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ON REGULARIZATION OF THE FORMAL FOURIER–WIENER TRANSFORM OF THE SELF-INTERSECTION LOCAL TIME OF A PLANAR GAUSSIAN PROCESS

The Fourier–Wiener transform of the formal expression for a multiple self-intersection local time is described in terms of an integral, which is divergent on the diagonals. The method of regularization we used in this work is related to the regularization of functions with nonintegrable singularities. The strong local nondeterminism property, which is more restrictive than the property of local nondeterminism introduced by S. Berman, is considered. Its geometrical meaning in the construction of the regularization is investigated. As an example, the problem of regularization is solved for a compact perturbation of the planar Wiener process.

The present paper considers the multiple self-intersection local time for a planar Gaussian process. To define it, we use the Fourier–Wiener transform. The Fourier–Wiener transform of a formal expression for multiple self-intersection local time is described in terms of the integral of the ratio of two functions, where the denominator turns to zero on the diagonals. That is why this integral must be regularized in some way. The method of regularization we use in this work is related to the regularization of functions with nonintegrable singularities [1] in the theory of generalized functions. To present such a regularization in the case of an arbitrary Gaussian process, we introduce the property of strong local nondeterminism, which plays the key role in the construction of a regularization. Among a large number of works devoted to the self-intersection local time for random processes, we recall the papers related to our work. The problem of regularization of the self-intersection local time for a planar Wiener process was described in [2, 3]. In [2], for $f_\varepsilon(x) = \frac{1}{2\pi\varepsilon}e^{-\frac{\|x\|^2}{2\varepsilon}}, \varepsilon > 0, x \in \mathbb{R}^2$, E. B. Dynkin considered the expression

$$T^w_{\varepsilon,k} = \int_{\Delta_k}\prod_{i=1}^{k-1} f_\varepsilon(w(s_{i+1}) - w(s_i))d\vec{s},\ \Delta_k = \{0 \le s_1 \le \ldots \le s_k \le 1\}$$

which “blows up” when $\varepsilon \to 0+$. He proved that, under the right choice of the coefficients $B^l_k(\varepsilon)$, the random variable

$$\mathcal{T}^w_k = L_p\text{-}\lim_{\varepsilon\to 0+}\left[T^w_{\varepsilon,k} + \sum_{l=1}^{k-1} B^l_k(\varepsilon)T^w_{\varepsilon,l}\right]$$

is well defined.

In [3], J. Rosen showed that there exists

$$\int_{\Delta_k}\prod_{i=1}^{k-1}(\delta_0(w(s_{i+1}) - w(s_i))) - E\delta_0(w(s_{i+1}) - w(s_i))d\vec{s} :=$$

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$$= L_2\text{-}\lim_{\varepsilon\to 0+}\left[\int_{\Delta_k}\prod_{i=1}^{k-1}(f_\varepsilon(w(s_{i+1})-w(s_i))-Ef_\varepsilon(w(s_{i+1})-w(s_i)))d\vec{s}\right].$$

The existence of multiple points of paths of the Brownian motion in the plane, Markov processes in a complete metric space, and Gaussian processes was proved in [4]–[6], respectively. The concept of local nondeterminism for a Gaussian process was considered in [7, 9]. The Fourier–Wiener transform of Brownian functionals was widely discussed in [10].

The work consists of three parts. The necessity of a regularization of a formal expression for the Fourier–Wiener transform of the self-intersection local time for a planar Gaussian process is established in Section 1. As an example, the problem of regularization is considered for the case of a planar Wiener process.

In Section 2, we introduce a modification of the local nondeterminism property which we call the strong local nondeterminism. Here, we consider the geometrical meaning of this property describing the joint behavior of increments of the process. The main example of a Gaussian process with strong local nondeterminism is a compact perturbation of the Wiener process. In Section 2, we present examples of such processes arising as a solution to the Sturm–Liouville problem with the white noise on the right-hand side.

In Section 3, we present our main result about the regularization of the Fourier–Wiener transform for the self-intersection local time of a planar Gaussian process.

1. Formal expression of the Fourier–Wiener transform for the self-intersection local time of a Gaussian process

Let $\{x(t); t\in[0,1]\}$ be a planar Gaussian process continuous in the square mean with the mean zero. The main object of our investigation is the expression

$$T_k^x = \int_{\Delta_k}\prod_{i=1}^{k-1}\delta_0(x(s_{i+1})-x(s_i))d\vec{s}, \tag{1}$$

where δ_0 is the delta-function at the point 0. Expression (1) is the formal definition of k-multiple self-intersection local time for the process x on the time interval $[0;1]$.

In (1), we will consider the action of δ_0 on the functionals of the white noise and use the well-developed tools from Gaussian analysis for its study. Suppose that H is a real separable Hilbert space. The inner product in H is denoted by $(\cdot,\cdot)$.

Let $g\in C([0;1],H)$ be a function such that the linear span of its values is dense in H. Consider two independent Gaussian white noises in H: ξ_1 and ξ_2 [12]. Recall that the Gaussian white noise ξ in H is a family of jointly Gaussian random variables $\{(h,\xi); h\in H\}$ linearly depending on h and such that

$$E(h,\xi)=0,\ E(h,\xi)^2=\|h\|.$$

Define the Gaussian process x as follows:

$$x(t)=((g(t),\xi_1),(g(t),\xi_2)).$$

To investigate (1), consider its Fourier–Wiener transform. For $h_1,h_2\in H$, we denote, by $\mathcal{E}(h_1,h_2)$, the stochastic exponent

$$\mathcal{E}(h_1,h_2)=e^{\langle h_1,\xi_1\rangle+\langle h_2,\xi_2\rangle-\frac{1}{2}(\|h_1\|^2+\|h_2\|^2)}.$$

We recall the following definition [13].

Definition 1. $\mathcal{T}(\alpha)(h_1,h_2):=E\alpha\mathcal{E}(h_1,h_2)$ is called the Fourier–Wiener transform of the random variable α.

We note that the delta-function of a Gaussian random variable as a generalized Gaussian functional was considered in [10, 14]. Let us give meaning to the expression $\prod_{i=1}^{k-1}\delta_0(x(s_{i+1})-x(s_i))$ by approximating the delta-function by the family of functions

$$f_\varepsilon(x)=\frac{1}{2\pi\varepsilon}e^{-\frac{\|x\|^2}{2\varepsilon}}, \varepsilon>0, x\in\mathbb{R}^2.$$

Consider approximating values $\prod_{i=1}^{k-1}f_\varepsilon(x(s_{i+1})-x(s_i))$. It is not difficult to prove that there exists a limit

$$\mathcal{T}\left(\prod_{i=1}^{k-1}\delta_0(x(s_{i+1})-x(s_i))\right)(h_1,h_2):=$$

$$=\lim_{\varepsilon\to 0+}\mathcal{T}\left(\prod_{i=1}^{k-1}f_\varepsilon(x(s_{i+1})-x(s_i))\right)(h_1,h_2)=$$

$$=\frac{e^{-\frac{1}{2}(A^{-1}_{t_1\ldots t_k}(\vec{u}_1,\vec{u}_1)+A^{-1}_{t_1\ldots t_k}(\vec{u}_2,\vec{u}_2))}}{\Gamma_{t_1\ldots t_k}}, \tag{2}$$

where

$$\Delta g(t_l)=g(t_{l+1})-g(t_l), l=1,\ldots,k-1,$$
$$\vec{u}_i=((\Delta g(t_1),h_i),\ldots,(\Delta g(t_{k-1}),h_i)), i=1,2,$$
$$A_{t_1\ldots t_k}=(\Delta g(t_l),\Delta g(t_j))_{lj=1}^{k-1}.$$

During the whole article, we use the following notations. $\Gamma_{t_1\ldots t_k}$ is a Gram determinant constructed on $\Delta g(t_1),\ldots,\Delta g(t_{k-1})$. We also suppose that the following condition is fulfilled. For any $0\le t_1<t_2<\ldots<t_k\le 1$,

$$\Gamma_{t_1\ldots t_k}\neq 0.$$

$P_{t_1\ldots t_k}$ is a projection on the linear span of $(\Delta g(t_1)\ldots,\Delta g(t_{k-1}))$. It can be verified that the following lemma holds.

Lemma 1.

$$A^{-1}_{t_1\ldots t_k}(\vec{u}_1,\vec{u}_1)=\|P_{t_1\ldots t_k}h_1\|^2,$$

if $A^{-1}_{t_1\ldots t_k}$ exists.

Proof.

$$A^{-1}_{t_1\ldots t_k}(\vec{u}_1,\vec{u}_1)=\frac{1}{\Gamma_{t_1\ldots t_k}}\sum_{ij=1}^{k-1}(-1)^{i+j}M_{ij}(\Delta g(t_i),h_1)(\Delta g(t_j),h_1),$$

where M_{ij} is the minor of the matrix $A_{t_1\ldots t_k}$ corresponding to a row i and a column j. Let us define $B_{t_1\ldots t_k}$ as follows:

$$B_{t_1\ldots t_k}h_1=\frac{1}{\Gamma_{t_1\ldots t_k}}\sum_{ij=1}^{k-1}(-1)^{i+j}M_{ij}(\Delta g(t_i),h_1)\Delta g(t_j).$$

It is not difficult to check that

1) For any $h_1\perp\Delta g(t_1),\ldots,\Delta g(t_{k-1})$,

$$B_{t_1\ldots t_k}h_1=0,$$

2) For any $i=\overline{1,k-1}$,

$$B_{t_1\ldots t_k}\Delta g(t_i)=\Delta g(t_i).$$

Conditions 1), 2) yield

$$B_{t_1\ldots t_k}=P_{t_1\ldots t_k}.$$

To finish the proof, it is enough to note that

$$A^{-1}_{t_1\ldots t_k}(\vec{u}_1,\vec{u}_1)=(B_{t_1\ldots t_k}h_1,h_1).$$

The lemma is proved. □

It follows from Lemma 1 that, for $\vec{t}=(t_1,\ldots,t_k)\in\Delta_k$,

$$\mathcal{T}\left(\prod_{i=1}^{k-1}\delta_0(x(s_{i+1})-x(s_i))\right)(h_1,h_2)=\frac{e^{-\frac{1}{2}(\|P_{t_1\ldots t_k}h_1\|^2+\|P_{t_1\ldots t_k}h_2\|^2)}}{\Gamma_{t_1\ldots t_k}}. \tag{3}$$

Consider expression (3) in the case of a planar Wiener process. Here, we use $H=L_2([0,1])$. Then one can define a Wiener process as $w(t)=((1\!\!1_{[0,t]},\xi_1),(1\!\!1_{[0,t]},\xi_2))$, where ξ_1 and ξ_2 are independent white noises in $L_2([0,1])$. Now, (3) has the form

$$\mathcal{T}\left(\prod_{i=1}^{k-1}\delta_0(w(s_{i+1})-w(s_i))\right)(h_1,h_2)=\frac{e^{-\frac{1}{2}(\sum_{i=1}^{k-1}\|P_{t_it_{i+1}}h_1\|^2+\sum_{i=1}^{k-1}\|P_{t_it_{i+1}}h_2\|^2)}}{\prod_{i=1}^{k-1}(t_{i+1}-t_i)}. \tag{4}$$

The next statement describes the regularization of (4).

Theorem 1. *The following integral is finite:*

$$\int_{\Delta_k}\frac{\sum_{M\subset\{1,\ldots,k-1\}}(-1)^{|M|}e^{-\frac{1}{2}(\sum_{i\in M}\|P_{t_it_{i+1}}h_1\|^2+\sum_{i\in M}\|P_{t_it_{i+1}}h_2\|^2)}}{\prod_{i=1}^{k-1}(t_{i+1}-t_i)}d\vec{t}.$$

Proof. It is enough to check that the following integral exists:

$$\int_{\Delta_k}\left|\frac{\sum_{M\subset\{1,\ldots,k-1\}}(-1)^{|M|}e^{-\frac{1}{2}(\sum_{i\in M}\|P_{t_it_{i+1}}h_1\|^2)}}{\prod_{i=1}^{k-1}(t_{i+1}-t_i)}\right|d\vec{t}=$$

$$\int_{\Delta_k}\left|\frac{\prod_{i=1}^{k-1}(e^{-\|P_{t_it_{i+1}}h_1\|^2}-1)}{\prod_{i=1}^{k-1}(t_{i+1}-t_i)}\right|dt\le$$

$$\le\int_{\Delta_k}\frac{\prod_{i=1}^{k-1}\|P_{t_it_{i+1}}h_1\|^2}{\prod_{i=1}^{k-1}(t_{i+1}-t_i)}d\vec{t}=\int_{\Delta_k}\frac{\prod_{i=1}^{k-1}\left(\int_{t_i}^{t_{i+1}}h_1(s)ds\right)^2}{\prod_{i=1}^{k-1}(t_{i+1}-t_i)^2}d\vec{t}. \tag{5}$$

Let us prove that integral (5) converges. It is sufficient to consider the case $h_1\ge0$. Let us check that

$$\int_{t_{k-1}}^1\frac{\left(\int_{t_{k-1}}^{t_k}h_1(s)ds\right)^2}{(t_k-t_{k-1})^2}dt_k\le C\|h_1\|^2.$$

It is not difficult to see

$$\int_{t_{k-1}}^1\frac{\left(\int_{t_{k-1}}^{t_k}h_1(s)ds\right)^2}{(t_k-t_{k-1})^2}dt_k=$$

$$=\int_{t_{k-1}}^1\iint_{t_{k-1}}^{t_k}h_1(s_1)h_1(s_2)ds_1ds_2\frac{1}{t_k-t_{k-1}}dt_k=$$

$$=\iint_{t_{k-1}}^1h_1(s_1)h_1(s_2)\int_{s_1\vee s_2}^1\frac{1}{(t_k-t_{k-1})^2}dt_k=dt_kds_1ds_2\le$$

$$\le\iint_{t_{k-1}}^{t_k}h_1(s_1)h_1(s_2)\frac{1}{s_1\vee s_2-t_{k-1}}ds_1ds_2=$$

$$=2\int_{t_{k-1}}^1h_1(s_1)\int_{s_1}^1\frac{h_1(s_2)}{s_2-t_{k-1}}ds_2ds_1.$$

In $L_2([t_{k-1};1])$, consider the integral operator with the kernel

$$k(s_1,s_2)=\frac{1}{s_2-t_{k-1}}1\!\!1_{\{s_2>s_1\}}.$$

Let us check that k defines a bounded operator in $L_2([t_{k-1};1])$, by using the Schur test [11]. If there exist the positive functions $p,q:[t_{k-1};1]\to(0,+\infty)$ and α,β such that

$$\int_{t_{k-1}}^{1}k(s_1,s_2)q(s_2)ds_2\le\alpha p(s_1),$$

$$\int_{t_{k-1}}^{1}k(s_1,s_2)p(s_1)ds_1\le\beta q(s_2),$$

then k corresponds to a bounded operator with the norm less or equal to $\alpha\beta$. Put

$$p(s_1)=\frac{1}{\sqrt{s_1-t_{k-1}}},\ q(s_2)=\frac{1}{\sqrt{s_2-t_{k-1}}}.$$

Then

$$\int_{t_{k-1}}^{1}k(s_1,s_2)q(s_2)ds_2=\int_{s_1}^{1}\frac{1}{(s_2-t_{k-1})^{3/2}}ds_2\le 2\frac{1}{\sqrt{s_1-t_{k-1}}},$$

$$\int_{t_{k-1}}^{1}k(s_1,s_2)p(s_1)ds_1=\int_{t_{k-1}}^{s_2}\frac{1}{\sqrt{s_1-t_{k-1}}}ds_1\frac{1}{s_2-t_{k-1}}=\frac{2}{\sqrt{s_2-t_{k-1}}}.$$

So, we get the estimation

$$2\int_{t_{k-1}}^{1}h(s_1)\int_{s_1}^{1}\frac{h(s_2)}{s_2-t_{k-1}}ds_2ds_1\le 8\|h\|^2.$$

It implies that

$$\int_{\Delta_k}\frac{\prod_{i=1}^{k-1}\left(\int_{t_i}^{t_{i+1}}h_1(s)ds\right)^2}{(t_{i+1}-t_i)^2}d\vec{t}=$$

$$=\int_{\Delta_{k-1}}\frac{\prod_{i=1}^{k-2}\left(\int_{t_i}^{t_{i+1}}h_1(s)ds\right)^2}{(t_{i+1}-t_i)^2}\cdot\int_{t_{k-1}}^{1}\frac{\left(\int_{t_k}^{t_{k+1}}h_1(s)ds\right)^2}{(t_k-t_{k-1})^2}dt_kd\vec{t}\le$$

$$\le 8\|h_1\|^2\int_{\Delta_{k-1}}\frac{\prod_{i=1}^{k-2}\left(\int_{t_i}^{t_{i+1}}h_1(s)ds\right)^2}{(t_{i+1}-t_i)^2}d\vec{t}. \tag{6}$$

By using the same arguments, it can be checked that expression (6) is less or equal than $(8\|h_1\|^2)^{k-1}$.

The theorem is proved. □

The main aim of the present work is to construct the regularization of expression (3) for a general Gaussian process x.

Let us describe the properties of the Gaussian process which are necessary for the application of the method of regularization considered in Theorem 1.

2. Strong local nondeterminism property

In this section, we introduce the condition under which we are able to prove the existence of a regularization for the Fourier–Wiener transform of the self-intersection local time of a Gaussian process. This property is a little bit stronger than the local nondeterminism introduced by S. Berman [7]. As before, for $g \in C([0;1], H)$, we define a Gaussian process $x(t) = (g(t), \xi)$ with the help of a white noise ξ in the Hilbert space H. In this section, we also suppose that,

for any $0 \le t_1 < t_2 < \ldots < t_k \le 1$,

$$\Gamma_{t_1 \ldots t_k} \neq 0.$$

Definition 2. The process x is strongly locally nondeterministic if, for any fixed k and an arbitrary $M \subset \{1, \ldots, k-1\}$,

$$\Gamma_{t_1 \ldots t_k} \sim G(\Delta g(t_i), i \notin M) \prod_{i \in M} \|\Delta g(t_i)\|^2, \tag{7}$$

when $\max_{i \in M}(t_{i+1} - t_i) \to 0$.

It is evident that the property of local nondeterminism follows from (7). But the condition of definition 2 is more restrictive. For example, the next lemma shows that the strong local nondeterminism is sufficient for a weak convergence to zero of projections on the small increments of the process.

Lemma 2. *Suppose that the process x is strongly locally nondeterministic. Then*

$$\forall\, h \in H: \ P_{t_1 t_2} h \to 0, \ t_2 - t_1 \to 0.$$

Proof. It is enough to consider $P_{t_1 t_2} g(t)$ for fixed t. Suppose that $t \le t_1 < t_2$. We now apply condition (7) to the points $0 < t < t_1 < t_2$ or $0 < t = t_1 < t_2$. Note that (7) means that the orthogonal component of $\Delta g(t_i)$ to the linear span of $\{\Delta g(t_j); 1 \le j \le k-1, j \neq i\}$ asymptotically coincides with $\Delta g(t_i)$. In particular, this means that, for arbitrary $j \neq i$,

$$\frac{(\Delta g(t_j), \Delta g(t_i))}{\|\Delta g(t_j)\| \|\Delta g(t_i)\|} \to 0, t_{i+1} - t_i \to 0.$$

In our case, we have

$$\frac{(g(t), \Delta g(t_1))}{\|\Delta g(t_1)\|} \to 0, t_2 - t_1 \to 0.$$

Now suppose that $0 \le t_1 < t_2 \le t$. Then it follows from (7) that

$$\frac{(g(t) - g(t_2) + g(t_1), \Delta g(t_1))}{\|\Delta g(t_1)\|} \to 0, t_2 - t_1 \to 0.$$

Since $\|\Delta g(t_1)\| \to 0, t_2 - t_1 \to 0$, we have again

$$\frac{(g(t), \Delta g(t_1))}{\|\Delta g(t_1)\|} \to 0, t_2 - t_1 \to 0.$$

The last case $t_1 \le t \le t_2$ can be considered in the same way. The lemma is proved. □

We recall that x is locally nondeterministic on some open interval J [7] if and only if

$$\lim_{c \to 0+} \inf_{t_m - t_1 \le c} G\left(\frac{x(t_1)}{(\operatorname{Var} x(t_1))^{1/2}}, \ldots, \frac{x(t_m) - x(t_{m-1})}{(\operatorname{Var}(x(t_m) - x(t_{m-1})))^{1/2}} \right) > 0,$$

for $m \ge 2$ and arbitrary points which are ordered according to their indices: $t_1 < \ldots < t_m$ in J. The next example shows that there exist the locally nondeterministic processes for which the statement of Lemma 2 does not hold. Consider

Example 1.

$$x(t) = w(t) + \sqrt{t}\xi,\ t \in [0,1],$$

where w is a one-dimensional Wiener process, and ξ is the standard Gaussian random variable. Suppose that w and ξ are independent. Let us check that x is locally nondeterministic. To prove this, we note that, for $0 < t_1 < \ldots < t_m \le 1$,

$$\mathrm{Var}(x(t_i) - x(t_{i-1})) = t_i - t_{i-1} + (\sqrt{t_i} - \sqrt{t_{i-1}})^2, i = \overline{2,m},$$

$$\left(\frac{x(t_i) - x(t_{i-1})}{(\mathrm{Var}(x(t_i) - x(t_{i-1})))^{1/2}}, \frac{x(t_i) - x(t_{i-1})}{(\mathrm{Var}(x(t_i) - x(t_{i-1})))^{1/2}}\right) = 1$$

and, for $k \neq l$,

$$\left(\frac{x(t_k) - x(t_{k-1})}{(\mathrm{Var}(x(t_k) - x(t_{k-1})))^{1/2}}, \frac{x(t_l) - x(t_{l-1})}{(\mathrm{Var}(x(t_l) - x(t_{l-1})))^{1/2}}\right) =$$

$$= \frac{(\sqrt{t_l} - \sqrt{t_{l-1}})(\sqrt{t_l} - \sqrt{t_{l-1}})}{\sqrt{t_k - t_{k-1} + (\sqrt{t_k} - \sqrt{t_{k-1}})^2}\sqrt{t_l - t_{l-1} + (\sqrt{t_l} - \sqrt{t_{l-1}})^2}} =$$

$$= \frac{(\sqrt{t_k} - \sqrt{t_{k-1}})(\sqrt{t_l} - \sqrt{t_{l-1}})}{\sqrt{(\sqrt{t_k} - \sqrt{t_{k-1}})2\sqrt{t_k}}\sqrt{(\sqrt{t_l} - \sqrt{t_{l-1}})2\sqrt{t_l}}} =$$

$$= \frac{1}{2}\left(1 - \frac{\sqrt{t_{k-1}}}{\sqrt{t_k}}\right)^{1/2}\left(1 - \frac{\sqrt{t_{l-1}}}{\sqrt{t_l}}\right)^{1/2} \to 0, t_m - t_1 \to 0.$$

This yields

$$\lim_{c \to 0+} \inf_{t_m - t_1 \le c} G\left(\frac{x(t_1)}{(\mathrm{Var}\, x(t_1))^{1/2}}, \ldots, \frac{x(t_m) - x(t_{m-1})}{(\mathrm{Var}(x(t_m) - x(t_{m-1})))^{1/2}}\right) = 1 > 0.$$

To check that the projection related to the increment of the process x on a small time interval does not tend to zero, we consider

$$g(t_1) = \sqrt{t_1}e + 1\!\!\mathrm{I}_{[0,t_1]}.$$

Then, for $h = e \oplus 0$, we get

$$\|P_{t_1} h\|^2 = \frac{(\sqrt{t_1})^2}{t_1 + t_1} = \frac{1}{2} \not\to 0, t_1 \to 0.$$

This example shows that the strong local nondeterminism property is more restrictive than the local nondeterminism.

In the present work, the main example of the process with strong local nondeterminism is the process

$$x(t) = ((I + S)g^0(t), \xi), \tag{8}$$

where I is the identity operator and S is a compact operator in $L_2([0;1])$ such that $\|S\| < 1, g^0(t) = 1\!\!\mathrm{I}_{[0;t]}$.

Example 2. Consider the process

$$x(t) = w(t) + u(t),$$

where w is a Wiener process in $\mathbb{R}$ such that $w(t) = (1\!\!\mathrm{I}_{[0;t]}, \xi)$, ξ is a white noise in $L_2([0, \frac{\pi}{2}])$, and u is a solution of the following Sturm–Liouville problem [8]:

$$\begin{cases} u'' + u = \xi \\ u(0) = 0 \\ u(\frac{\pi}{2}) = 0. \end{cases} \tag{9}$$

The solution of (9) is given by the formula

$$u(t) = (g(t, \cdot), \xi),$$

where g is the Green function.

It is not difficult to check that

$$g(t,s) = -\cos t \sin s 1\!\!1_{\{s<t\}} - \sin t \cos s 1\!\!1_{\{s>t\}}.$$

We want that g describe the law of action of the operator S on $g^0(t)$. We have

$$(Sg^0(t))(u) = (S1\!\!1_{[0,t]})(u) = -\cos t \sin u 1\!\!1_{\{u<t\}} - \sin t \cos u 1\!\!1_{\{u>t\}}. \tag{10}$$

By using (10), we get

$$(Sf)(s) = \int_0^{\frac{\pi}{2}} f(u)[1\!\!1_{[s,\frac{\pi}{2}]}(u) \sin u \sin s - 1\!\!1_{[0,s]}(u) \cos u \cos s]du. \tag{11}$$

It follows from (11) that S is a compact operator.

The following lemma describes one of the properties of the process x.

Lemma 3. *x is strongly locally nondeterministic.*

Proof. To prove the lemma, let us check that

$$\lim_{\max_{i\in M}(t_{i+1}-t_i)\to 0} \frac{\Gamma_{t_1\ldots t_k}}{G(\Delta g(t_i), i \notin M)\prod_{i\in M}\|\Delta g(t_i)\|^2} = 1.$$

For an arbitrary $q \in L_2([0;1])$, we denote $\widetilde{q} = \frac{q}{\|q\|}$.

The properties of the Gram determinant imply that

$$\frac{\Gamma_{t_1\ldots t_k}}{G(\Delta g(t_i), i \notin M)\prod_{i\in M}\|\Delta g(t_i)\|^2} = \frac{G\Big(\widetilde{\Delta g}(t_1), \ldots, \widetilde{\Delta g}(t_{k-1})\Big)}{G(\widetilde{\Delta g}(t_i), i \notin M)}.$$

Check that, for $m \in M, l = \overline{1, k-1}, m \neq l$,

$$\Big(\widetilde{\Delta g}(t_m), \widetilde{\Delta g}(t_l)\Big) \to 0,$$

when $\max_{i\in M}(t_{i+1} - t_i) \to 0$.

Note that, for any $h \in L_2([0,1])$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $i \in M,\ t_{i+1} - t_i < \delta$.

$$\Big|(h, \widetilde{\Delta g^0}(t_i))\Big| < \varepsilon. \tag{12}$$

This implies that $\|S\widetilde{\Delta g^0}(t_m)\| \to 0$, when $\max_{i\in M}(t_{i+1} - t_i) \to 0$, since S is a compact operator.

By using (12), we get

$$\begin{aligned} &\frac{|((I+S)\Delta g^0(t_m), (I+S)\Delta g^0(t_l))|}{\|(I+S)\Delta g^0(t_m)\|\|(I+S)\Delta g^0(t_l)\|} = \\ &= (S\widetilde{\Delta g^0}(t_m), \widetilde{\Delta g^0}(t_l)) + (\widetilde{\Delta g^0}(t_m), S\widetilde{\Delta g^0}(t_l)) + (S\widetilde{\Delta g^0}(t_m), S\widetilde{\Delta g^0}(t_l))\times \\ &\times (1 + 2(S\widetilde{\Delta g^0}(t_m), \widetilde{\Delta g^0}(t_m)) + (S\widetilde{\Delta g^0}(t_m), S\widetilde{\Delta g^0}(t_m)))^{-1/2}\times \\ &\times (1 + 2(S\widetilde{\Delta g^0}(t_l), \widetilde{\Delta g^0}(t_l)) + (S\widetilde{\Delta g^0}(t_l), S\widetilde{\Delta g^0}(t_l)))^{-1/2} \to 0, \end{aligned} \tag{13}$$

when $\max_{i\in M}(t_{i+1} - t_i) \to 0$.

Since a value of the determinant does not change under an even number of transpositions of rows and columns, we suppose that $M = \{1, \ldots, l\}$. Then

$$\frac{G\left(\widetilde{\Delta g}(t_1), \ldots, \widetilde{\Delta g}(t_{k-1})\right)}{G\left(\widetilde{\Delta g}(t_i), i \notin M\right)} =$$

$$= \frac{G\left(\widetilde{\Delta g}(t_i), i \notin M\right) + F\left(\widetilde{\Delta g}(t_i), M_{1j}, m_{ij}^n, i, j = \overline{1, k-1}, n = \overline{1, l-1}\right)}{G\left(\widetilde{\Delta g}(t_i), i \notin M\right)},$$

where

$$F\left(\widetilde{\Delta g}(t_i), M_{1j}, m_{ij}^n, i, j = \overline{1, k-1}, n = \overline{1, l-1}\right) =$$
$$= \sum_{j=2}^{k-1} (-1)^{1+j} \left(\widetilde{\Delta g}(t_1), \widetilde{\Delta g}(t_j)\right) M_{1j} +$$
$$+ \sum_{j=3}^{k-1} (-1)^{1+j} \left(\widetilde{\Delta g}(t_2), \widetilde{\Delta g}(t_j)\right) m_{2j}^1 + \ldots +$$
$$+ \sum_{j=l+1}^{k-1} (-1)^{l+j} \left(\widetilde{\Delta g}(t_l), \widetilde{\Delta g}(t_j)\right) m_{lj}^{l-1}.$$

Here, M_{1j} is the minor of the matrix $\left(\left(\widetilde{\Delta g}(t_i), \widetilde{\Delta g}(t_j)\right)\right)_{ij=1}^{k-1}$,

and m_{ij}^k is the minor of the same matrix after the deleting of k rows and k columns.
Since $\|S\| < 1$, $(I + S)$ has a continuous inverse operator.
This and the compactness of S imply that $\inf_{\vec{t}} G(\widetilde{\Delta g}(t_i), i \notin M) > 0$. Consequently,

$$\frac{G(\widetilde{\Delta g}(t_i), i \notin M) + F\left(\widetilde{\Delta g}(t_i), M_{1j}, m_{ij}^n, i, j = \overline{1, k-1}, n = \overline{1, l-1}\right)}{G(\widetilde{\Delta g}(t_i), i \notin M)} \to 1,$$

when $\max_{i \in M}(t_{i+1} - t_i) \to 0$.
The lemma is proved. □

The strong local nondeterminism property can be reformulated in terms of the conditional variance.

Definition 3. A Gaussian process x has the strong local nondeterminism property if and only if $t_1 < t_2 < \ldots < t_k$

$$\frac{\operatorname{Var}(\Delta x(t_i)/\Delta x(t_j), 1 \le j \le k-1, j \neq i)}{\operatorname{Var}(\Delta x(t_i))} \to 1, t_{i+1} - t_i \to 0.$$

The strong local nondeterminism property can be used to describe the asymptotic behavior of $\Gamma_{t_1 \ldots t_k}$ when some of differences $t_{i+1} - t_i$ converge to zero. Note that this convergence holds for every $i = 1, \ldots, k-1$ in contrast to the Berman definition [7], where $i = k - 1$.

3. Regularization for the Fourier–Wiener transform of the self-intersection local time

As was shown in the previous section, the Fourier–Wiener transform of the formal expression for the self-intersection local time contains the function $\Gamma^{-1}_{t_1\ldots t_k}$ which has singularities along the diagonals. Here, we present the way of regularization of the integral with $\Gamma^{-1}_{t_1\ldots t_k}$ for the processes which are compact perturbations of the Wiener process. Let us suppose in this section that the Gaussian process x has the form

$$x(t)=((g(t),\xi_1),(g(t),\xi_2))$$

with the independent Gaussian white noises ξ_1,ξ_2 in $L_2([0;1])$ and

$$g(t)=g^0(t)+Sg^0(t),$$

where $g^0(t)=1\!\!\mathrm{I}_{[0;t]}$, S is a compact operator in $L_2([0;1])$ with $\|S\|<1$. For $0\le t_1<\ldots<t_k\le 1$, we denote, by $\widetilde{\Delta g}(t_1),\ldots,\widetilde{\Delta g}(t_{k-1})$, the orthonormal system which is obtained from $\Delta g(t_1),\ldots,\Delta g(t_{k-1})$ via the orthogonalization procedure. Since the elements $\Delta g(t_1),\ldots,\Delta g(t_{k-1})$ are linearly independent (see Section 2), all the elements $\widetilde{\Delta g}(t_1),\ldots,\widetilde{\Delta g}(t_{k-1})$ are nonzero. For $M\subset\{1,\ldots,k-1\}$, we denote, by P_M, the projection on $\widetilde{\Delta g}(t_i), i\in M$.

Theorem 2. *The following integral converges for arbitrary $h\in L_2([0;1])$:*

$$\int_{\Delta_k}\Gamma^{-1}_{t_1\ldots t_k}(\sum_{M\subset\{1,\ldots,k-1\}}(-1)^{|M|}e^{-\frac{1}{2}\|P_Mh\|^2})d\vec{t}.$$

Proof. It is enough to check the convergence of the integral

$$\int_{\Delta_k}|\prod_{j=1}^{k-1}(t_{j+1}-t_j)^{-1}(\sum_{M\subset\{1,\ldots,k-1\}}(-1)^{|M|}e^{-\frac{1}{2}\|P_Mh\|^2})|d\vec{t}=$$

$$=\int_{\Delta_k}\prod_{j=1}^{k-1}\frac{1-e^{-\frac{1}{2}(h,\widetilde{\Delta g}(t_j))^2}}{t_{j+1}-t_j}d\vec{t}\le\int_{\Delta_k}\frac{1}{2^{k-1}}\prod_{j=1}^{k-1}\frac{(h,\widetilde{\Delta g}(t_j))^2}{t_{j+1}-t_j}d\vec{t}.$$

Let us consider

$$\int_{t_{k-1}}^1\frac{(h,\widetilde{\Delta g}(t_{k-1}))^2}{t_k-t_{k-1}}dt_k.$$

By $f(t_k)$, we denote the difference

$$f(t_k)=\Delta g(t_{k-1})-P_{t_1\ldots t_{k-1}}\Delta g(t_{k-1}).$$

As was proved above, the process x is strongly locally nondeterministic. Hence, uniformly with respect to $t_1,\ldots,t_{k-1}$, the following relations hold:

$$\|P_{t_1\ldots t_{k-1}}\frac{\Delta g(t_{k-1})}{\sqrt{t_k-t_{k-1}}}\|\to 0,\ t_k\to t_{k-1},$$

$$\frac{\|f(t_k)\|^2}{t_k-t_{k-1}}\to 1,\ t_k\to t_{k-1}.$$

Consequently, it is enough to consider the integral

$$\int_{t_{k-1}}^1\frac{(h,\Delta g(t_{k-1}))^2}{(t_k-t_{k-1})^2}dt_k=\int_{t_{k-1}}^1\frac{(h+S^*h,\Delta g^0(t_{k-1}))^2}{(t_k-t_{k-1})^2}dt_k.$$

It can be shown that the last integral can be estimated above by $C\|h\|^2$ for some absolute constant C, as it was done in Section 1. Consequently, the initial integral absolutely converges. The theorem is proved. □

As a corollary, one can obtain the regularization for the formal expression of the self-intersection local time for the process x. For $0 \le t_1 < \ldots < t_k \le 1$, we define the random vectors $\overline{\Delta x}(t_1), \ldots, \overline{\Delta x}(t_{k-1})$ as follows:

$$\overline{\Delta x}(t_1) = \Delta x(t_1),$$

$$\overline{\Delta x}(t_j) = \Delta x(t_j) - E(\Delta x(t_j)/\Delta x(t_1), \ldots, \Delta x(t_{j-1})), \ j = 2, \ldots, k-1.$$

The following statement holds.

Theorem 3. *The following integral of a generalized Gaussian functional has a well-defined Fourier–Wiener transform:*

$$\int_{\Delta_k} \sum_{M \subset \{1,\ldots,k-1\}} (-1)^{|M|+(k-1)} \prod_{j \in M} \delta_0(\overline{\Delta x}(t_j)) \prod_{j=1}^{k-1} \frac{1}{t_{j+1} - t_j} (E \prod_{j \in M} \delta_0(\overline{\Delta x}(t_j)))^{-1} d\vec{t}.$$

The proof of this theorem is a straightforward application of Theorem 2.

Remark. Note that, for the Wiener process, $\overline{\Delta x}(t_j)$ coincide with $\Delta x(t_j), j = 1, \ldots, k-1$, and we obtain a regularization described in Section 1.

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