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RANDOM MAPS AND KOLMOGOROV WIDTHS

In this paper we consider strong random operators. We present the sufficient conditions on a compact set in a Hilbert space under which its image under a Gaussian strong random operator is well-defined and compact. In addition, we investigate the behavior of Kolmogorov widths of some compacts under a Gaussian strong random operator.

1. Introduction

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_H$, $(\Omega,\mathcal{F},P)$ be a probability space, and $L_0(\Omega,P,H)$ be a space of random elements in H.

Definition 1 ([1]). An operator $A : H \to L_0(\Omega,P,H)$ is said to be a strong random operator (SRO) if the following conditions hold

1) for any $\alpha,\beta\in\mathbb{R}$ and $f,g\in H$

$$P\{A(\alpha f+\beta g)=\alpha Af+\beta Ag\}=1;$$

2) for any $\varepsilon>0$

$$\lim_{f_n\to f} P\{\|Af_n-Af\|_H>\varepsilon\}=0.$$

The main feature of an SRO is that it can be unbounded for any $\omega\in\Omega$.

Example 1. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in H, $\xi_1,\xi_2,\dots$ be independent standard normal variables. Define the SRO A as follows

$$Af=\sum_{n=1}^{\infty}\xi_n(f,e_n)e_n,\ \ f\in H.$$

The operator A is unbounded almost surely because $\sup_n|\xi_n|=+\infty$ a.s.

This example shows that for any $f\in H$ Af is defined on the set Ω_f of a full measure that, generally speaking, depends on f. So the images of uncountable sets under operator A may be undefined.

Example 2. Let the SRO A be the same as in the Example 1. The set $K\subset H$ is defined as the image of the closed unit ball with center 0 in H under the compact operator S

$$Sg=\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{\ln(n+2)}}(g,e_n)e_n,\ \ g\in H.$$

Since $P\Big\{\overline{\lim_n}\frac{\xi_n^2}{\sqrt{\ln(n+2)}}<+\infty\Big\}=0$, then

$$P\{\text{for any } f\in K\ \ \sum_{n=1}^{\infty}\xi_n(f,e_n)e_n\ \text{ converges in } H\}=0.$$

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The operator A from the previous two examples has a following property. For every $h \in H$ Ah is a Gaussian random element in H. This naturally leads to the following definition.

Definition 2 ([2])**.** An SRO A is said to be a Gaussian SRO (GSRO) if for any $f \in H$ the distribution of Af is the Gaussian one.

As was shown before, the image of a compact set under a GSRO in general is not defined. The aim of this paper is to present sufficient conditions on a compact $K \subset H$ and a GSRO A under which $A(K)$ is a random compact. Also, we study the width of $A(K)$. These questions are inspired by the studying of stochastic flows. To explain our motivation more precisely consider the Harris flow [3] (its particular case is the Arratia flow [4]), which consists of Brownian particles with spatial correlation that depends on the difference between the positions of the particles. Such flows can lose the homeomorphic property. Thus, for the investigation of their geometry we can not apply the tools from differential geometry. In [5] it was proposed to study the geometry of random operators describing shifts of functions along a flow (in case of the Harris flow they are SROs). In addition, it was noticed that the geometry of semigroup of such operators can be characterized in terms of widths of compact sets with respect to it. Consequently, we need to know when the image of a compact set under the SRO that is generated by the Harris flow exists. In this paper we study the case of GSRO which seems to be simpler than the case of operators generated by the Harris flow.

2. The existence of the images of compact sets in Hilbert space under Gaussian strong random operators

Let $N_K(u)$ denote the smallest number of the closed balls with radii u that cover a compact K. Note that for GSRO A one can define its expectation EA as a bounded linear operator acting as

$$(EA)h := E(Ah),\ \ h \in H.$$

In what follows we will consider the centered GSRO A, i.e. such that $EA = 0$.

Theorem 1. *Let A be a centered GSRO and K be a compact set such that*

$$\int_{N_K(u)>1} (\ln N_K(u))^{1/2} du < +\infty. \tag{1}$$

Then with probability 1 $A(K)$ is well-defined and is a compact set.

Proof. Let us consider a Hilbert-valued Gaussian random process $\{Af, f \in K\}$. By Theorem 1.4.11 [6], for continuous modification of considered process to exist it is sufficient that there exist a constant $\alpha \in (0;1]$ and a convex, even, continuous function φ such that

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty,\ \lim_{x\to 0} \frac{\varphi(x)}{x} = 0$$

for which the following relations hold:

$$E\varphi\Big(\frac{\|Af - Ag\|_H^\alpha}{\|f-g\|_H}\Big) \le 1; \tag{2}$$

$$\int_{N_K(u)>1} \varphi^{-1}(N_K(u)) du < +\infty. \tag{3}$$

Let us check that for GSRO A there exists a constant $a > 0$ for which relation (2) holds with the function $\varphi(x) = e^{ax^2} - 1$ and $\alpha = 1$. Since A is a linear operator it is sufficient to show that there exists a constant $a > 0$ such that for any $h \in H$ with $\|h\| = 1$ $Ee^{a\|Ah\|_H^2} \le 2$. By [2] there exists a constant $c > 0$ such that for any $f \in H$ $E\|Af\|_H^2 \le$

$c\|f\|_H^2$. Take $a > 0$ for which $e^{2ac} \le 2$. Fix arbitrary $h \in H$ with $\|h\|_H = 1$. Denote by $\{\lambda_k\}_{k=1}^{\infty}$ the eigenvalues of the correlation operator S_h (S_h is a nuclear operator) of Ah [7]. Then

$$Ee^{a\|Ah\|_H^2} = \prod_{k=1}^{\infty} \frac{1}{\sqrt{1-2a\lambda_k}} = e^{-\frac{1}{2}\sum\limits_{k=1}^{\infty}\ln(1-2a\lambda_k)}.$$

Since $\sum_{k=1}^{\infty} \lambda_k = E\|Ah\|_H^2 \le c$ then for any $k \ge 1$ $\ 2a\lambda_k \le 2ac < \frac{1}{2}$. Consequently, for any $k \ge 1$ $\ -\ln(1-2a\lambda_k) \le 4a\lambda_k$ and

$$Ee^{a\|Ah\|_H^2} \le e^{2a\sum\limits_{k=1}^{\infty}\lambda_k} = e^{2aE\|Ah\|_H^2} \le e^{2ac} \le 2.$$

Thus, for GSRO A relation (2) is true with $\alpha = 1$ and $\varphi(x) = e^{ax^2} - 1$. Consequently, GSRO A has continuous modification on K and the image $A(K)$ is defined and is a compact set. □

Example 3. Let us consider the set $K = \{f \in C[0;2\pi] : f(0) = 0$ and for any $t_1, t_2 \in [0;2\pi]$ $\ |f(t_1) - f(t_2)| \le |t_1 - t_2|\}$. K is a compact set in $C[0;2\pi]$ with uniform metric and, consequently, in $L_2[0;2\pi]$. As a u-net of K we can consider the set of polygonal lines that take value 0 at the origin, have vertexes at points $\{\frac{2\pi k}{u}, k = \overline{1, [\frac{2\pi}{u}]}\}$, and have one-sided derivatives at these points ± 1. The number of elements of this u-net equals $2^{[\frac{2\pi}{u}]}$. Thus, for the set K

$$\int_{N_K(u)>1} (\ln N_K(u))^{1/2} du < +\infty.$$

The investigation of SROs was originated from work of A.V.Skorokhod [1], where he introduced the following GSRO A

$$Af(t) = \int_0^t f(s)dw(s), \ f \in L_2[0;2\pi].$$

For considered compact $K \subset L_2[0;2\pi]$ relation (1) holds. Consequently, $A(K)$ is a compact set. Thus, for $f, f_n \in K$ such that $f_n \xrightarrow[n\to\infty]{L_2} f$

$$\int_0^{2\pi} f_n(s)dw(s) \underset{n\to\infty}{\longrightarrow} \int_0^{2\pi} f(s)dw(s) \text{ a.s.}$$

and $\sup_{f\in K} \int_0^{2\pi} f(s)dw(s) < +\infty$.

3. The Kolmogorov widths of compact sets and their image under GSRO

Let $C \subseteq H$ be a subset of Hilbert space $H, L \subseteq H$ be a subspace of H.

Definition 3 ([8])**.** The Kolmogorov n-width of a set C in the space H is given by

$$d_n(C) := \inf_{\dim L \le n} \sup_{f\in C} \inf_{g\in L} \|f - g\|_H.$$

For the compact set from the Example 3 and the GSRO A

$$Af = \sum_{k=1}^{\infty} \xi_k (f, e_k) e_k, \ f \in L_2[0;2\pi],$$

where $\xi_1, \xi_2, \ldots$ are independent standard normal variables, $\{e_k\}_{k=1}^{\infty} = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos kt, \frac{1}{\sqrt{\pi}}\sin kt\}_{k=1}^{\infty}$ is an orthonormal basis in $L_2[0;2\pi]$, the following statement holds.

Proposition 1. *A.s.* $d_n(A(K)) = O\big((\frac{1}{n})^{\frac{1}{2}-\varepsilon}\big), n \to \infty$ *for any* $\varepsilon > 0$.

Proof. Let L_n be a space of polygonal lines that take value 0 at the origin, have vertexes at points $\{\frac{2\pi k}{n}, k=\overline{1,[\frac{2\pi}{n}]}\}$ and have one-sided derivatives at these point ± 1.

Then, for fixed $n \in \mathbb{N}$

$$d_n(A(K))^2 \leq \sup_{f \in A(K)} \inf_{g \in L_n} \|f-g\|^2_{L_2[0;2\pi]} \leq$$

$$\leq \sup_{f \in A(K)} \sum_{k=1}^{n} \frac{2\pi}{n} \Big(\sup_{t_1,t_2 \in [\frac{2\pi(k-1)}{n};\frac{2\pi k}{n}]} |f(t_1)-f(t_2)| \Big)^2.$$

Let us consider a new stochastic process $\widetilde{\varphi}(t) = \{A\varphi(t), \varphi \in K\}$, where $t \in [0;2\pi]$ is fixed. Since A is a continuous operator on K $\widetilde{\varphi}(t)$ is a continuous stochastic process on K. Consequently,

$$\sup_{f \in A(K)} \sum_{k=1}^{n} \frac{2\pi}{n} \Big(\sup_{t_1,t_2 \in [\frac{2\pi(k-1)}{n};\frac{2\pi k}{n}]} |f(t_1)-f(t_2)| \Big)^2 \leq$$

$$\leq \sum_{k=1}^{n} \frac{2\pi}{n} \Big(\sup_{t_1,t_2 \in [\frac{2\pi(k-1)}{n};\frac{2\pi k}{n}]} \|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|_{C(K)} \Big)^2.$$

To get upper estimation for $\sup_{t_1,t_2 \in [\frac{2\pi(k-1)}{n};\frac{2\pi k}{n}]} \|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|_{C(K)}$ let us consider the stochastic process $\widetilde{\varphi} = \{\widetilde{\varphi}(t), t \geq 0\}$ with values in $C(K)$. Use the Kolmogorov theorem about the sufficient condition for existence of Hölder continuous modification.

Lemma 1. *For any $m \in \mathbb{N}$ there exists a constant $c_m > 0$ such that*

$$E\|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|^{2m}_{C(K)} \leq c_m |t_1-t_2|^m.$$

Proof. Let $m=1$. Since $K = \{f \in L_2[0;2\pi] : f(0)=0$ and for any $t_1,t_2 \in [0;2\pi]$

$$|f(t_1)-f(t_2)| \leq |t_1-t_2|\}$$

then there exists a constant $c>0$ such that for any $f \in K$

$$\sum_{k=1}^{\infty} (f,e_k)^2 k^2 \leq c.$$

Thus,

$$\|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|^2_{C(K)} \leq \Big(\sup_{a: \sum_{k=1}^{\infty} a_k^2 k^2 \leq c} \sum_{k=1}^{\infty} \frac{\xi_k}{k} a_k k (e_k(t_1)-e_k(t_2)) \Big)^2 \leq$$

$$\leq c \cdot \sum_{k=1}^{\infty} \frac{\xi_k^2}{k^2} (e_k(t_1)-e_k(t_2))^2.$$

Since $\{e_k\}_{k=1}^{\infty} = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos kt, \frac{1}{\sqrt{\pi}}\sin kt\}_{k=1}^{\infty}$ is an orthonormal basis in $L_2[0;2\pi]$ there exists a constant $c_1>0$ for which

$$E\|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|^2_{C(K)} \leq c_1|t_1-t_2|.$$

By Corollary 3.2 [9], for any $m>1$ there exists constant $c_{m,2}>0$ such that

$$(E\|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|^{2m}_{C(K)})^{\frac{1}{2m}} \leq c_{m,2}(E\|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|^2_{C(K)})^{\frac{1}{2}}.$$

Consequently, for any $m>1$

$$E\|\widetilde{\varphi}(t_1)-\widetilde{\varphi}(t_2)\|^{2m}_{C(K)} \leq c_m|t_1-t_2|^m$$

with some constant $c_m > 0$. □

Thus, by Lemma 1 and the Kolmogorov theorem, for any $r \in (0; m-1)$ there exists $\gamma > 0$ for which

$$\sup_{t_1,t_2 \in [\frac{2\pi(k-1)}{n}; \frac{2\pi k}{n}]} \|\widetilde{\varphi}(t_1) - \widetilde{\varphi}(t_2)\|_{C(K)} \le \gamma |t_1 - t_2|^{\frac{r}{2m}} \text{ a.s.}$$

By the latter inequality for any $\varepsilon > 0$

$$d_n(A(K)) \le \widetilde{c}\Big(\frac{1}{n}\Big)^{\frac{1}{2}-\varepsilon} \text{ a.s.}$$

□

There exist compact sets such that behavior of their Kolmogorov widths under the GSRO from Example 1 when n tends to infinity doesn't change.

For the compact set $K = \{f \in H : (f, e_n)^2 \le \frac{1}{n^2}, n \ge 1\}$ the following statement holds.

Lemma 2. *Let A be the GSRO from Example 1. Then*

$$d_n(K) = \Big(\sum_{k=n+1}^{+\infty} \frac{1}{k^2}\Big)^{1/2} \text{ and } d_n(A(K)) \asymp \frac{1}{\sqrt{n}} \text{ a.s..}$$

Proof. Let $L_n := span\{e_k, k = \overline{1,n}\}$. Then one has

$$d_n(K) \le \sup_{f\in K} \inf_{g \in L_n} \|f - g\|_H = \Big(\sum_{k=n+1}^{+\infty} \frac{1}{k^2}\Big)^{1/2}.$$

Assume that $K_1 = K$ and σ is a distribution of $(\eta_1, \eta_2, \ldots)$, where $\eta_1, \eta_2, \ldots$ are independent random variables such that

$$\eta_n = \begin{cases} \frac{1}{n}, & \frac{1}{2} \\ -\frac{1}{n}, & \frac{1}{2} \end{cases}, \ n \ge 1.$$

Then, by Theorem 1 from [10] one has

$$d_n(K) \ge \Big(\sum_{k=n+1}^{+\infty} \frac{1}{k^2}\Big)^{1/2}.$$

To get upper estimation for $d_n(A(K))$ let us consider the set C_0 that is countable and dense in $A(K)$

$$C_0 = \cup_{m\ge 1}\big\{(\xi_1 r_1, \ldots, \xi_m r_m, 0, 0, \ldots),\ r_k \in \mathbb{Q},\ |r_k| \le \frac{1}{k^2}\big\}.$$

Then

$$d_n(A(K)) = d_n(C_0) \le \Big(\sum_{k=n+1}^{+\infty} \frac{\xi_k^2}{k^2}\Big)^{1/2}.$$

Since $\sum_{k=n+1}^{+\infty} \frac{\xi_k^2 - 1}{k^2} \underset{n\to\infty}{\longrightarrow} 0$ a.s. then there exists constant $c_1 > 0$ for which

$$d_n(A(K)) \le c_1 \frac{1}{\sqrt{n}} \text{ a.s.}$$

To get lower estimation for $d_n(A(K))$ let us use the theorem about the width of unit ball [8].

Show that with probability 1 $A(K)$ contains an n-dimensional ball with radius $\frac{1}{2\sqrt{n}}$.

Consider $\tau = \min\{k \ge 1 : |\xi_k| \ge \sqrt{n}, \ldots, |\xi_{k+n}| \ge \sqrt{n}\}$. Then for any $m \in \mathbb{N}$

$$p_m := P\{\tau = m\} = \Big(1 - \Big(\frac{2}{\sqrt{2\pi}} \int_{\sqrt{n}}^{+\infty} e^{-\frac{x^2}{2}} dx\Big)^n\Big)^{m-1} \Big(\frac{2}{\sqrt{2\pi}} \int_{\sqrt{n}}^{+\infty} e^{-\frac{x^2}{2}} dx\Big)^n$$

and with probability p_m $A(K)$ contains the set

$$C := \underbrace{0 \times \ldots \times 0}_{m-1} \times \Big[-\frac{\sqrt{n}}{m}; \frac{\sqrt{n}}{m}\Big] \times \Big[-\frac{\sqrt{n}}{m+1}; \frac{\sqrt{n}}{m+1}\Big] \times \ldots \times \Big[-\frac{\sqrt{n}}{m+n}; \frac{\sqrt{n}}{m+n}\Big] \times 0 \times 0 \times \ldots$$

C contains n-dimensional ball with radius $\frac{\sqrt{n}}{m+n}$. Thus, with probability p_m

$$d_{n-1}(A(K)) \geq \frac{\sqrt{n}}{m+n}.$$

Consequently, with probability

$$\sum_{m=1}^{n} P\{\tau = m\} = 1 - \Big(1 - \Big(\frac{2}{\sqrt{2\pi}} \int_{\sqrt{n}}^{+\infty} e^{-\frac{x^2}{2}} dx\Big)^n\Big)^n$$

the following relation holds

$$d_n\left(A\left(K\right)\right) \geq \frac{1}{2\sqrt{n}}.$$

Since $\sum_{n=1}^{\infty} \Big(1 - \Big(\frac{2}{\sqrt{2\pi}} \int_{\sqrt{n}}^{+\infty} e^{-\frac{x^2}{2}} dx\Big)^n\Big)^n$ converges, the Borel–Cantelli lemma gives

$$d_{n-1}(A(K)) \geq c_2 \frac{1}{\sqrt{n}} \text{ a.s.}$$

Taking into account the lower estimation for $d_n(A(K))$ one has

$$d_n(A(K)) \asymp \frac{1}{\sqrt{n}} \text{ a.s.}$$

Consequently, for considered compact set K and GSRO A the widths $d_n(A(K))$ and $d_n(K)$ behave identically when $n \to \infty$. $\square$

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