabla u(\mu))\#\mu) \right\rangle_x \mu(dx) \\
& + \int_M \langle \nabla(\langle \nabla v(\mu), \nabla w(\mu)\rangle_x), \nabla u(\mu)\rangle_x \\
= & \int_M \langle \nabla T(v).U, \nabla w(\mu)\rangle_x \, \mu(dx) + \int_M \langle \nabla v(\mu), \nabla T(w).U\rangle_x \, \mu(dx) \\
& \quad + \int_M \langle \nabla\nabla v, \nabla w\rangle_x, \nabla u\rangle_x \, \mu + \int_M \langle\langle \nabla v, \nabla\nabla w\rangle_x, \nabla u\rangle_x \, \mu \\
= & -\langle \mathrm{div}(\mu \nabla T(v).U), w\rangle_\mu - \langle \mathrm{div}(\mu \nabla T(w).U), v\rangle_\mu \\
& \quad + \langle \mathrm{div}(\mathrm{div}\, \mu \nabla u)\nabla v), w\rangle_\mu + \langle \mathrm{div}(\mathrm{div}\, \mu \nabla v)\nabla u), v\rangle_\mu \\
= & \langle \widetilde{\nabla}_U V, W\rangle_\mu + \langle V, \widetilde{\nabla}_U W\rangle_\mu.
\end{aligned}$$

Taking some Riemannian connection $\overline{\nabla}$ definied in terms of the Koszul formula

$$\begin{aligned}
2\langle \overline{\nabla}_U V, W\rangle_\mu \quad = \quad & U\langle V, W\rangle_\mu + V\langle W, U\rangle_\mu - W\langle U, V\rangle_\mu + \langle W, [U, V]\rangle_\mu \\
& -\langle V, [U, W]\rangle_\mu - \langle U, [V, W]\rangle_\mu
\end{aligned}$$

and substituting the Lie bracket and the calculations of $U\langle V, W\rangle_\mu$ into this formula shows that $\overline{\nabla} = \widetilde{\nabla}$. It is the Levi-Civitá connection since $\widetilde{\nabla}$ is torsion-free by definition. □

2. Second order calculus

In [6] the Hessian of the entropy functional $\mathrm{Ent}(\mu) = \int_{\mathbb{R}^n} \mu \log(\mu)\, \mathrm{vol}(dx)$ with respect to Kantorovich-Rubinstein metric was calculated by second order variation of the entropy functional along constant speed geodesics. This is possible for any smooth functional $E : P_{ac} \to \mathbb{R}$ of the type

$$E(\mu) = \int_M e(m(x))\, \mathrm{vol}(dx), \quad \frac{d\mu}{d\mathrm{vol}}(x) = m(x), \quad e : \mathbb{R}_+ \to \mathbb{R} \quad C^2.$$

The calculation of the Hessian will be done in normal coordinates, i.e. covariant derivatives are calculated in directions $U \in \Gamma(T\mathrm{P}^\infty)$ giving rise to geodesics:

$$U(\mu) = -\mathrm{div}(\mu \nabla u)$$

for some $u \in C_c^\infty(M)$ depending not on μ:

Proposition 2.1 (The Hessian: a variational approach)**.** *Let M have Ricci curvature bounded from below. Given a functional $E : P_{ac}(M) \to \mathbb{R}$ of the type*

$$E(\mu) = \int_M e(\mu(x)) vol(dx)$$

where $e : \mathbb{R}_+ \to \mathbb{R}$ is twice differentiable we define

$$\begin{aligned} p(\mu) &= \mu e'(\mu) - e(\mu) \\ p_2(\mu) &= \mu p'(\mu) - p(\mu). \end{aligned}$$

By $Hess\,^{var}E(\dot\mu, \dot\mu)$ we denote the second order variation of E along a geodesic path $t \mapsto \mu_t$ in P_{ac} of the form

$$\begin{cases} \partial_t \mu + \nabla.(\mu \nabla\varphi) = 0 \\ \partial_t \varphi + \frac{|\nabla\varphi|^2}{2} = 0. \end{cases}$$

for φ a time-dependent compactly supported function on M. By Γ resp. Γ_2 we denote the square field resp. the iterated square field operator with respect to the Laplace-Beltrami opererator Δ: $\Gamma(f,g) := \frac{1}{2}\{\Delta(fg) - g\Delta f - f\Delta g\}$ and $\Gamma_2(f) \equiv \Gamma_2(f,f) := \frac{1}{2}\Delta\Gamma(f,f) - \Gamma(f, \Delta f)$. Then

$$Hess\,^{var}E(\dot\mu, \dot\mu) = \int_M \Gamma_2(\varphi_0)\, p(\mu)\, vol + \int_M (\Delta\varphi_0)^2\, p_2(\mu)\, vol. \tag{1}$$

Proof. See [9], p441f. □

3. A regularized Laplacian on $\mathrm{P}^\infty(\mathbb{T}^1)$

Proposition 3.1 (Renormalized Laplacian on $\mathrm{P}^\infty(\mathbb{T}^1)$)**.** *Given a functional $E : P^\infty(\mathbb{T}^1) \to \mathbb{R}$ of the type*

$$E(\mu) = \int_{\mathbb{T}^1} e(\mu(x)) vol(dx),$$

where $e : \mathbb{R}_+ \to \mathbb{R}$ is C^3. For an orthonormal system $\{e_k(\mu)\}_{k\in\mathbb{N}}$ of

$$T_\mu P^\infty(\mathbb{T}^1) := \overline{C^\infty(\mathbb{T}^1)/\mathbb{R}}^{L^2(\mu)}$$

we define an operator A on $T_\mu P^\infty(\mathbb{T}^1)$ by diagonalization in the basis $\{e_k(\mu)\}_{k\in\mathbb{N}}$:

$$A : e_k(\mu) \mapsto \lfloor k/2 \rfloor^{-a} e_k(\mu);\ k \in \{2,3,\dots\},\ a > \frac{3}{2}$$

For the first mode we define $A : e_1(\mu) \mapsto 2\pi\sqrt{2} e_1(\mu)$.

Let $\widetilde{HessE}$ be the Hessian operator associated to the the (variational) Hessian $Hess\,^{var}E(.,.)(\mu)$. The **renormalized Wasserstein Laplacian** *in an open neighbourhood of μ as defined below is finite:*

$$\Delta^a{}_{P^\infty(\mathbb{T}^1)} E(\mu) := \sum_{k=1}^\infty \langle \widetilde{HessE} A e_k(\mu), A e_k(\mu)\rangle_\mu < \infty$$

Proof. For the inner product

$$\langle e_k, e_k \rangle_{\mathrm{vol}} \equiv \langle e_k, e_k \rangle_{H^1(\mathrm{vol})} := \frac{1}{(2\pi)^2} \langle e_k', e_k' \rangle_{L^2}$$

on $T_{\mathrm{vol}}\mathrm{P}^\infty(\mathbb{T}^1)$, we are given a complete orthonormal system on $T_{\mathrm{vol}}\mathrm{P}^\infty(\mathbb{T}^1)$ by

$$\begin{cases} e_{2k}(x) = \sqrt{2}\, k^{-1} \sin 2\pi kx, & k \in \mathbb{N} \\ e_{2k+1}(x) = \sqrt{2}\, k^{-1} \cos 2\pi kx, & k \in \mathbb{N} \\ e_1(x) = 1. \end{cases}$$

Likewise by

$$\begin{cases} e_{2k}(\mu)(x) \text{ such that} & \frac{d}{dx}e_{2k}(\mu)(x) = \frac{1}{\sqrt{\mu(x)}}\frac{d}{dx}e_{2k}(x), & k \in \mathbb{N} \\ e_{2k+1}(\mu)(x) \text{ such that} & \frac{d}{dx}e_{2k+1}(\mu)(x) = \frac{1}{\sqrt{\mu(x)}}\frac{d}{dx}e_{2k+1}(x), & k \in \mathbb{N} \\ e_1(\mu)(x) = 1. \end{cases}$$

with inital data

$$\begin{cases} e_{2k}(\mu)(0) = 0, & k \in \mathbb{N} \\ e_{2k+1}(\mu)(0) = 0, & k \in \mathbb{N} \end{cases}$$

we are given a complete orthonormal system of $T_\mu \mathrm{P}^\infty(\mathbb{T}^1)$: On the torus we can solve the defining differential equation by integration and orthonormality of $\{e_k(\mu)\}_{k\in\mathbb{N}}$ is given by definition. To show that $\{e_k(\mu)\}_{k\in\mathbb{N}} \subset T_\mu \mathrm{P}^\infty(\mathbb{T}^1)$ we consider a vector field u such that $\mathrm{div}_\mu u = 0$. We have to show that $e_k(\mu) \perp u$ with respect to $\langle .,. \rangle_\mu$ for all $k \in \mathbb{N}$:

$$\begin{aligned} \int_{\mathbb{T}^1} e_k'(\mu).u\mu &= \int_{\mathbb{T}^1} e_k'.u\sqrt{\mu} \\ &= -\int_{\mathbb{T}^1} \underbrace{(\sqrt{\mu}/\mu)e_k}_{\varphi_1 \in C_c^\infty(\mathbb{T}^1)}.(u\mu)' + \int_{\mathbb{T}^1} (e_k u\mu)\frac{\mu'}{2\sqrt{\mu^3}} \\ &= 0 + \int_{\mathbb{T}^1} \underbrace{\left(e_k \frac{\mu'}{2\sqrt{\mu^3}}\right)}_{\varphi_2' \text{ for } \varphi_2 \in C_c^\infty(\mathbb{T}^1)} u\mu = 0 \end{aligned}$$

since $\int \mu u.\varphi' = 0$ for any $\varphi \in C_c^\infty(S^1)$. Note that at this place it is crucial to deal with differentiable densities with full support. The function φ_2 is obtained by integration.

Given a functional $E : \mathrm{P}^\infty \to \mathbb{R}$ and a distribution $U \in T\mathrm{P}^\infty$ such that $(U(\mu)|\varphi) = \int_{\mathbb{T}^1} u'\varphi'\mu$ for smooth, compactly supported functions u and φ. According to ([9]):

$$\mathrm{Hess}^{\mathrm{var}} E(U,U)(\mu) = \int_{\mathbb{T}^1} \Gamma_2^\Delta(u)(\mu e'(\mu) - e(\mu))\mathrm{vol} + \int_{\mathbb{T}^1} (\Delta u)^2(\mu p'(\mu) - p(\mu))\mathrm{vol},$$

with

$$p(x) = xe'(x) - e(x) \text{ and } p'(x) = xe''(x) + e'(x) - e'(x) = xe''(x)$$

and Γ_2^Δ the iterated carré du champ operator with respect to the flat Laplacian $\Delta = \Delta_{\mathbb{T}^1} = \frac{d^2}{dx^2}$. Then

$$\begin{aligned} \mathrm{Hess}^{\mathrm{var}} E(U,U)(\mu) &= \int_{\mathbb{T}^1} (u'')^2(\mu e'(\mu) - e(\mu))\mathrm{vol} \\ &\quad + \int_{\mathbb{T}^1} (u'')^2(\mu^2 e''(\mu) - \mu e'(\mu) + e(\mu))\mathrm{vol} \\ &= \int_{\mathbb{T}^1} (u'')^2 \mu^2 e''(\mu)\mathrm{vol} \end{aligned}$$

$$\begin{aligned}\Delta^a_{\mathrm{P}^\infty(\mathbb{T}^1)}E(\mu) &= \int_{\mathbb{T}^1} 2(2\pi)^2((e_1(\mu))'')^2\mu^2 e''(\mu)\mathrm{vol}(dx) \\ &\quad + \sum_{k=2}^{\infty}\int_{\mathbb{T}^1}\lfloor k/2\rfloor^{-2a}((e_k(\mu))'')^2\mu^2 e''(\mu)\mathrm{vol}(dx) \\ &= 2(2\pi)^2\int_{\mathbb{T}^1}\frac{((\log\mu)')^2}{4\mu}\mu^2 e''(\mu)\mathrm{vol}(dx) \\ &\quad + \sum_{k=1}^{\infty}\int_{\mathbb{T}^1} k^{-2a}(((\mu^{-1/2}e'_{2k})')^2 + ((\mu^{-1/2}e'_{2k+1})')^2)\mu^2 e''(\mu)\mathrm{vol}(dx)\end{aligned}$$

Since $e''_{2k} = 2\pi k e'_{2k+1}$ resp. $e''_{2k+1} = -2\pi k e'_{2k}$ and $(e'_{2k})^2 + (e'_{2k+1})^2 = 2(2\pi)^2$ it follows that

$$\begin{aligned}((\mu^{-1/2}e'_{2k})')^2 + ((\mu^{-1/2}e'_{2k+1})')^2 &= ((e'_{2k})^2 + (e'_{2k+1})^2)(1/4\mu^{-3}(\mu')^2) \\ &\quad + ((e'_{2k})^2 + (e'_{2k+1})^2)(\mu^{-1}(2\pi k)^2) \\ &\quad + e'_{2k}e'_{2k+1}(-\mu^{-3/2}\mu'\mu^{-1/2}2\pi k + \mu^{-3/2}\mu'\mu^{-1/2}2\pi k) \\ &= 2(2\pi)^2\{1/4\mu^{-3}(\mu')^2 + \mu^{-1}(2\pi k)^2\}.\end{aligned}$$

Consequently

$$\begin{aligned}\Delta^a_{\mathrm{P}^\infty(\mathbb{T}^1)}E(\mu) &= \sum_{k=1}^{\infty}\int_{\mathbb{T}^1} k^{-2a}2(2\pi)^2\left\{1/4\mu^{-3}(\mu')^2 + \mu^{-1}(2\pi k)^2\right\}\mu^2 e''(\mu)\mathrm{vol}(dx) \\ &\quad +2(2\pi)^2\int_{\mathbb{T}^1}\frac{((\log\mu)')^2}{4\mu}\mu^2 e''(\mu)\mathrm{vol}(dx) \\ &= 2(2\pi)^2\sum_{k=1}^{\infty}k^{-2a}\int_{\mathbb{T}^1}\left\{1/4((\log\mu)')^2 + (2\pi k)^2\right\}\mu e''(\mu)\mathrm{vol}(dx) \\ &\quad +2(2\pi)^2\int_{\mathbb{T}^1}\frac{((\log\mu)')^2}{4}\mu e''(\mu)\mathrm{vol}(dx) \\ &< \infty\end{aligned}$$

since

$$\begin{aligned}&||((\log\mu)')^2\mu e''(\mu)||_\infty < +\infty \\ &||\mu e''(\mu)||_\infty < +\infty,\end{aligned}$$

which is guaranteed since the densities are supposed to have full support and to be sufficiently regular. □

For the Riemann zeta function $\zeta^R(s) = \sum_{k=1}^{\infty}\frac{1}{k^s}, \Re(s) > 1$ there exists a meromorphic continuation (see [10]) to the complex plane with single pole at $s = 1$ which was proved by Riemann in 1859 by the following functional equation:

$$\zeta^R(s) = 2^s\pi^{s-1}\sin\frac{s\pi}{2}\Gamma(1-s)\zeta^R(1-s);\;\; s\in\mathbb{C}\setminus\{1\}$$

which enables us to calculate a specific value:

$$\begin{aligned}\zeta^R(0) &= \frac{1}{\pi}\lim_{s\to 0}\sin\frac{s\pi}{2}\Gamma(1-s)\zeta^R(1-s) \\ &= \frac{1}{\pi}\lim_{s\to 0}\left(\frac{s\pi}{2} - \frac{s^3\pi^3}{48} + \dots\right)\left(-\frac{1}{s} + \dots\right) = -\frac{1}{2}.\end{aligned}$$

We used that $\mathrm{Res}(\zeta, 1) = \lim_{s\to 1}(s-1)\zeta^R(s) = 1 = a_{-1}$ and the Laurent series reads $\zeta^R(s) = \sum_{n=-1}^{\infty}a_n(s-1)^n$ i.e. $\zeta^R(1-s) = -\frac{1}{s} + \dots$.

Definition 3.1 (Zeta function regularized Laplacian)**.**

$$\Delta_{\mathrm{P}^\infty(\mathbb{T}^1)} E(\mu) := \lim_{a\to 0} \Delta^a_{\mathrm{P}^\infty(\mathbb{T}^1)} E(\mu)$$

is called (Zeta function) regularized Laplacian.

Proposition 3.2. *Let $e:\mathbb{R}_+\to\mathbb{R}$ is C^3. Given a functional $E: P^\infty(\mathbb{T}^1)\to\mathbb{R}$ of the type*

$$E(\mu) = \int_{\mathbb{T}^1} e(\mu(x))vol(dx).$$

Then

$$\begin{aligned}\Delta_{P^\infty(\mathbb{T}^1)} E(\mu) &= 2(2\pi)^2(\zeta^R(0)+1)\int_{\mathbb{T}^1}\left\{((\log\mu)'/2)^2\}\mu e''(\mu)\right\} vol(dx)\\ &= \pi^2\int_{\mathbb{T}^1}\{(\log\mu)')\}^2\,\mu e''(\mu)vol(dx).\end{aligned}$$

We used additionally the fact that for the analytical continuation of the Zeta function $\zeta^R(-2)=0$ holds.

Example 3.1. For $\mathrm{Ent}(\mu) = \int_{\mathbb{T}^1}\mu(x)\log\mu(x)\,\mathrm{vol}(dx)$ we have

$$\Delta_{\mathrm{P}^\infty(\mathbb{T}^1)}\mathrm{Ent}(\mu) = \pi^2||(\log\mu)'||^2_{L^2(\mathrm{vol})}$$

Example 3.2. For functionals $E(\mu)=\int_{\mathbb{T}^1} f(x)\mathrm{vol}(dx)$ with f a measurable function on $\mathbb{T}^1$ we have $\Delta_{\mathrm{P}^\infty(\mathbb{T}^1)}E(\mu)=0$ for all $\mu\in\mathrm{P}^\infty$.

Example 3.3. Set $E(\mu)=\frac{1}{2}\int_{\mathbb{T}^1}\mu^2\mathrm{vol}$, then

$$\Delta_{\mathrm{P}^\infty} E(\mu) = \pi^2||\nabla^{\mathrm{P}^\infty}\mathrm{Ent}(\mu)||^2_\mu$$

for all $\mu\in\mathrm{P}^\infty$.

In the following proposition we denote the $L^2(\mu)$ inner product by $\langle .,.\rangle_\mu$, if no measure is specified we consider the inner product on $L^2(\mathrm{vol})$, furthermore $\langle f,\mu\rangle := \int f\mu(dx)$.

Proposition 3.3. *Given a functional $F: P^\infty(\mathbb{T}^1)\to\mathbb{R}$ of the type*

$$F(\mu) = \Phi\left(\langle f,\mu\rangle\right),$$

where $f\in C_b(\mathbb{T}^1)$ and $\Phi\in C_b(\mathbb{R})$. Then

$$\Delta_{P^\infty}F(\mu) = 2(2\pi)^2\Phi''(\langle f,\mu\rangle)||f'\sqrt{\mu}||^2_{L^2}$$

and the square-field operator with respect to Δ_{P^∞} applied to functionals F reads:

$$\Gamma(F) = 2(2\pi)^2||\nabla^{P^\infty}F(\mu)||^2_\mu$$

Proof. Following ([4]) a geodesic $(\mu_t)_{t\in[0,T]}$ in P^∞ starting at $\mu_0=\mu$ satisfies

$$\dot\mu_t = -\mathrm{div}(\mu_t\nabla v_t)$$

where the smooth function v_t satisfies

$$\dot v_t = \frac{-|\nabla v_t|^2}{2}.$$

The second order variation of F along $(\mu_t)_{t\in[0,T]}$ reads

$$\begin{aligned}\frac{d^2}{dt^2}\Phi(\langle f,\mu_t\rangle) &= \frac{d}{dt}(\Phi'(\langle f,\mu_t\rangle)\langle f,\dot\mu_t\rangle)\\ &= \Phi''(\langle f,\mu_t\rangle)\langle f,\dot\mu_t\rangle^2 + \Phi'(\langle f,\mu_t\rangle)\frac{d}{dt}\langle f,\dot\mu_t\rangle\end{aligned}$$

Remark since $\dot{v}_t' = -v_t' v_t''$ and $\dot{\mu}_t = -(\mu_t v_t')'$

$$\begin{aligned} \frac{d}{dt}\langle f, \dot{\mu}_t\rangle &= \frac{d}{dt}\int f' v_t' \mu_t \mathrm{vol} \\ &= \langle f', -v_t' v_t'' \mu_t + v_t' \dot{\mu}_t\rangle \\ &= -\langle f', ((v_t')^2)' \mu_t + \mu_t' (v_t')^2\rangle \\ &= -\langle f', ((v_t')^2 \mu_t)'\rangle \end{aligned}$$

$$\begin{aligned} \mathrm{Hess}(F)(v', v')(\mu) &= \frac{d^2}{dt^2}_{t=0} \Phi(\langle f, \mu_t\rangle) \\ &= \Phi''(\langle f, \mu\rangle)\langle f', v'\rangle_\mu^2 + \Phi'(\langle f, \mu\rangle)\langle f'', (v')^2 \mu\rangle \end{aligned}$$

In this formula at the place of v' we plug in (here for $s > 1/2$) $k^{-s} e_k(\mu)'$ as in the proof of Proposition 3.1 in order to calculate

$$\Delta^s_{\mathrm{P}^\infty(\mathbb{T}^1)} F(\mu) = \sum_{k=1}^{\infty} \Phi''(\langle f, \mu\rangle)\langle f', k^{-s} e_k(\mu)'\rangle_\mu^2 + \Phi'(\langle f, \mu\rangle)\langle f'', (k^{-s} e_k(\mu)')^2 \mu\rangle$$

which equals

$$\Phi''(\langle f, \mu\rangle)\sum_{k=1}^{\infty}\langle f', k^{-s}\mu^{-1/2} e_k'\rangle_\mu^2 + 2(2\pi)^2 \Phi'(\langle f, \mu\rangle)\langle f'', 1\rangle \zeta(2s)$$

Since $\zeta(2s)$ is finite for $2s > 1$ and we now that $\langle f'', 1\rangle = 0$ the second term vanishes and by the functional equation for ζ we define again

$$\Delta_{\mathrm{P}^\infty(\mathbb{T}^1)} F(\mu) := \lim_{s\to 0} \Delta^s_{\mathrm{P}^\infty(\mathbb{T}^1)} F(\mu)$$

which equals

$$\lim_{s\to 0} \Phi''(\langle f, \mu\rangle) 2(2\pi)^2 ||f'\mu^{1/2}||^2_{H^{-s}} = \Phi''(\langle f, \mu\rangle) 2(2\pi)^2 ||f'\mu^{1/2}||^2_{L^2}.$$

Note that the limit is taken for $s \in \mathbb{C}$.

The square-field operator $\Gamma^s(F)$ with respect to $\Delta^s_{\mathrm{P}^\infty}$ is defined by

$$\frac{1}{2}\Delta^s_{\mathrm{P}^\infty}(F^2) - F\Delta^s_{\mathrm{P}^\infty}(F).$$

In a first step we remark that

$$\frac{1}{2}\frac{d^2}{dt^2}(F(\mu_t))^2 = (\frac{d}{dt}F(\mu_t))^2 + F(\mu_t)\frac{d^2}{dt^2}F(\mu_t)$$

and so

$$\frac{1}{2}\frac{d^2}{dt^2}(F(\mu_t))^2 - F\frac{d^2}{dt^2}(F(\mu_t)) = (\frac{d}{dt}F(\mu_t))^2 = (\Phi'(\langle f, \mu\rangle)\langle f', v'\rangle_\mu)^2$$

which entails

$$\Gamma^s(F) = \sum_{k=1}^{\infty}(\Phi'(\langle f, \mu\rangle)\langle f', k^{-s} e_k(\mu)'\rangle_\mu)^2 = 2(2\pi)^2(\Phi'(\langle f, \mu\rangle))^2 ||f'||^2_{H_\mu^{-s}}$$

But

$$\lim_{s\to 0} ||f'||^2_{H_\mu^{-s}} = ||f'||^2_{L^2(\mu)}$$

and consequently

$$\lim_{s\to 0}\Gamma^s(F) = 2(2\pi)^2 ||\nabla^{\mathrm{P}^\infty} F(\mu)||^2_\mu$$

□

Remark 3.1. By the chain rule the formulas for the regularized Wasserstein Laplacian can be extended to the set of test functions

$$\mathfrak{Z} = \left\{ F(\mu) \equiv \Phi\left(\langle f, \mu \rangle\right); \Phi \in C^2(\mathbb{R}^d), \quad f = (f_1, \ldots, f_d) \in C^2(\mathbb{T}^1; \mathbb{R}^d); \mu \in \mathrm{P}^\infty(\mathbb{T}^1) \right\},$$

i.e.

$$\Delta_{\mathrm{P}^\infty} F(\mu) = \sum_{i,j=1}^{d} \partial_i \partial_j \Phi\left(\langle f, \mu \rangle\right) \int_0^1 f_i' f_j' \mu$$

which equals the generator of the Wasserstein diffusion (see [8]) on $\mathrm{P}^\infty(\mathbb{T}^1)$ with inverse temperature $\beta = 0$.

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Université du Luxembourg, Unité de Recherche en Mathématiques, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Grand-Duchy of Luxembourg

E-mail address: christian.selinger@gmail.com