Український математичний вісник
Том 17 (**2020**), № 4, 594 – 600
https://doi.org/10.37069/1810-3200-2020-17-4-8

Mean value theorems for polynomial solutions of linear elliptic equations with constant coefficients in the complex plane

Olga D. Trofymenko

(*Presented by V. Gutlyanskii*)

Abstract. We characterize solutions of the mean value linear elliptic equation with constant coefficients in the complex plane in case of regular polygon.

2010 MSC. 35B05.

Key words and phrases. Mean value theorem, regular polygon, harmonic polynomial.

Introduction

Let $n \in \mathbb{N}$, $n \geq 3$. There is well-known result in theory of harmonic functions, that was proved independently in works of Kakutani and Nagumo [1], Walsh [2] and Privalov [3, Chapter 3, §11]: a function $f \in C(\mathbb{C})$ is a harmonic polynomial of order $\leq n-1$ if and only if the mean value of the function f taken over vertices of any regular n-gon equals to the value of this function at its center.

Ramsey and Weit obtained the similar result with the radius $R = \infty$ of the disk: the function $f \in C(\mathbb{R}^2)$ satisfies

$$\frac{1}{N+1}\sum_{k=0}^{N} f\left(z + e^{\frac{2\pi k i}{N+1}}\xi\right) = f(z)$$

for all $z \in \mathbb{C}$ and $|\xi| = R$ for some fixed $R > 0$ if and only if f is a harmonic polynomial of degree not exceeding N.

Then Volchkov V. [5, Part 5, Chapter 5, assertion (1) of Theorem 5.9] obtained the following local variant of their result.

Received 08.10.2020

The author was supported by the Fundamental Research Programme funded by the Ministry of Education and Science of Ukraine (project 0118U003138).

Let $V = \{0, \nu_1, ..., \nu_n\}$, λ_r are motions of the regular n-gon in the disk B_r.

Let $R > inf\{r > 0 : \mathrm{cl}\{\cup\lambda_r \mathrm{V}\} = \overline{\mathrm{B}}_\mathrm{r}\}$. Then set of the functions, which satisfy the equality

$$f(\lambda 0) = \frac{1}{n}\sum_{j=1}^{n} f\left(\lambda_R \nu_j\right)$$

for almost all motions λ_R, coincides with a harmonic polynomial of degree at most $n-1$.

In present paper we prove the new similar theorem, which describes classes of polynomial solutions for homogeneous linear elliptic equations with constant coefficients on the complex plane. The left hand side of these equations is representable in the form of the product of some non-negative operators of complex differentiation. In case of Laplace operator this theorem coincides with above Volchkov result [5] and hence contains the all above results of the works [1,...,4]. The analogue of this result for the circle domains is represented in [6].

The author is grateful to A. V. Pokrovskii for attention to this work.

1. Main results

Let $B_R := \{z \in \mathbb{C} : |z| < R\}$, $m, n \in \mathbb{N}$, $s \in \mathbb{N}_0$, $n \geq 3$, $s < m < n+1$, $d_n := 2(5 + 4\cos\frac{\pi}{n})^{-1/2}$ for odd n, $d_n := 2(4 + 5\cos^2\frac{\pi}{n})^{-1/2}$ for even n. Denote by $E(n, m, s)$ the set of all pairs of integer nonnegative numbers (k, l), such that the following conditions hold: $k < m - s$ or $l < m$; $k < n + s$; $l < n - s$.

Theorem 1. *Let $R > 0$, $f \in C^{2m-s-2}(B_R)$, $r \in (0,\ d_n R)$. Then the following assertions are equivalent:*
1) for all $z \in B_R$ and $\alpha \in [0,\ 2\pi)$ such that $\{z + re^{i\alpha + i\frac{2\pi\nu}{n}}\}_{\nu=0}^{n-1} \subset B_R$ we have the equality

$$\sum_{p=s}^{m-1} \frac{nr^{2p}}{(p-s)!p!} \partial^{p-s}\bar{\partial}^{p} f(z) = \sum_{\nu=0}^{n-1} (re^{i\alpha + i\frac{2\pi\nu}{n}})^s f(z + re^{i\alpha + i\frac{2\pi\nu}{n}}); \qquad (1.1)$$

2) the function f is represented in the form

$$f(z) = \sum_{(k,l)\in E(n,m,s)} c_{k,l} z^k \bar{z}^l, \quad c_{k,l} \in \mathbb{C}. \qquad (1.2)$$

It follows from the definition of the set $E(n, m, s)$ that functions satisfying the condition(1.1) form a finite-dimensional linear space over field $\mathbb{C}$, whose elements are polynomial solutions of the equation $\partial^{m-s}\overline{\partial}^m f = 0$.

2. Auxiliary results and constructions

Let $U \in \mathbb{R}^n$, $\varphi \in \mathcal{E}'(\mathbb{R}^n)$ and $A_\varphi(U) = \bigcap\limits_{\tau \in SO(n)} \mathcal{D}'_{\tau\varphi}(U)$. Let d_k be a dimension (over $\mathbb{C}$) of the vector space of all spherical harmonics of degree k. We associate with each function $f \in L_{loc}(U)$ its Fourier series into spherical harmonics and $F_{k,l,p}(x)$ denote the term of the Fourier series, where $F^{k,l}(x) = F_{k,l,l}(x)$, $x \in U$.

Consider the following Proposition (see p. 34 in monograph [5]).

Proposition 1. *Let $f \in A_\varphi(U)$. Then the following assertions hold.*
1) $\partial f/dx_j \in A_\varphi(U)$ for all $j \in 1, ..., n$.
2) $D_{i,j} f \in A_\varphi(U)$ for all $i, j \in 1, ..., n$.
3) $F^{k,l} \in A_\varphi(U)$ for all $k \in \mathbb{Z}_+$, $l \in 1, ..., d_k$. 4) If $n \geq 3$, and $f \in L_{loc}(U)$ then $F_{k,l,p} \in A_\varphi(U)$ for all $k \in \mathbb{Z}$, $l, p \in 1, ..., d_k$.
5) If $n = 2$, $f \in L_{loc}(U)$, and

$$\langle \varphi, g(x_1, -x_2) \rangle = \langle \varphi, g(x_1, x_2) \rangle$$

for each $g \in \mathcal{E}(\mathbb{R}^n)$, then $F_{k,l,p} \in A_\varphi(U)$ for all $k \in \mathbb{Z}$, $l, p \in 1, ..., d_k$.

To each function $f \in C(B_R)$ we assign its Fourier series

$$f(z) \sim \sum_{k=-\infty}^{+\infty} f_k(\rho) e^{ik\varphi}, \tag{2.3}$$

where $z = \rho e^{i\varphi}$ is the trigonometric form of z,

$$f_k(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{it}) e^{-itk} dt \qquad (z \neq 0, \quad k \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}).$$

For $z = 0$ we define the Fourier coefficients by continuity, i.e., $f_0(0) = f(0)$, $f_k(0) = 0$ for all integer $k \neq 0$.

Formulate the following well-known property of the Fourier coefficients [5, Part 1, §5.1].

Lemma 1. *If $f \in C^\infty(B_R)$, then for any $k \in \mathbb{Z}$ the function $f_k(\rho)e^{ik\varphi}$ is infinitely differentiable with respect to x and y ($z = x + iy = \rho e^{i\varphi}$) and the Fourier series (3) converges to the function f in the space $\mathcal{E}(B_R)$, i.e., converges uniformly together with its all partial derivatives of arbitrary order on each compact subset of the ball B_R.*

Let $\nu_1, .., \nu_m \in \mathbb{R}^n$, where $\nu_i \neq \nu_j$ for $1 \leq i$, $j \leq m$, $i \neq j$ and let $\varepsilon > 0$. For $\nu = 1, ..., m$ we set $\Omega_{\nu,\varepsilon} = \{x \in \mathbb{R}^n : |v_\nu| - \varepsilon < |x| < |v_\nu| + \varepsilon\}$,

$\nu = 1, ..., m$. In addition, we set $\mathfrak{H}_k(U) = span_{\mathbb{C}}\{\mathfrak{H}_{k,1}(U), ..., \mathfrak{H}_{k,d_k}(U)\}$, where $\mathfrak{H}_{k,l}(U) = \left(L_{loc} \bigcap \mathcal{D}'_{k,l}(U)\right)$ $(l \in 1, ..., d_k)$. Consider the following Lemma (see theorem 3.1 on p. 250 from the monograph [5]).

Lemma 2. *Assume that the functions $F_\nu \in L_{loc}(\Omega_{\nu,\varepsilon})$, $\nu = 1, ..., m$ satisfy the following conditions:*

(1) there exists $q \in \mathbb{Z}_+$ such that

$$F_\nu \in span_{\mathbb{C}}\{\mathfrak{H}_0(\Omega_{\nu,\varepsilon}), ..., \mathfrak{H}_q(\Omega_{\nu,\varepsilon})\}$$

for all $\nu = 1, ..., m$;

(2)

$$\sum_{\nu=1}^{m} F_\nu(x + v_\nu) = 0$$

for almost all $x \in B_\varepsilon$.

Then F_ν is a polynomial in $\Omega_{\nu,\varepsilon}$ for all $\nu = 1, ..., m$.

3. Proof of Theorem 1

Proof. Let $R > 0$, and $r \in (0, R)$ with following property: set of points $\cup_z(\{z + re^{i\alpha + i\frac{2\pi\nu}{n}}\}_{\nu=0}^{n-1} \cup \{z\})$ with all such $z \in B_R$ and $\alpha \in [0,\ 2\pi)$, that $\{z + re^{i\alpha + i\frac{2\pi\nu}{n}}\}_{\nu=0}^{n-1} \subset B_R$, coincides with the disk B_R. It is clear that all sufficiently small values r satisfy this condition. Assume that $f \in C^\infty(B_R)$. Let $F_q(z) = f_q(\rho)e^{iq\varphi}$ is Fourier series q-summand of the function $f(z)$ $(z = x + iy = re^{i\varphi},\ 0 < \rho < R,\ q \in \mathbb{Z})$. Then $F_q \in C^\infty(B_R)$ (see Lemma 1), and from Proposition 1 function $F_q(z)$ satisfies the general mean value condition on polygonals (1.1) for all $z \in B_R$ and $\alpha \in [0,\ 2\pi)$ such, that $\{z + re^{i\alpha + i\frac{2\pi\nu}{n}}\}_{\nu=0}^{n-1} \subset B_R$.

Let $t_\nu = re^{i\frac{2\pi\nu}{n}}$, $\nu = 0, 1, \ldots, n-1$, $t_n = 0$. Define the functions

$$F_{q,\nu}(z) := t_\nu^s F_q(z), \qquad \nu = 0, 1, \ldots, n-1,$$

$$F_{q,n}(z) := -\sum_{p=s}^{m-1} \frac{nr^{2p}}{(p-s)!p!} \partial^{p-s}\bar{\partial}^p F_q(z).$$

Then we have from (1.1)

$$\sum_{\nu=0}^{n} t_\nu^s F_{q,\nu}(z + t_\nu) \equiv 0$$

Using for these functions Lemma 2, similarly to constructions on p. 406 of the monograph [5], we obtain, that $F_q(z)$ is a polynomial. Therefore it follows from uniqueness theorem that we can consider $R = \infty$. Using again Proposition 1 we conclude that all functions $\partial^k \overline{\partial}^l F_q(z)$, $k, l \in \mathbb{N}_0$ satisfy general mean value condition on polygons for all $z \in \mathbb{C}$ and $\alpha \in [0,\ 2\pi)$. This implies that all elements of the expansion for polynomial $F_q(z)$ in powers z, $\bar{z}$ satisfy equation (1.1) for all z and α too.

Suppose that polynomial $z^k \bar{z}^l$ satisfies general mean value condition on polygons. Then all polynomials $z^j \bar{z}^q$ with $j \leq k$, $q \leq l$ also satisfy this condition. Hence we have from binomial formula that polynomial $z^k \bar{z}^l$ satisfies general mean value condition on polygons if and only if all polynomials $f(z) = z^j \bar{z}^q$ where $j \leq k$, $q \leq l$ satisfy condition

$$\sum_{p=s}^{m-1} \frac{nr^{2p}}{(p-s)!p!} \partial^{p-s} \bar{\partial}^p f(0) = \sum_{\nu=0}^{n-1} (re^{i\alpha + i\frac{2\pi\nu}{n}})^s f(re^{i\alpha + i\frac{2\pi\nu}{n}}). \tag{3.4}$$

This implies that or $k < m - s$ or $l < m$, because otherwise function $z^{m-s} \bar{z}^m$ satisfies general mean value condition on polygons, but left side (3.4) is equal zero and right side is not equal. On the other hand we have from (3.4), that all functions $z^{p-s} \bar{z}^p$, $s \leq p < m$ satisfy this condition.

Consider the case $k + s \neq l$. Then left side of (3.4) equals to zero. Then we have from geometric progression formula that right side of (3.4) equals to zero only if $(k - l + s)/n$ is integer. This implies $k + s < n$, $l - s < n$, because otherwise for functions z^{n-s} and $\bar{z}^{n+s}$ left side of (3.4) is equal zero and right side is not equal. Thus polynomial $f(z) := z^k \bar{z}^l$ satisfies general mean value condition on polygons if and only if $k < m - s$ or $l < m$, and $k < n - s$, $l < n + s$.

From the above we have statement from our theorem for the function $f \in C^\infty(B_R)$ and for the following r: $A_r := \cup_z(\{z + re^{i\alpha + i\frac{2\pi\nu}{n}}\}_{\nu=0}^{n-1} \cup \{z\} = B_R$ with all such $z \in B_R$ and $\alpha \in [0,\ 2\pi)$, that $\{z + re^{i\alpha + i\frac{2\pi\nu}{n}}\}_{\nu=0}^{n-1} \subset B_R$. Consider the least upper bond for r.

Let regular n-gon $P_n \subset \overline{B}_R$ with the center O_1 and the radius r of the circumscribed circle is a symmetric with respect to axis Ox and has two vertices on the circle $|z| = R$ in right half-plane. Then we have condition from geometric considerations for $A_r = B_R$. There are

$$r < 2|OO_1| \tag{3.5}$$

for odd n, and

$$r \cos \frac{\pi}{n} < 2|OO_1| \tag{3.6}$$

where $|OO_1|$ is the Euclidean distance between center O_1 of n-gon P_n and origin O. Using elementary calculations we have

$$|OO_1| = \sqrt{R^2 - r^2 \sin^2 \frac{\pi}{n}} - r \cos \frac{\pi}{n}.$$

Then for odd n we obtain from (3.5):

$$r < 2\left(\sqrt{R^2 - r^2\sin^2\frac{\pi}{n}} - r\cos\frac{\pi}{n}\right),$$

$$r\left(1 + 2\cos\frac{\pi}{n}\right) < \sqrt{R^2 - r^2\sin^2\frac{\pi}{n}},$$

$$r^2\left(1 + 4\cos\frac{\pi}{n} + 4\cos^2\frac{\pi}{n}\right) < 4R^2 - 4r^2\sin^2\frac{\pi}{n},$$

$$r^2\left(5 + 4\cos\frac{\pi}{n}\right) < 4R^2,$$

$$r < d_nR, \qquad d_n = 2\left(5 + 4\cos\frac{\pi}{n}\right)^{-1/2}.$$

Similarly, for even n we obtain from (3.6)

$$2r\cos\frac{\pi}{n} < \left(\sqrt{R^2 - r^2\sin^2\frac{\pi}{n}} - r\cos\frac{\pi}{n}\right),$$

$$r^2\left(9\cos^2\frac{\pi}{n} + 4\sin^2\frac{\pi}{n}\right) < 4R^2,$$

$$r < d_nR, \qquad d_n = 2\left(4 + 5\cos\frac{\pi}{n}\right)^{-1/2}.$$

Thus we proof the Theorem for $f \in C^\infty(B_R)$.

For the general case we for $R > 0$ and $r \in (0,\ d_nR)$ fix $\varepsilon_0 \in (0, R)$ $(r < d_n(R - \varepsilon_0))$ and real nonnegative function $\varphi \in C_0^\infty(B_1)$ such, that

$$\iint_{\mathbb{C}} \varphi(z)\,dxdy = 1.$$

Let

$$\varphi_\varepsilon(z) = \varepsilon^{-n}\iint_{\mathbb{C}} \varphi(z/\varepsilon)\,dxdy \qquad 0 < \varepsilon < \varepsilon_0.$$

Using convolution's standard properties we have, that function

$$f_\varepsilon(x) := f * \varphi_\varepsilon(x) = \int f(x - y)\varphi_\varepsilon(y)\,dy$$

is defined in the disk $B_{R-\varepsilon}$, belongs to the class $C^\infty(B_{R-\varepsilon})$ and $f_\varepsilon \to f$ for $\varepsilon \to 0$ uniformly on compact subsets of the disk B_R. On the other hand, we have a proof, that every function f_ε has the form (1.2). Thus we justify that function f has this form. $\square$

References

[1] Kakutani, S., Nagumo, M. (1935). On the functional equation $\sum_{\nu=0}^{n-1} f(z + e^{2\nu\pi i/n}\xi) = nf(z)$. *Zenkoku Sûgaku Danwakai, 66*, 10–12.

[2] Walsh, J. L. (1936). A mean value theorem for polynomials and harmonic polynomials. *Bull. Amer. Math. Soc., 42*, 923–930.

[3] Privalov, I.I. (1937). *Subharmonic functions.* Moscow–Leningrad: ONTI NKTP SSSR.

[4] Ramsey, T., Weit, Y. (1984). Mean values and classes of harmonic functions. *Math. Proc. Camb. Phil. Soc., 96*, 501–505.

[5] Volchkov, V.V. (2003). *Integral Geometry and Convolution Equations.* Dordrecht: Kluwer Academic Publishers.

[6] Trofymenko, O. (2018). Convolution equations and mean value theorems for solutions of linear elliptic equations with constant coefficients in the complex plane. *Journal of Mathematical Sciences, 229*(1), 96–107.

Contact information

Olga D. Trofymenko
Vasyl' Stus Donetsk National University,
Vinnytsia, Ukraine
E-Mail: odtrofimenko@gmail.com