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APPROXIMATION BY RATIONAL FUNCTIONS ON DOUBLY CONNECTED DOMAINS IN WEIGHTED GENERALIZED GRAND SMIRNOV CLASSES

НАБЛИЖЕННЯ РАЦІОНАЛЬНИМИ ФУНКЦІЯМИ ДЛЯ ДВОЗВ'ЯЗНИХ ОБЛАСТЕЙ У ЗВАЖЕНИХ УЗАГАЛЬНЕНИХ ВЕЛИКИХ КЛАСАХ СМІРНОВА

Let $G\subset\mathbb{C}$ be a doubly connected domain bounded by two rectifiable Carleson curves. In this work, we use the higher modulus of smoothness in order to investigate the approximation properties of $(p-\varepsilon)$-Faber–Laurent rational functions in the subclass of weighted generalized grand Smirnov classes $E^{p),\theta}(G,\omega)$ of analytic functions.

Нехай $G\subset\mathbb{C}$ — двозв'язна область, що обмежена двома спрямлюваними кривими Карлесона. У цій роботі за допомогою вищого модуля гладкості вивчаються апроксимаційні властивості раціональних $(p-\varepsilon)$-функцій Фабера–Лорана у підкласах зважених узагальнених великих класів Смірнова $E^{p),\theta}(G,\omega)$ аналітичних функцій.

1. Introduction. We assume that B is a simply connected domain, bounded by a rectifiable Jordan curve Γ. We denote by $L^p(\Gamma)$ and $E^p(B)$, $1\le p<\infty$, the set of all measurable complex valued functions such that $|f|^p$ is Lebesgue integrable with respect to arclength on Γ, and the Smirnov class of analytic functions in B, respectively. We recall that if there exists a sequence (γ_n), $n=1,2,\ldots,$ of rectifiable Jordan curves in B, which converges to Γ as $n\to\infty$ such that

$$\sup_n\left\{\int\limits_{\gamma_n}\big|f(z)\big|^p|dz|\right\}<\infty,$$

we say that f belongs to Smirnov class $E^p(B)$ [24, p. 168]. Each function $f\in E^p(B)$ has nontangential limit almost everywhere (a.e.) on Γ and if we use the same notation for the limit function of f, then $f\in L^p(\Gamma)$. $L^p(\Gamma)$ and $E^p(B)$ are Banach spaces with respect to the norm

$$\|f\|_{E^p(B)}:=\|f\|_{L^p(\Gamma)}:=\left(\int\limits_{\Gamma}\big|f(z)\big|^p|dz|\right)^{1/p},\quad 1\le p<\infty.$$

Let $G\subset\mathbb{C}$ be a doubly connected domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curves Γ_1 and Γ_2 which Γ_2 is in Γ_1.

Let $G_1^-:=\operatorname{Ext}\Gamma_1$, $G_1:=\operatorname{Int}\Gamma_1$ and $G_2^-:=\operatorname{Ext}\Gamma_2$, $G_2:=\operatorname{Int}\Gamma_2$. Without loss of generality we suppose that $0\in G_2$.

Let also $\mathbb{T}:=\big\{w\in\mathbb{C}\colon|w|=1\big\}$, $\mathbb{U}:=\operatorname{Int}\mathbb{T}$ and $\mathbb{U}^-:=\operatorname{Ext}\mathbb{T}$. We denote by φ and φ_1 the conformal mappings of G_1^- and G_2 onto $\mathbb{U}^-$, respectively, normalized by

$$\varphi(\infty)=\infty,\qquad \lim_{z\to\infty}\frac{\varphi(z)}{z}>0,\qquad\text{and}\qquad \varphi_1(0)=\infty,\qquad \lim_{z\to0}z\varphi_1(z)>0.$$

Let ψ and ψ_1 be the inverse mappings of φ and φ_1, respectively. The functions φ and ψ have continuous extensions to Γ_1 and $\mathbb{T}$, their derivatives φ' and ψ' have definite nontangential limit values a.e. on Γ_1 and $\mathbb{T}$, they are integrable with respect to Lebesgue measure on Γ_1 and $\mathbb{T}$, respectively. Similarly, the functions φ_1 and ψ_1 have continuous extensions to Γ_2 and $\mathbb{T}$, their derivatives φ_1' and ψ_1' have definite nontangential limit values a.e. on Γ_2 and $\mathbb{T}$, which are integrable with respect to Lebesgue measure on Γ_2 and $\mathbb{T}$ [16, p. 19–438].

We set

$$L_r := \{z \in G_1^- : |\varphi(z)| = r > 1\} \qquad \text{and} \qquad L_R := \{z \in G_2 : |\varphi(z)| = R > 1\}.$$

Let $G_r^- := \operatorname{Ext} L_r$, $G_r := \operatorname{Int} L_r$ and $G_R^- := \operatorname{Ext} L_R$, $G_R := \operatorname{Int} L_R$.

φ is analytic function in G_r^- and

$$[\varphi(z)]^k \, [\varphi'(z)]^{1/(p-\varepsilon)}$$

has a pole with kth degree at ∞, where $0 < \varepsilon < p-1$. Besides φ_1 is analytic function in G_R and

$$[\varphi_1(z)]^{k-\frac{2}{p-\varepsilon}} \, [\varphi_1'(z)]^{1/(p-\varepsilon)}$$

has a pole with kth degree at 0, where $0 < \varepsilon < p-1$. For construction of polynomials of approximation process, we need some expansions. For this purpose, applying the same technic used in [5], for $1 < p < \infty$ and $0 < \varepsilon < p-1$, we obtain

$$\frac{[\psi'(w)]^{1-\frac{1}{p-\varepsilon}}}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_{k,p,\varepsilon}(z)}{w^{k+1}}, \quad z \in G_r, \quad w \in \mathbb{U}^-,$$

$$\frac{w^{\frac{-2}{p-\varepsilon}} [\psi_1'(w)]^{1-\frac{1}{p-\varepsilon}}}{\psi_1(w) - z} = \sum_{k=0}^{\infty} -\frac{\widetilde{F}_{k,p,\varepsilon}(1/z)}{w^{k+1}}, \quad z \in G_R^-, \quad w \in \mathbb{U}^-,$$

where $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ are polynomials with respect to z and $1/z$, respectively. Note that, firstly, $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ were stated in [14]. As in the classical case $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ have the following integral representations for every $k = 0, 1, 2, \ldots$:

$$F_{k,p,\varepsilon}(z) = \frac{1}{2\pi i} \int\limits_{L_r} \frac{[\varphi(\zeta)]^k (\varphi'(\zeta))^{\frac{1}{p-\varepsilon}}}{\zeta - z} d\zeta, \qquad z \in G_r, \quad r > 1, \tag{1}$$

$$\widetilde{F}_{k,p,\varepsilon}(1/z) = -\frac{1}{2\pi i} \int\limits_{L_R} \frac{[\varphi_1(\xi)]^{k-2/p-\varepsilon} (\varphi_1'(\xi))^{\frac{1}{p-\varepsilon}}}{\xi - z} d\xi, \qquad z \in G_R^-, \quad R > 1. \tag{2}$$

The polynomials $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ are called the $(p-\varepsilon)$-Faber polynomials for G_r and G_R^-, respectively.

If f is analytic function in doubly connected domain bounded by curves L_r and L_R, then, for $k = 0, 1, 2, \ldots$, using Cauchy's integral formulae and the expansions given for $F_{k,p,\varepsilon}$ and $\widetilde{F}_{k,p,\varepsilon}$, we have the $(p-\varepsilon)$-Faber–Laurent series expansion

$$f(z)=\sum_{k=0}^{\infty}a_k\left(f\right)F_{k,p,\varepsilon}(z)+\sum_{k=1}^{\infty}\widetilde{a}_k\left(f\right)\widetilde{F}_{k,p,\varepsilon}(1/z),$$

where

$$a_k(f):=\frac{1}{2\pi i}\int\limits_{|w|=r_1}\frac{f\left[\psi(w)\right]\left(\psi'(w)\right)^{1/(p-\varepsilon)}}{w^{k+1}}dw,\quad 1<r_1<r,$$

and

$$\widetilde{a}_k(f):=\frac{1}{2\pi i}\int\limits_{|w|=R_1}\frac{f\left[\psi_1(w)\right]\left(\psi_1'(w)\right)^{1/(p-\varepsilon)}w^{2/(p-\varepsilon)}}{w^{k+1}}dw,\quad 1<R_1<R.$$

The rational function

$$R_n(f)(z):=\sum_{k=0}^{n}a_k(f)F_{k,p,\varepsilon}(z)+\sum_{k=1}^{n}\widetilde{a}_k(f)\widetilde{F}_{k,p,\varepsilon}(1/z)$$

is called the $(p-\varepsilon)$-Faber – Laurent rational function of degree n of f.

Definition 1. *A rectifiable Jordan curve Γ is called Carleson curve if the condition*

$$\sup_{z\in\Gamma}\sup_{r>0}\frac{\left|\Gamma(z,r)\right|}{r}<\infty$$

holds, where $\Gamma(z,r)$ is portion of Γ in the open disk of radius r centered at z and $\left|\Gamma(z,r)\right|$ is it's length. We denote by S the set of all Carleson curves.

Under the various conditions on boundary of simply connected domains the direct and converse theorems of approximation theory in weighted and non weighted Smirnov classes have been investigated widely. When Γ is an analytic curve some results were obtained by Walsh and Russel in [19]. In the case of Γ is Dini-smooth curve the direct and inverse theorems were proved by S. Y. Alper in [27]. In Smirnov classes when Γ is Carleson curve these results are generalized in [18] and in weighted Smirnov classes some similar results for Carleson curves were obtained in [5 – 9, 17]. Similar theorems of approximation theory in Smirnov – Orlicz classes were studied in [26, 30, 31, 34].

When Γ is Dini-smooth curve direct and inverse theorems of approximation theory in the Smirnov classes with variable exponent were proved in [10, 12] and earlier similar results are stated without proof in [15, 25]. The approximation properties of Faber – Laurent series in Lebesgue space with variable exponent were investigated in [11].

On doubly connected domain, bounded by two Carleson curve, the rate of approximation by p-Faber – Laurent rational functions in Smirnov classes was studied in [29]. On doubly connected domain, bounded by Dini-smooth curves, the rate of approximation by Faber rational functions in Smirnov – Orlicz classes and Smirnov classes with variable exponent were investigated in [28] and [3], respectively.

The direct and inverse theorems of approximation theory in the weighted generalized grand Lebesgue spaces were proved in [13]. After that in weighted generalized grand Smirnov classes, defined on simple connected domain bounded by Carleson curve, some approximation theorems were proved in [14]. In this work, we investigate the approximation property of so-called $(p-\varepsilon)$-Faber – Laurent rational functions in the weighted generalized grand Smirnov classes, defined on doubly connected domains.

The set of all measurable functions f such that

$$\sup_{0<\varepsilon<p-1} \left\{ \varepsilon^{\theta} \frac{1}{|\Gamma|} \int\limits_{\Gamma} |f(x)|^{p-\varepsilon} \, \omega(x) dx \right\}^{1/(p-\varepsilon)} < \infty$$

constitute weighted generalized grand Lebesgue space $L^{p),\theta}(\Gamma,\omega)$. It becomes a Banach space equipped with the norm

$$\|f\|_{L^{p),\theta}(\Gamma,\omega)} := \sup_{0<\varepsilon<p-1} \left\{ \varepsilon^{\theta} \frac{1}{|\Gamma|} \int\limits_{\Gamma} \big|f(x)\big|^{p-\varepsilon} \omega(x) dx \right\}^{1/(p-\varepsilon)}.$$

If $\theta = 0$, then $L^{p),\theta}(\Gamma)$ turns into classical Lebesgue space $L^{p}(\Gamma)$. In nonweighted case, when $\theta = 1$, $L^{p),\theta}(\Gamma)$ is called grand Lebesgue space and it is denoted by $L^{p)}(\Gamma)$. The spaces $L^{p),\theta}(\Gamma)$ were introduced for $\theta = 1$ in [32] and for $\theta > 1$ in [21]. Dual spaces of $L^{p)}(\Gamma)$ were characterized in [1] and at the same work was given that $L^{p)}(\Gamma)$ is rearrangement invariant and Banach function space but it is not reflexive. We can show that

$$L^{p}(\Gamma) \subset L^{p)}(\Gamma) \subset L^{p-\varepsilon}(\Gamma).$$

We can say that similar embedding relations hold in case of weighted generalized grand Lebesgue space: if $\theta_1 < \theta_2$ and $1 < p < \infty$, then the embeddings

$$L^{p}(\Gamma,\omega) \subset L^{p),\theta_1}(\Gamma,\omega) \subset L^{p),\theta_2}(\Gamma,\omega) \subset L^{p-\varepsilon}(\Gamma,\omega)$$

are valid.

$L^{p}(\Gamma,\omega)$ is not dense in $L^{p),\theta}(\Gamma,\omega)$. We denote by $\mathcal{L}^{p),\theta}(\Gamma,\omega)$ the closure of $L^{p}(\Gamma,\omega)$ with respect to the norm of $L^{p),\theta}(\Gamma,\omega)$. We state that (see [20, 22]) $\mathcal{L}^{p)}(\Gamma)$ is the set of functions satisfying the condition

$$\lim_{\varepsilon\to 0} \left(\varepsilon^{\theta} \frac{1}{|\Gamma|} \int\limits_{\Gamma} \big|f(x)\big|^{p-\varepsilon} \omega(x) dx \right) = 0.$$

Now we construct the Smirnov class defined on the doubly connected domains. Let G^{*} be a doubly connected domain in $\mathbb{C}$ and f be an analytic function in G^{*}. If there exists a sequence $(\Delta_\nu)_{\nu=1}^{\infty}$ of domains whose boundaries $(\Gamma_\nu)_{\nu=1}^{\infty}$ consist of two rectifiable Jordan curves, lengths of $(\Gamma_\nu)_{\nu=1}^{\infty}$ are bounded, such that the domain Δ_n contains each compact subset of G^{*} for every $n \geq N$, for some $n \in \mathbb{N}$ and

$$\limsup_{\nu\to\infty} \left\{ \int\limits_{\Gamma_\nu} |f(z)|^{p} |dz| \right\} < \infty,$$

then it is said that f belongs to Smirnov classes $E^{p}(G^{*})$, $p \geq 1$ [24, p. 182].

Definition 2. *Let* $\Gamma := \Gamma_1 \cup \Gamma_2^-$ *and* G *be a doubly connected domain bounded by* Γ_1 *and* $\Gamma_2 \in S$, *where* Γ_2 *is in* Γ_1. *Let* ω *be a weight function on* Γ. *The set*

$$E^{p),\theta}(G,\omega) := \left\{ f \in E^1(G) : f \in L^{p),\theta}(\Gamma,\omega) \right\}$$

is called the weighted generalized grand Smirnov class of analytic functions in G.

For $f \in E^{p),\theta}(G,\omega)$ norm is defined by $\|f\|_{E^{p),\theta}(G,\omega)} := \|f\|_{L^{p),\theta}(\Gamma,\omega)}$. We denote by $\mathcal{E}^{p),\theta}(G,\omega)$ the closure of Smirnov class $E^p(G,\omega)$ of analytic function with respect to norm $E^{p),\theta}(G,\omega)$.

The Cauchy singular integral $S_\Gamma(f)$ and the Hardy–Littlewood maximal function $M_\Gamma(f)$ for $f \in L^1(\Gamma)$ for almost all $z_0 \in \Gamma$ are defined as following:

$$S_\Gamma(f)(z_0) := \lim_{r\to 0} \int\limits_{\Gamma\setminus\Gamma(z_0,r)} \frac{f(z)}{z - z_0} dz \qquad \text{and} \qquad M_\Gamma(f)(z_0) := \sup_{r>0} \frac{1}{r} \int\limits_{\Gamma(z_0,r)} |f(z)|\,|dz|.$$

Definition 3. *Let* ω *be weight function on* Γ *such that* $\Gamma \in S$. *Let* $1 < p < \infty$ *and* $1/p + 1/q = 1$. *We say that* ω *satisfies Muckenhoupt's* A_p *condition on* Γ *if*

$$\sup_{z_0\in\Gamma} \sup_{r>0} \left(\frac{1}{r} \int\limits_{\Gamma(z_0,r)} \omega(z)|dz| \right) \left(\frac{1}{r} \int\limits_{\Gamma(z_0,r)} \left[\omega(z)\right]^{-1/(p-1)} |dz| \right)^{p-1} < \infty.$$

Theorem A [33]. *Let* $\Gamma \in \mathcal{S}$, $1 < p < \infty$ *and* $\theta > 0$. *The operators* $S_\Gamma : f \to S_\Gamma(f)$ *and* $M_\Gamma : f \to M_\Gamma(f)$ *are bounded in* $L^{p),\theta}(\Gamma,\omega)$ *if and only if* $\omega \in A_p(\Gamma)$.

The norm in $L^{p),\theta}(\mathbb{T},\omega)$ space of 2π-periodic functions f is defined as

$$\|f\|_{L^{p),\theta}(\mathbb{T},\omega)} := \sup_{0<\varepsilon<p-1} \left\{ \frac{\varepsilon^\theta}{2\pi} \int\limits_0^{2\pi} \left| f\left(e^{it}\right) \right|^{p-\varepsilon} \omega\left(e^{it}\right) dt \right\}^{1/(p-\varepsilon)}.$$

Let $f \in L^{p),\theta}(\mathbb{T},\omega)$, $1 < p < \infty$, $\theta > 0$, and, for $r = 1, 2, 3, \ldots$,

$$\Delta_t^r f(w) = \sum_{s=0}^{r} (-1)^{r+s+1} \binom{r}{s} f\left(we^{ist}\right), \quad t > 0.$$

We define an operator $\sigma_h^r f(w) := \dfrac{1}{h}\displaystyle\int_0^h \left|\Delta_t^r f(w)\right| dt$. Let now $0 < h < \infty$. For a given $\omega \in A_p(\mathbb{T})$, $1 < p < \infty$, $\theta > 0$, by using Theorem A, we get that $\sup_{|h|\le\delta} \left\|\sigma_h^r f(w)\right\|_{L^{p),\theta}(\mathbb{T},\omega)} \le c\|f\|_{L^{p),\theta}(\mathbb{T},\omega)} < \infty$ which implies the correctness of the following definition.

Definition 4. *Let* $1 < p < \infty$, $\theta > 0$ *and let* $f \in L^{p),\theta}(\mathbb{T},\omega)$, $\omega \in A_p(\mathbb{T})$, $\delta > 0$. *The function* $\Omega_r(f,.)_{p),\theta,\omega} : [0,\infty) \to [0,\infty)$ *defined by*

$$\Omega_r(f,\delta)_{p),\theta,\omega} := \sup_{|h|\le\delta} \left\|\sigma_h^r f(w)\right\|_{L^{p),\theta}(\mathbb{T},\omega)}$$

is called rth mean modulus of f.

Let $\Gamma_1, \Gamma_2 \in \mathcal{S}$ and ω be a weight function on $\Gamma_1 \cup \Gamma_2^-$. We can consider ω as a weight on Γ_1 and Γ_2, separately.

For any $f \in L^{p),\theta}(\Gamma_1, \omega)$ and $\omega \in A_p(\Gamma_1)$, we set

$$f_0(w) := f\big[\psi(w)\big]\big(\psi'(w)\big)^{1/(p-\varepsilon)}, \qquad \omega_0(w) := \omega\big[\psi(w)\big], \tag{3}$$

and, for any $f \in L^{p),\theta}(\Gamma_2, \omega)$ and $\omega \in A_p(\Gamma_2)$, we set

$$f_1(w) := f\big[\psi_1(w)\big]\big(\psi_1'(w)\big)^{\frac{1}{p-\varepsilon}} w^{\frac{2}{p-\varepsilon}}, \qquad \omega_1(w) := \omega\big[\psi_1(w)\big]w^{-2}. \tag{4}$$

In this case obviously we have $f_0 \in L^{p),\theta}(\mathbb{T}, \omega_0)$ and $f_1 \in L^{p),\theta}(\mathbb{T}, \omega_1)$.

Let $f \in E^1(B)$, where B is a simply connected domain bounded with the rectifiable Jordan curve Γ^*. Then f has a nontangential limit a.e. on Γ^* and the boundary function belongs to $L^1(\Gamma^*)$. For a given $f \in L^{p),\theta}(\Gamma^*, \omega)$ the functions f^+ and f^- defined by

$$f^+(z) := \frac{1}{2\pi i}\int\limits_{\Gamma^*}\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{\big[\psi'(w)\big]^{1-\frac{1}{p-\varepsilon}}}{\psi(w) - z}f_0(w)dw, \quad z \in B,$$

$$f^-(z) := \frac{1}{2\pi i}\int\limits_{\Gamma^*}\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{w^{\frac{-2}{p-\varepsilon}}\big[\psi_1'(w)\big]^{1-\frac{1}{p-\varepsilon}}}{\psi_1(w) - z}f_1(w)dw, \quad z \in B^-,$$

are analytic in B and B^-, respectively, and $f^-(\infty) = 0$. The functions f^+ and f^- have the nontangential limits a.e. on Γ and the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z) \qquad \text{and} \qquad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z) \tag{5}$$

hold. Hence,

$$f(z) = f^+(z) - f^-(z) \tag{6}$$

holds a.e. on Γ [16].

The main result of this paper is the following theorem.

Theorem 1. *Let $\Gamma_1, \Gamma_2 \in \mathcal{S}$ and G be a finite doubly connected domain bounded by Γ_1 and Γ_2 such that the curve Γ_2 is inside of Γ_1. Let $\Gamma := \Gamma_1 \cup \Gamma_2^-$ and $\omega \in A_p(\Gamma)$, $\omega_0 \in A_p(\mathbb{T})$, $\omega_1 \in A_p(\mathbb{T})$, $1 < p < \infty$, $\theta > 0$. If $f \in \mathcal{E}^{p),\theta}(G, \omega)$, then there is a positive constant c independent of n such that*

$$\big\|f - R_n(f)\big\|_{L^{p),\theta}(\Gamma,\omega)} \le c\left[\Omega_r\left(f_0, \frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1, \frac{1}{n}\right)_{p),\theta,\omega_1}\right]$$

for $r = 1, 2, 3, \ldots$, where $R_n(f)$ is the nth partial sum of the $(p-\varepsilon)$-Faber–Laurent series of f.

2. Auxiliary results. We shall denote by $c,\ c_1,\ldots,$ the constants (in general, different in different relations) depending only on numbers that are not important for the questions of our interest.

The some properties of Faber polynomials were investigated in [2, 4, 23]. Similarly, to p-Faber polynomials (see [5]) we express some integral representations of $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$:

If $z\in G_r^-$, then

$$F_{k,p,\varepsilon}(z)=\left[\varphi(z)\right]^k\left[\varphi'(z)\right]^{1/(p-\varepsilon)}+\frac{1}{2\pi i}\int\limits_{L_r}\frac{\left[\varphi(\zeta)\right]^k\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}}{\zeta-z}d\zeta, \tag{7}$$

and if $z\in G_R$, then

$$\widetilde{F}_{k,p,\varepsilon}\left(\frac{1}{z}\right)=\left[\varphi_1(z)\right]^{k-\frac{2}{p-\varepsilon}}\left[\varphi_1'(z)\right]^{1/(p-\varepsilon)}-$$

$$-\frac{1}{2\pi i}\int\limits_{L_R}\frac{\left[\varphi_1(\xi)\right]^{k-\frac{2}{p-\varepsilon}}\left[\varphi_1'(\xi)\right]^{1/(p-\varepsilon)}}{\xi-z}d\xi. \tag{8}$$

By using Cauchy integral formulae, we have

$$f(z)=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi-z}d\xi,\quad z\in G.$$

If $z\in G_2$ or $z\in G_1^-$, then

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi-z}d\xi=0. \tag{9}$$

We define

$$I_1(z):=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta\qquad\text{and}\qquad I_2(z):=\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi-z}d\xi.$$

The function I_1 determines the analytic functions I_1^+ and I_1^- for $z\in G_1$ and $z\in G_1^-$, respectively, while the function I_2 determines the analytic functions I_2^+ and I_2^- for $z\in G_2$ and $z\in G_2^-$, respectively.

Lemma 1 [14]. *Let* $\Gamma\in\mathcal{S}$, $\omega\in A_p(\Gamma)$, $1<p<\infty$ *and* $\theta>0$. *If* $f\in L^{p),\theta}(\Gamma,\omega)$, *then* $f^+\in E^{p),\theta}(G,\omega)$ *and* $f^-\in E^{p),\theta}(G^-,\omega)$.

For $f_0\in L^{p),\theta}(\mathbb{T},\omega)$ and $\omega_0\in A_p(\mathbb{T})$, Lemma 1 implies that $f_0^+\in E^{p),\theta}(\mathbb{U},\omega_0)$ and $f_0^-\in$ $\in E^{p),\theta}(\mathbb{U}^-,\omega_0)$ such that $f_0^-(\infty)=0$. Similarly, for $f_1\in L^{p),\theta}(\mathbb{T},\omega)$ and $\omega_1\in A_p(\mathbb{T})$, Lemma 1 implies that $f_1^+\in E^{p),\theta}(\mathbb{U},\omega_1)$ and $f_1^-\in E^{p),\theta}(\mathbb{U}^-,\omega_1)$ such that $f_0^-(\infty)=0$. Then by (6), for $k=0,1,2,3,\ldots,$ we have

$$a_k(f)=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^+(w)}{w^{k+1}}dw-\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^-(w)}{w^{k+1}}dw=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^+(w)}{w^{k+1}}dw$$

and

$$\widetilde{a}_k(f)=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_1^+(w)}{w^{k+1}}dw-\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_1^-(w)}{w^{k+1}}dw=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_1^+(w)}{w^{k+1}}dw.$$

Hence, a_k and $\widetilde{a}_k$, $k=1,2,\ldots$, are Taylor coefficients of $f_0^+\in E^{p),\theta}(\mathbb{U},\omega_0)$ and $f_1^+\in E^{p),\theta}(\mathbb{U},\omega_1)$, respectively.

Lemma 2. *Let* $\omega\in A_p(\mathbb{T})$, $1<p<\infty$, *and* $\theta>0$. *If* $g\in\mathcal{L}^{p),\theta}(\mathbb{T},\omega)$, *then* $\Omega_r(g^+,\cdot)_{p),\theta,\omega}\leq$ $\leq c\Omega_r(g,\cdot)_{p),\theta,\omega}$ *for* $r=1,2,3,\ldots$.

Proof. Let $g\in\mathcal{L}^{p),\theta}(\mathbb{T},\omega)$. Firstly, we show that

$$\Omega_r\big(S_{\mathbb{T}}(g),\cdot\big)_{p),\theta,\omega}\leq c\Omega_r(g,\cdot)_{p),\theta,\omega}.$$

By using variate transformation $\zeta=ue^{ist}$ and Fubini theorem, we get

$$\sigma_h^r\big[S_{\mathbb{T}}(g)(w)\big]=\frac{1}{h}\int\limits_0^h\Delta_t^rS_{\mathbb{T}}(g(w))dt=$$

$$=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}S_{\mathbb{T}}(g\left(we^{ist}\right)dt=$$

$$=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}\left\{\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{g(\zeta)}{\zeta-we^{ist}}d\zeta\right\}dt=$$

$$=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}\left\{\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{g(ue^{ist})}{ue^{ist}-we^{ist}}e^{ist}du\right\}dt=$$

$$=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}\left\{\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{g(ue^{ist})}{u-w}du\right\}dt=$$

$$=\frac{1}{2\pi i}(P.V)\int\mathbb{T}\frac{\left\{\dfrac{1}{h}\displaystyle\int_0^h\sum\nolimits_{s=0}^r(-1)^{r+s+1}\binom{r}{s}g(ue^{ist})dt\right\}}{u-w}du=$$

$$=\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{\left\{\dfrac{1}{h}\displaystyle\int_0^h\Delta_t^r(g(u)dt\right\}}{u-w}du=S_{\mathbb{T}}\big[\sigma_h^rg(w)\big].$$

Taking norm and supremum over $h\leq\delta$ and applying Theorem A, we have

$$\Omega_r\big(S_{\mathbb{T}}(g),\cdot\big)_{p),\theta,\omega}=\sup_{h\leq\delta}\big\|\sigma_h^r[S_{\mathbb{T}}(g)(w)]\big\|_{L^{p),\theta}(\mathbb{T},\omega)}=$$

$$=\sup_{h\leq\delta}\big\|S_{\mathbb{T}}[\sigma_h^rg(w)]\big\|_{L^{p),\theta}(\mathbb{T},\omega)}\leq$$

$$\leq \sup_{h\leq\delta} c\left\|\sigma_h^r g(w)\right\|_{L^{p),\theta}(\mathbb{T},\omega)} \leq$$

$$\leq c\sup_{h\leq\delta}\left\|\sigma_h^r g(w)\right\|_{L^{p),\theta}(\mathbb{T},\omega)} = \Omega_r(g,\cdot)_{p),\theta,\omega}. \tag{10}$$

Hence, by (5) and (10), we obtain

$$\Omega_r\big(g^+,\cdot\big)_{p),\theta,\omega} \leq c\left\{\Omega_r\big(g,\cdot\big)_{p),\theta,\omega} + \Omega_r\big(S_{\mathbb{T}}(g),\cdot\big)_{p),\theta,\omega}\right\} \leq c\Omega_r\big(g,\cdot\big)_{p),\theta,\omega}.$$

Lemma 2 is proved.

Lemma 3 [13]**.** *Let $g\in\mathcal{E}^{p),\theta}(\mathbb{U},\omega)$, $\omega\in A_p(\mathbb{T})$, $1<p<\infty$, and $\theta>0$. If $\sum_{k=0}^{n}\gamma_k(g)w^k$ is the nth partial sum of Taylor series of g at the origin, then there exists a positive constant c independent of $n=1,2,\ldots$ such that*

$$\left\|g(w)-\sum_{k=0}^{n}\gamma_k(g)w^k\right\|_{L^{p),\theta}(\mathbb{T},\omega)} \leq c\Omega_r\left(g,\frac{1}{n}\right)_{p),\theta,\omega}, \quad r=1,2,3,\ldots.$$

3. Proof of Theorem 1. Let $\omega\in A_p(\Gamma)$, $\omega_0\in A_p(\mathbb{T})$, $\omega_1\in A_p(\mathbb{T})$, $1<p<\infty$, $\theta>0$. Let $\Gamma:=\Gamma_1\cup\Gamma_2^-$, where $\Gamma_1,\Gamma_2\in\mathcal{S}$ and $f\in\mathcal{E}^{p),\theta}(G,\omega)$. We get

$$\left\|f-R_n(f)\right\|_{L^{p),\theta}(\Gamma,\omega)} \leq \left\|f-R_n(f)\right\|_{L^{p),\theta}\left(\Gamma_1,\omega\right)} + \left\|f-R_n(f)\right\|_{L^{p),\theta}(\Gamma_2,\omega)}.$$

Since $f\in\mathcal{E}^{p),\theta}(G,\omega)$ we have $f_0\in\mathcal{L}^{p),\theta}\big(\Gamma_1,\omega\big)$ and $f_1\in\mathcal{L}^{p),\theta}(\Gamma_2,\omega)$. For $\zeta\in\Gamma_1$ and $\xi\in\Gamma_2$, by means of (3), (4) and (6), we have

$$f(\zeta)=\left[f_0^+\big(\varphi(\zeta)\big)-f_0^-\big(\varphi(\zeta)\big)\right]\big(\varphi'(\zeta)\big)^{1/(p-\varepsilon)} \tag{11}$$

and

$$f(\xi)=\left[f_1^+\big(\varphi_1(\xi)\big)-f_1^-\big(\varphi_1(\xi)\big)\right]\big(\varphi_1(\xi)\big)^{-2/(p-\varepsilon)}\big(\varphi_1'(\xi)\big)^{1/(p-\varepsilon)}. \tag{12}$$

It suffices to prove validity the inequalities

$$\left\|f-R_n(f)\right\|_{L^{p),\theta}\left(\Gamma_1,\omega\right)} \leq c\left\{\Omega_r\left(f_0,\frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1,\frac{1}{n}\right)_{p),\theta,\omega_1}\right\} \tag{13}$$

and

$$\|f-R_n(f)\|_{L^{p),\theta}(\Gamma_2,\omega)} \leq c\left\{\Omega_r\left(f_0,\frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1,\frac{1}{n}\right)_{p),\theta,\omega_1}\right\}. \tag{14}$$

Firstly, we prove the estimation (13). Let us take a $z'\in G_1^-$. Then by relations (7) and (11) we have

$$\sum_{k=0}^{n}a_kF_{k,p,\varepsilon}(z')=\left[\varphi'(z')\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi(z')\right]^k +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi(\zeta)\right]^k}{\zeta-z'}d\zeta=$$

$$= \left[\varphi'(z')\right]^{1/(p-\varepsilon)} \sum_{k=0}^{n} a_k \left[\varphi\left(z'\right)\right]^k +$$

$$+ \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)} \sum_{k=0}^{n} a_k \left[\varphi(\zeta)\right]^k}{\zeta - z'} d\zeta -$$

$$- \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)} f_0^{+}\left(\varphi(\zeta)\right)}{\zeta - z'} d\zeta +$$

$$+ \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)} f_0^{-}\left(\varphi(\zeta)\right)}{\zeta - z'} d\zeta + \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z'} d\zeta.$$

Since $[\varphi'(\zeta)]^{1/(p-\varepsilon)}\, f_0^{-}\left(\varphi(\zeta)\right) \in E^1(G_1^{-})$, we get

$$-\left[\varphi'(z')\right]^{1/(p-\varepsilon)} f_0^{-}\left(\varphi(z')\right) = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)} f_0^{-}\left(\varphi(\zeta)\right)}{\zeta - z'} d\zeta,$$

therefore,

$$\sum_{k=0}^{n} a_k F_{k,p,\varepsilon}(z') = \left[\varphi'(z')\right]^{1/(p-\varepsilon)} \sum_{k=0}^{n} a_k \left[\varphi(z')\right]^k -$$

$$- \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)} \left[f_0^{+}\left(\varphi(\zeta)\right) - \sum_{k=0}^{n} a_k \left[\varphi(\zeta)\right]^k\right]}{\zeta - z'} d\zeta -$$

$$- \left[\, \varphi'(z')\right]^{1/(p-\varepsilon)} f_0^{-}\left(\varphi(z')\right) + \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z'} d\zeta. \qquad (15)$$

If $z' \in G_2^{-}$, then by the relations (2) and (12) we have

$$\sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z') = -\frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{1/(p-\varepsilon)} \left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(\xi)\right]^k}{\xi - z'} d\xi =$$

$$= \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{1/(p-\varepsilon)} \left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)} \left(f_1^{+}\left(\varphi_1(\xi)\right) - \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(\xi)\right]^k\right)}{\xi - z'} d\xi =$$

$$= \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{1/(p-\varepsilon)} \left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)} f_1^{-}\left(\varphi_1(\xi)\right)}{\xi - z'} d\xi - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z'} d\xi$$

and by using Cauchy integral formula for $\left(\varphi_1(\xi)\right)^{-2/(p-\varepsilon)} \left(\varphi_1'(\xi)\right)^{1/(p-\varepsilon)} f_1^{-}\left(\varphi_1(\xi)\right) \in E^1(G_2)$, we obtain

$$\sum_{k=1}^{n}\widetilde{a}_k\widetilde{F}_{k,p,\varepsilon}(1/z') = -\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi - z'}d\xi +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\left[\varphi_1(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\left(\varphi_1(\xi)\right) - \sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi - z'}d\xi. \tag{16}$$

Then, for $z' \in G_1^-$, by (15), (16) and (9), we get

$$\sum_{k=0}^{n}a_kF_{k,p,\varepsilon}(z') + \sum_{k=1}^{n}\widetilde{a}_k\widetilde{F}_{k,p,\varepsilon}(1/z') = \left[\varphi'(z')\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi(z')\right]^k -$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}\left[f_0^+\left(\varphi(\zeta)\right) - \sum_{k=0}^{n}a_k\left[\varphi(\zeta)\right]^k\right]}{\zeta - z'} - d\zeta -$$

$$-\left[\varphi'(z')\right]^{1/(p-\varepsilon)}f_0^-\left(\varphi(z')\right) +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\left[\varphi_1(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\left(\varphi_1(\xi)\right) - \sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi - z'}d\xi.$$

Now taking the limit as $z' \to z \in \Gamma_1$ along all nontangential paths outside Γ_1, we obtain

$$f(z) - \sum_{k=0}^{n}a_kF_{k,p,\varepsilon}(z) - \sum_{k=1}^{n}\widetilde{a}_k\widetilde{F}_{k,p,\varepsilon}(1/z) =$$

$$= \left[\varphi'(z)\right]^{1/(p-\varepsilon)}\left[f_0^+\left(\varphi(z)\right) - \sum_{k=0}^{n}a_k\left[\varphi(z)\right]^k\right] -$$

$$-\frac{1}{2}\left[\varphi'(z)\right]^{1/(p-\varepsilon)}\left[f_0^+\left(\varphi(z)\right) - \sum_{k=0}^{n}a_k\left[\varphi(z)\right]^k\right] +$$

$$+ S_{\Gamma_1}\left[\left[\varphi'\right]^{1/(p-\varepsilon)}\left(\left(f_0^+\circ\varphi\right) - \sum_{k=0}^{n}a_k\left[\varphi\right]^k\right)\right](z) -$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\left[\varphi_1(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(\sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k - f_1^+\left(\varphi_1(\xi)\right)\right)}{\xi - z}d\xi \tag{17}$$

a.e. on Γ_1. Since $\omega \in A_p(\Gamma)$, applying Theorem A for Γ_1 and using (17) and the Minkowski inequality, we have

$$\left\|f - R_n(f)\right\|_{L^{p),\theta}(\Gamma_1,\omega)} \le c\left\{\left\|f_0^+(w) - \sum_{k=0}^{n}a_kw^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_0)} + \left\|f_1^+(w) - \sum_{k=1}^{n}\widetilde{a}_kw^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_1)}\right\}.$$

The Faber coefficients a_k and $\widetilde{a}_k$ are Taylor coefficients of f_0^+ and f_1^+, respectively, at origin. Since $\omega_0 \in A_p(\mathbb{T})$ and $\omega_1 \in A_p(\mathbb{T})$, by using Lemmas 3 and 2, we get

$$\left\|f - R_n(f)\right\|_{L^{p),\theta}(\Gamma_1,\omega)} \leq c\left\{\Omega_r\left(f_0, \frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1, \frac{1}{n}\right)_{p),\theta,\omega_1}\right\}. \qquad (18)$$

Let $z'' \in G_2$. Then by (8) and (12) we obtain

$$\sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z'') = \left[\varphi_1'(z'')\right]^{\frac{1}{p-\varepsilon}} \left[\varphi_1(z'')\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z'')\right]^k -$$

$$- \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1'(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k}{\xi - z''}d\xi =$$

$$= \left[\varphi_1'(z'')\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(z)\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z'')\right]^k +$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1'(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\left(\varphi_1(\xi)\right) - \sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi - z''}d\xi -$$

$$- \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1'(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}} f_1^-\left(\varphi_1(\xi)\right)}{\xi - z''}d\xi - \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi - z''}d\xi.$$

Since $\left(\varphi_1(\xi)\right)^{-2/(p-\varepsilon)}\left(\varphi_1'(\xi)\right)^{1/(p-\varepsilon)} f_1^-\left(\varphi_1(\xi)\right) \in E^1(G_2)$, we have

$$\left[\varphi_1'(z'')\right]^{1/(p-\varepsilon)}\left[\varphi_1\left(z''\right)\right]^{-2/(p-\varepsilon)} f_1^-\left(\varphi_1(z'')\right) =$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1'(\xi)\right]^{1/(p-\varepsilon)}\left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)} f_1^-\left(\varphi_1(\xi)\right)}{\xi - z''}d\xi.$$

This equality implies that

$$\sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z'') = \left[\varphi_1'(z'')\right]^{1/(p-\varepsilon)} \left[\varphi_1(z'')\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z'')\right]^k +$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1'(\xi)\right]^{1/(p-\varepsilon)}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\left(\varphi_1(\xi)\right) - \sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi - z''}d\xi -$$

$$- \left[\,\varphi_1'(z'')\right]^{1/(p-\varepsilon)}\left[\varphi_1\left(z''\right)\right]^{-\frac{2}{p-\varepsilon}} f_1^-\left(\varphi_1(z'')\right) - \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi - z''}d\xi. \qquad (19)$$

Let $z'' \in G_1$. Then by (1) and (11) we get

$$\sum_{k=0}^{n} a_k F_{k,p,\varepsilon}\left(z''\right) = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{(\varphi'(\zeta))^{\frac{1}{p-\varepsilon}} \sum_{k=0}^{n} a_k\big[\varphi(\zeta)\big]^k}{\zeta - z''} d\zeta =$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{(\varphi'(\zeta))^{\frac{1}{p-\varepsilon}} \left(\sum_{k=0}^{n} a_k\big[\varphi(\zeta)\big]^k - f_0^{+}(\varphi(\zeta))\right)}{\zeta - z''} d\zeta +$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} f_0^{-}(\varphi(\zeta))}{\zeta - z''} d\zeta + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z''} d\zeta,$$

and by using the Cauchy integral formula for $\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} f_0^{-}(\varphi(\zeta)) \in E^1(G_1^-)$, we obtain

$$\sum_{k=0}^{n} a_k F_{k,p,\varepsilon}\big(z''\big) = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z''} d\zeta +$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} \left(\sum_{k=0}^{n} a_k\big[\varphi(\zeta)\big]^k - f_0^{+}(\varphi(\zeta))\right)}{\zeta - z''} d\zeta. \qquad (20)$$

For $z'' \in G_2$, the relations (19), (20) and (9) imply that

$$\sum_{k=0}^{n} a_k F_{k,p,\varepsilon}\left(z''\right) + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z'') =$$

$$= \big[\varphi_1'(z'')\big]^{1/(p-\varepsilon)} \left[\varphi_1\left(z''\right)\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z'')\right]^k +$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\big[\varphi_1'(\xi)\big]^{1/(p-\varepsilon)} \big[\varphi_1(\xi)\big]^{-\frac{2}{p-\varepsilon}} \left(f_1^{+}(\varphi_1(\xi)) - \sum_{k=1}^{n} \widetilde{a}_k\big[\varphi_1(\xi)\big]^k\right)}{\xi - z''} d\xi -$$

$$- \big[\varphi_1'(z'')\big]^{1/(p-\varepsilon)} \left[\varphi_1\left(z''\right)\right]^{-\frac{2}{p-\varepsilon}} f_1^{-}\left(\varphi_1(z'')\right) +$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} \left(\sum_{k=0}^{n} a_k\big[\varphi(\zeta)\big]^k - f_0^{+}(\varphi(\zeta))\right)}{\zeta - z''} d\zeta.$$

Taking the limit as $z'' \to z \in \Gamma_2$ along all nontangential paths inside Γ_2, by (11) we obtain

$$f(z) - \sum_{k=0}^{n} a_k F_{k,p,\varepsilon}(z) - \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z) =$$

$$= \big[\varphi_1'(z)\big]^{1/(p-\varepsilon)} \big[\varphi_1(z)\big]^{-\frac{2}{p-\varepsilon}} f_1^{+}\left(\varphi_1(z)\right) -$$

$$-\frac{1}{2}\left[\varphi_1'(z)\right]^{1/(p-\varepsilon)}\left[\varphi_1(z)\right]^{-\frac{2}{p-\varepsilon}}\left[\sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(z)\right]^k - f_1^+(\varphi_1(z))\right] -$$

$$- S_{\Gamma_2}\left[\left[\varphi_1'\right]^{1/(p-\varepsilon)}\left[\varphi_1\right]^{-\frac{2}{p-\varepsilon}}\left(\sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1\right]^k - \left(f_1^+\circ\varphi_1\right)\right)\right](z) -$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}}\left(\sum_{k=0}^{n}a_k\left[\varphi(\zeta)\right]^k - f_0^+(\varphi(\zeta))\right)}{\zeta - z}d\zeta \tag{21}$$

a.e. on Γ_2. Since $\omega \in A_p(\Gamma)$, applying Theorem A for Γ_2 and by using (17) and Minkowski inequality, we obtain

$$\left\|f - R_n(f)\right\|_{L^{p),\theta}(\Gamma_2,\omega)} \le c\left\{\left\|f_0^+(w) - \sum_{k=0}^{n}a_k w^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_0)} + \left\|f_1^+(w) - \sum_{k=1}^{n}\widetilde{a}_k w^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_1)}\right\}.$$

The Faber coefficients a_k and $\widetilde{a}_k$ are Taylor coefficients of f_0^+ and f_1^+, respectively, at origin. Since $\omega_0 \in A_p(\mathbb{T})$ and $\omega_1 \in A_p(\mathbb{T})$, finally by using Lemmas 3 and 2, we have

$$\left\|f - R_n(f)\right\|_{L^{p),\theta}(\Gamma_2,\omega)} \le c\left\{\Omega_r\left(f_0, \frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1, \frac{1}{n}\right)_{p),\theta,\omega_1}\right\}. \tag{22}$$

Hence, (18) and (22) complete the proof of Theorem 1.

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