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ON β-DUAL OF BANACH-SPACE-VALUED DIFFERENCE SEQUENCE SPACES

ПРО β-ДУАЛЬНІ ПРОСТОРИ ДИФЕРЕНЦІАЛЬНИХ ПОСЛІДОВНОСТЕЙ ІЗ ЗНАЧЕННЯМИ У БАНАХОВИХ ПРОСТОРАХ

The main object of the paper is to introduce Banach-space-valued difference sequence spaces $\ell_\infty(X,\Delta)$, $c(X,\Delta)$, and $c_0(X,\Delta)$ as a generalization of the well-known difference sequence spaces of Kizmaz. We obtain a set of sufficient conditions for $(A_k)\in E^\beta(X,\Delta)$, where $E\in\{\ell_\infty,c,c_0\}$ and (A_k) is a sequence of linear operators from a Banach space X into another Banach space Y. Necessary conditions for $(A_k)\in E^\beta(X,\Delta)$ are also investigated.

Основна мета статті — ввести простори диференціальних послідовностей $\ell_\infty(X,\Delta)$, $c(X,\Delta)$, значення яких лежать у банаховому просторі, і $c_0(X,\Delta)$, як узагальнення добре відомих просторів диференціальних послідовностей Кізмаза. Встановлено низку достатніх умов для $(A_k)\in E^\beta(X,\Delta)$, де $E\in\{\ell_\infty,c,c_0\}$, а (A_k) — послідовність лінійних операторів із банахового простору X в інший банахів простір Y. Досліджено також і необхідні умови для $(A_k)\in E^\beta(X,\Delta)$.

1. Introduction and background. Let X, Y be Banach spaces with zero element θ and $\|\cdot\|$ denote the norm in either X or Y. Let $B(X,Y)$ be the Banach space of bounded linear operators on X into Y with the usual operator norm. $S=\{x\in X\colon \|x\|\le 1\}$ is the closed unit sphere in X. By $s(X)$ we mean the space of all X-valued sequences $x=(x_k)$, where $x_k\in X$, for each $k\in\mathbf{N}$, the set of positive integers. In case $X=\mathbf{C}$, the space of complex numbers, $s(X)$ reduces to s, the space of all scalar sequences. $\ell_\infty(X)$, $c(X)$ and $c_0(X)$ denote the Banach spaces of bounded, convergent and null X-valued sequences respectively, normed by $\|x\|_\infty=\sup_k\|x_k\|$. Let $A=(A_k)$ denote a sequence of linear but not necessarily bounded operators on X into Y. If E is any nonempty subset of $s(X)$, then the α- and β-duals of E were defined by Maddox [11] as follows:

$$E^\alpha=\left\{(A_k):\sum_k\|A_kx_k\|<\infty,\quad \text{for all}\quad x=(x_k)\in E\right\},$$

$$E^\beta=\left\{(A_k):\sum_k A_kx_k\quad\text{converges in}\quad \mathrm{Y}\quad\text{for all}\quad x=(x_k)\in E\right\}.$$

All sums without limits will be taken from $k=1$ to $k=\infty$. Since Y is complete, we have $E^\alpha\subset E^\beta$. The α- and β- duals of E may be regarded as generalized Köthe – Toeplitz duals, since in case $X=Y=\mathbf{C}$, when the (A_k) may be identified with complex numbers a_k, the duals reduce to the classical spaces first considered by Köthe and Toeplitz (see, for instance, [8]).

Maddox [11] determined Köthe – Toeplitz duals, in the operator case, for the sequence spaces $\ell_\infty(X)$, $c(X)$ and $c_0(X)$. The results indicate the gap between the operator and the ordinary scalar case. For example, in the scalar case, it is well known that $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$ (the space of absolutely summable sequences of scalars). However, for the operator case it is possible only to assert that $\ell_\infty^\beta(X) \subset c^\beta(X) \subset c_0^\beta(X)$. But, as far as, α-duals are concerned, Maddox [11] showed that $\ell_\infty^\alpha(X) = c^\alpha(X) = c_0^\alpha(X)$ which is natural extension of the scalar case where $\ell_\infty^\alpha = c^\alpha = c_0^\alpha = \ell_1$.

Inspired from the work of Maddox, many mathematicians have contributed in the determination of generalized Köthe – Toeplitz duals of various vector valued sequence spaces (see, for instance, [14, 16, 17] where many more references can be found).

The concept of difference sequence spaces was introduced by Kizmaz [9] as follows:

$$E(\Delta) = \Big\{x = (x_k) \in s\colon (\Delta x_k) \in E\Big\},$$

where $E \in \{\ell_\infty, c, c_0\}$ and $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbf{N}$. For a detailed account of difference sequence spaces one may refer to [1 – 7, 9, 12, 13]. It is well known [3, 4, 6, 9, 13] that $\ell_\infty^\alpha(\Delta) = c^\alpha(\Delta) = c_0^\alpha(\Delta) = D_1$, where $D_1 = \Big\{a = (a_k)\colon \sum k|a_k| < \infty\Big\}$, and $\ell_\infty^\beta(\Delta) = c^\beta(\Delta) = D_2$, where $D_2 = \Big\{a = (a_k)\colon \sum k a_k$ is convergent, $\sum \Big|\sum_{v=k+1}^\infty a_v\Big| < \infty\Big\}$ whereas $c_0^\beta(\Delta) = \Bigg\{a = (a_k)\colon \sum a_k\Big(\sum_{j=0}^k v_j\Big)$ converges for all $v \in c_0^+\} \bigcap \{a = (a_k)\colon \sum \Big|\sum_{j=k}^\infty a_j\Big| < \infty\Bigg\}$, where c_0^+ denotes the set of all positive sequences in c_0. Thus [13] (Theorem 3) $\ell_\infty^\beta(\Delta) = c^\beta(\Delta) \neq c_0^\beta(\Delta)$.

The main object of this paper is to introduce the Banach-space-valued difference sequence spaces $\ell_\infty(X, \Delta)$, $c(X, \Delta)$, and $c_0(X, \Delta)$ as a generalization of the classical difference sequence spaces of Kizmaz. We obtain a set of sufficient conditions for $(A_k) \in E^\beta(X, \Delta)$, where $E \in \{\ell_\infty, c, c_0\}$. Necessary conditions for $(A_k) \in E^\beta(X, \Delta)$ have also been investigated.

The following definition and well-known lemmas are required for establishing the results of this paper.

Let $(T_k) = (T_1, T_2, T_3, \ldots)$ be a sequence in $B(X, Y)$. Then the group norm of (T_k) is defined by $\|(T_k)\| = \sup \Big\|\sum_{k=1}^n T_k x_k\Big\|$ where the supremum is taken over all $n \in \mathbf{N}$ and all x_k in S. This concept was introduced by Robinson [15] and was termed as group norm by Lorentz and Macphail [10].

We write R_n for the n th tail of the sequence (T_k), i. e., $R_n = (T_n, T_{n+1}, T_{n+2}, \ldots)$.

Lemma 1.1 [11]. *If* (T_k) *be a sequence in* $B(X, Y)$, *then*

$$\left\|\sum_{k=n}^{n+p} T_k x_k\right\| \leq \|R_n\| \max\Big\{\|x_k\|\colon n \leq k \leq n+p\Big\}$$

for any x_k *and all* $n \in \mathbf{N}$ *and all nonnegative integers* p.

Lemma 1.2 [16]. *If* (T_k) *is a sequence in* $B(X, Y)$, *then exactly one of the following is true*:

(i) $\|R_n\| = \infty$ *for all* $n \geq 1$,

(ii) $\|R_n\| < \infty$ *for all* $n \geq 1$.

We now introduce the following sequence spaces:

$$c_0(X,\Delta) = \Big\{x = (x_k) \in s(X)\colon (\Delta x_k) \in c_0(X)\Big\},$$

$$c(X,\Delta) = \Big\{x = (x_k) \in s(X)\colon (\Delta x_k) \in c(X)\Big\},$$

$$\ell_\infty(X,\Delta) = \Big\{x = (x_k) \in s(X)\colon (\Delta x_k) \in \ell_\infty(X)\Big\}.$$

If we take $X = \mathbf{C}$, then we obtain the familiar difference sequence spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ of Kizmaz [9], respectively.

It is easy to see that these sequence spaces are BK spaces with the norm $\|x\|_\Delta = \|x_1\| + \|\Delta x\|_\infty$, $x = (x_k) \in E(X,\Delta)$, $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for $E \in \{\ell_\infty, c, c_0\}$.

2. Main results. We start this section by investigating necessary conditions for $(A_k) \in E^\beta(X,\Delta)$ where $E \in \{\ell_\infty, c, c_0.\}$. It is also shown that these conditions do not turn out to be sufficient.

Theorem 2.1 (*Necessity*). *If* $(A_k) \in c_0^\beta(X,\Delta)$, *then there exists* $m \in \mathbf{N}$ *such that*:

(i) $A_k \in B(X,Y)$ *for all* $k \geq m$,

(ii) $\big\|R_m(\lambda)\big\| < \infty$ *for some* $\lambda > 1$, *where* $R_m(\lambda) = \big(m\lambda^{-m}A_m, (m+1)\lambda^{-(m+1)}A_{(m+1)}, \ldots\big)$,

(iii) $\big\|R_n(\lambda)\big\| \to 0$ *as* $n \to \infty$.

Proof. Suppose that $(A_k) \in c_0^\beta(X,\Delta)$ but no $m \in \mathbf{N}$ exists for which $A_k \in B(X,Y)$ for all $k \geq m$. Proceeding as in [11] (Proposition 3.1), we get a strictly increasing sequence (k_i) of natural numbers and a sequence (z_i) in S such that $\|A_{k_i} z_i\| > i$ for each $i \geq 1$.

Define

$$x_k = \begin{cases} \dfrac{z_i}{i}, & \text{for } k = k_i, \quad i \geq 1, \\ \theta, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in c_0(X) \subset c_0(X,\Delta)$ but $\|A_k x_k\| > 1$ for infinitely many k, which is a contradiction to the fact that $\sum A_k x_k$ converges. Hence the A_k's are ultimately bounded.

Next suppose that (ii) fails, i.e., $\|R_m(\lambda)\| = \infty$ for all $\lambda > 1$. By Lemma 1.2, we have

$$\big\|R_n(\lambda)\big\| = \sup_{p\in\mathbf{N}, z_k\in S} \left\|\sum_{k=n}^{n+p} k\lambda^{-k}A_k z_k\right\| = \infty$$

for all $n \geq m$ and for all $\lambda > 1$. Then there exists a subsequence $m = n(1) < n(2) < \ldots$ of natural numbers and a sequence (z_k) in S such that $\Big\|\sum_{k=1+n(i)}^{n(i+1)} k\lambda^{-k}A_k z_k\Big\| > 1$ for each $i \geq 1$ and for all $\lambda > 1$.

Define

$$x_k = \begin{cases} k\lambda^{-k}z_k, & \text{for } n(i) < k \leq n(i+1), \quad i \geq 1, \\ \theta, & \text{otherwise.} \end{cases}$$

Then we have $(x_k) \in c_0(X) \subset c_0(X,\Delta)$ but $\Big\|\sum_{k=1+n(i)}^{n(i+1)} A_k x_k\Big\| > 1$ for each $i \geq 1$ showing that $\sum A_k x_k$ does not converge in Y which is again a contradiction.

Let, if possible, (iii) fails, say $\limsup_n \left\|R_n(\lambda)\right\| = 3p > 0$. Following Maddox [11] (Proposition 3.3), there exist natural numbers $n(1) \geq m(1) > m$ and $z_{m(1)}, \ldots, z_{n(1)}$ in S such that $\left\|\sum_{k=m(1)}^{n(1)} k\lambda^{-k} A_k z_k\right\| > p$. Choose $m(2) > n(1)$ such that $\left\|R_{m(2)}(\lambda)\right\| > 2p$. Then there exist $n(2) \geq m(2)$ and $z_{m(2)}, \ldots, z_{n(2)}$ in S such that $\left\|\sum_{k=m(2)}^{n(2)} k\lambda^{-k} A_k z_k\right\| > p$. Proceeding in this way, we define $x_k = \theta\ \big(1 \leq k < m(1)\big)$, $x_k = k\lambda^{-k} z_k\ \big(m(1) \leq k \leq n(1)\big)$, $x_k = \theta\ \big(n(1) < k < m(2)\big)$, $x_k = k\lambda^{-k} z_k\ \big(m(2) \leq k \leq n(2)\big)$, etc. Then $(x_k) \in c_0(X) \subset c_0(X, \Delta)$ but $\sum A_k x_k$ diverges, which gives a contradiction.

***Remark* 2.1.** It is clear that the conditions of Theorem 2.1 are also necessary for $(A_k) \in E^\beta(X, \Delta)$, where $E = \ell_\infty$ or c.

***Remark* 2.2.** The conditions of Theorem 2.1 are not sufficient for $(A_k) \in c_0^\beta(X, \Delta)$ and hence for $(A_k) \in E^\beta(X, \Delta)$, where $E = \ell_\infty$ or c, as is clear from the following example.

***Example* 2.1.** Let $X = Y = c_0$. Define $A_k \colon X \to Y$ as $A_k(x) = (0, 0, \ldots, k^{-1}x_k, 0, 0, \ldots)$ with $k^{-1}x_k$ in the k-position, where $x = (x_k) \in c_0$. Then $A_k \in B(X, Y)$ for all $k \in \mathbf{N}$, and for any $n \in \mathbf{N}$ and $\lambda > 1$,

$$\|R_n(\lambda)\| = \left\|(n\lambda^{-n} A_n, (n+1)\lambda^{-(n+1)} A_{(n+1)}, \ldots)\right\| =$$

$$= \sup_{x_k \in S, p \in \mathbf{N}} \left\| \sum_{k=n}^{n+p} (k\lambda^{-k} A_k) x_k \right\| \leq \frac{1}{\lambda^n}$$

so that $\left\|R_n(\lambda)\right\| \to 0$ as $n \to \infty$.

Let $0 \neq x \in \mathbf{C}$ and define a sequence (x_n) whose each term x_n is itself the sequence $(x, 2x, 3x, \ldots)$. Then $(\Delta x_n) = (x_n - x_{n+1})_{n \in \mathbf{N}} = \big((0, 0, \ldots), (0, 0, \ldots), (0, 0, \ldots) \ldots\big)$ which converges to $(0, 0, \ldots)$ as $n \to \infty$ so that $(\Delta x_n) \in c_0(c_0)$ and hence $(x_n) \in c_0(c_0, \Delta)$. However $\sum_{k=1}^n A_k x_k = (x, x, x, \ldots x, 0, \ldots)$ with entry x in the first n positions and 0 elsewhere and so $\sum_k A_k x_k$ is not convergent.

Although the conditions of Theorem 2.1 are not sufficient for $(A_k) \in E^\beta(X, \Delta)$ where $E \in \{\ell_\infty, c, c_0\}$, it is quite interesting to note that if we take $\lambda = 1$ in condition (ii) and conditions (i) and (iii) remaining the same, we get a set of sufficient conditions as proved below.

Theorem 2.2 (*Sufficiency*)**.** $(A_k) \in c_0^\beta(X, \Delta)$ *if there exists* $m \in \mathbf{N}$ *such that*:

(i) $A_k \in B(X, Y)$ *for all* $k \geq m$,

(ii) $\|R_m\| < \infty$, *where* $R_m = \big(mA_m, (m+1)A_{(m+1)}, \ldots\big)$,

(iii) $\|R_n\| \to 0$ *as* $n \to \infty$.

Proof. Let $x = (x_k) \in c_0(X, \Delta)$. Then $(x_k - x_{k+1}) \in c_0(X)$ and so $\sup_k \|x_k - x_{k+1}\| < \infty$. Now

$$\left\|x_1 - x_{k+1}\right\| = \left\| \sum_{v=1}^{k} (x_v - x_{v+1}) \right\| \leq \sum_{v=1}^{k} \|x_v - x_{v+1}\| = O(k)$$

and so $\|x_k\| \leq \|\Delta x_k\| + \|x_{k+1} - x_1\| + \|x_1\|$, for every k, which implies that $\sup_k k^{-1}\|x_k\| < \infty$. Let $\epsilon > 0$ be given. For $n \geq m$ and a nonnegative integer p, by Lemma 1.1 we have

$$\left\| \sum_{k=n}^{n+p} A_k x_k \right\| = \left\| \sum_{k=n}^{n+p} k A_k (k^{-1} x_k) \right\| \leq \|R_n\| \max_{n \leq k \leq n+p} k^{-1} \|x_k\| \leq \|R_n\| M,$$

where $M = \sup_k k^{-1} \|x_k\|$.

We can choose $k_1 \geq m$ such that $\|R_k\| < \dfrac{\epsilon}{M}$ for all $k \geq k_1$. Consequently we have $\left\| \sum_{k=n}^{n+p} A_k x_k \right\| < \epsilon$ for all $n \geq k_1$ and a nonnegative integer p, whence the completeness of Y implies that $\sum A_k x_k$ converges.

***Remark* 2.3.** It is clear that the conditions of Theorem 2.2 are also sufficient for $(A_k) \in$ $\in E^\beta(X, \Delta)$, where $E = \ell_\infty$ or c.

***Remark* 2.4.** The conditions of Theorem 2.2 are not necessary for $(A_k) \in c_0^\beta(X, \Delta)$ otherwise $\ell_\infty^\beta(X, \Delta) = c^\beta(X, \Delta) = c_0^\beta(X, \Delta)$, contrary to the case when $X = \mathbf{C}$ since $\ell_\infty^\beta(\Delta) = c^\beta(\Delta) \neq$ $\neq c_0^\beta(\Delta)$, as mentioned before.

***Remark* 2.5.** Although it seems that the conditions of the Theorem 2.2 are also not necessary for $(A_k) \in E^\beta(X, \Delta)$ where $E = \ell_\infty$ or c, but we have not been able to prove it and hence is an open problem.

3. Some further generalizations. The difference sequence spaces of Kizmaz were generalized by Et and Çolak [5] as follows:

Let r be a nonnegative integer. Then $E(\Delta^r) = \{x = (x_k) \colon (\Delta^r x_k) \in E\}$ for $E \in \{\ell_\infty, c, c_0\}$, where $\Delta^0 x = (x_k)$ and $\Delta^r x_k = \Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}$, for all $k \in \mathbf{N}$. The sequence spaces $E(\Delta^r)$ are BK spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{r} |x_i| + \|\Delta^r x\|_\infty, \quad E \in \{\ell_\infty, c, c_0\}.$$

Analogously, we define the following X-valued generalized difference sequence spaces $E(X, \Delta^r) = \{x = (x_k) \in s(X) \colon (\Delta^r x_k) \in E(X)\}$ for $E \in \{\ell_\infty, c, c_0\}$. Obviously, taking $X = \mathbf{C}$ we have $E(X, \Delta^r) = E(\Delta^r)$. Proceeding on the lines similar to the scalar case it is not a big issue to see that $E(X, \Delta^r)$ are BK spaces with norm $\|x\|_\Delta = \sum_{i=1}^{r} \|x_i\| + \|\Delta^r x\|_\infty$, $E \in \{\ell_\infty, c, c_0\}$ and to have the following simple but useful lemma.

Lemma 3.1. $\sup_k \|\Delta^r x_k\| < \infty$ *implies* $\sup_k k^{-r} \|x_k\| < \infty$.

Theorem 3.1 (*Necessity*). *If* $(A_k) \in c_0^\beta(X, \Delta^r)$, *then there exists* $m \in \mathbf{N}$ *such that*:

(i) $A_k \in B(X, Y)$ *for all* $k \geq m$,

(ii) $\|R_m(\lambda)\| < \infty$ *for some* $\lambda > 1$, *where* $R_m(\lambda) = \big(m^r \lambda^{-m} A_m, (m+1)^r \lambda^{-(m+1)} A_{(m+1)}, \ldots\big)$,

(iii) $\|R_n(\lambda)\| \to 0$ *as* $n \to \infty$.

The proof is similar to that of Theorem 2.1 and hence is omitted.

***Remark* 3.1.** The conditions of Theorem 3.1 are also necessary for $(A_k) \in E^\beta(X, \Delta^r)$, where $E = \ell_\infty$ or c.

***Remark* 3.2.** From Example 2.1, it is clear that the conditions of Theorem 3.1 are not sufficient for $(A_k) \in E^\beta(X, \Delta^r)$, where $E \in \{\ell_\infty, c, c_0\}$.

Using Lemma 3.1 and applying the same technique as in Theorem 2.2, we have the following theorem.

Theorem 3.2 (*Sufficiency*). $(A_k) \in c_0^\beta(X, \Delta^r)$ *if there exists* $m \in \mathbf{N}$ *such that*:

(i) $A_k \in B(X,Y)$ *for all* $k \geq m$,

(ii) $\|R_m\| < \infty$, *where* $R_m = \big(m^r A_m, (m+1)^r A_{(m+1)}, \ldots\big)$,

(iii) $\|R_n\| \to 0$ *as* $n \to \infty$.

Remark **3.3.** The conditions of Theorem 3.2 are not necessary for $(A_k) \in c_0^\beta(X, \Delta^r)$ otherwise $c_0^\beta(X,\Delta^r) = c^\beta(X,\Delta^r) = \ell_\infty^\beta(X,\Delta^r)$, contrary to the case when $X = \mathbf{C}$ and $r = 1$.

Remark **3.4.** To see that the conditions of Theorem 3.2 are not necessary for $(A_k) \in E^\beta(X,\Delta^r)$, where $E = \ell_\infty$ or c, is an open problem.

The difference sequence spaces of Kizmaz were also generalized by Gnanaseelan and Srivastava [7] as follows:

Let $v = (v_k)$ be a sequence of non-zero complex numbers such that

$$\frac{|v_k|}{|v_{k+1}|} = 1 + O\Big(\frac{1}{k}\Big) \quad \text{for each} \quad k,$$

$$k^{-1}|v_k| \sum\nolimits_{i=1}^{k} |v_i^{-1}| = O(1),$$

$\big(k|v_k^{-1}|\big)$ is a sequence of positive numbers increasing monotonically to infinity.

Then $E(\Delta_v) = \{x = (x_k)\colon (\Delta_v x_k) \in E\}$ for $E \in \{\ell_\infty, c, c_0\}$, where $\Delta_v x_k = v_k(x_k - x_{k+1})$, for all $k \in \mathbf{N}$.

We define $E(X,\Delta_v) = \{x = (x_k) \in s(X) : (\Delta_v x_k) \in E(X)\}$ for $E \in \{\ell_\infty, c, c_0\}$. Obviously, taking $X = \mathbf{C}$ and $v = (v_k) = (1,1,1,\ldots)$, we get back the classical spaces of Kizmaz.

The following extension of Lemma 1 of [7] is a useful tool for obtaining the sufficient conditions for $(A_k) \in E^\beta(X,\Delta_v)$, where $E \in \{\ell_\infty, c, c_0\}$.

Lemma 3.2. $\sup_k \|v_k(x_k - x_{k+1})\| < \infty$ *implies* $\sup_k k^{-1}\|v_k x_k\| < \infty$.

Arguing in the same way as in Theorem 2.1, we have the following theorem.

Theorem 3.3 (*Necessity*). *If* $(A_k) \in c_0^\beta(X,\Delta_v)$, *then there exists* $m \in \mathbf{N}$ *such that*

(i) $A_k \in B(X,Y)$ *for all* $k \geq m$,

(ii) $\|R_m(\lambda,v)\| < \infty$ *for some* $\lambda > 1$, *where* $R_m(\lambda,v) = \big(m\lambda^{-m} v_m^{-1} A_m, (m+1)\lambda^{-(m+1)} \times$ $\times\, v_{m+1}^{-1} A_{(m+1)}, \ldots\big)$,

(iii) $\|R_n(\lambda,v)\| \to 0$ *as* $n \to \infty$.

Remark **3.5.** The conditions of Theorem 3.3 are also necessary for $(A_k) \in E^\beta(X,\Delta_v)$, where $E = \ell_\infty$ or c.

Remark **3.6.** In view of Example 2.1, we see that the conditions of Theorem 3.3 are not sufficient for $(A_k) \in E^\beta(X,\Delta_v)$ where $E \in \{\ell_\infty, c, c_0\}$.

Arguing in the same way as in Theorem 2.2 and using Lemma 3.2, we have the following theorem.

Theorem 3.4 (*Sufficiency*). $(A_k) \in c_0^\beta(X,\Delta_v)$ *if there exists* $m \in \mathbf{N}$ *such that*:

(i) $A_k \in B(X,Y)$ *for all* $k \geq m$,

(ii) $\big\|R_m(v)\big\| < \infty$, *where* $R_m(v) = \big(m v_m^{-1} A_m, (m+1) v_{m+1}^{-1} A_{(m+1)}, \ldots\big)$,

(iii) $\big\|R_n(v)\big\| \to 0$ *as* $n \to \infty$.

***Remark* 3.7.** The conditions of Theorem 3.4 are not necessary for $(A_k) \in c_0^\beta(X, \Delta_v)$ otherwise $c_0^\beta(X, \Delta_v) = c^\beta(X, \Delta_v) = \ell_\infty^\beta(X, \Delta_v)$, contrary to the case where $X = \mathbf{C}$ and $v = (v_k) = (1, 1, 1, \ldots)$.

***Remark* 3.8.** It is an open problem to see the necessity of conditions of Theorem 3.4 for $(A_k) \in E^\beta(X, \Delta_v)$, where $E = \ell_\infty$ or c.

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