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Критерій перевірки гіпотези про вигляд коваріаційної функції Гауссового стохастичного процесу

Метод мажоруючих мір застосовується для побудови критерію перевірки гіпотези про вигляд коваріаційної функції Гауссового стохастичного процесу.

Ключові слова: метод мажоруючих мір, простір Орлича, коваріаційна функція.

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How to test the hypothesis concerning the form of covariance function of Gaussian stochastic process

Method of majorizing measures is applied for constructing the criterion for testing the hypothesis concerning the form of covariance function of Gaussian stochastic process.

Key Words: method of majorizing measures, Orlicz spase, covariance function.

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Introduction

In this paper we propose some inequalities of the distribution of supremum of some correlogram form of the stochastic processes. These estimates are applying for constructing the criterion of hypothesis testing on covariance functions of these processes.

1 Definitions and properties of some elements from the Orlicz space

Definition 1. *A function $U(x),\ x \in \mathbf{R}$ is called an Orlicz N-function if it is continuous, even, convex and*

1) $U(0) = 0,\ U(x) > 0$ *при* $x \neq 0$,

2) $\lim_{x\to 0} \frac{U(x)}{x} = 0$,

3) $\lim_{x\to\infty} \frac{U(x)}{x} = \infty$.

Let $(\mathbf{T}, \rho)$ be a compact metric space and let $\mathcal{A}$ be a Borel $\sigma-$algebra on $(\mathbf{T}, \rho)$. Let $\mu(\cdot)$ be a finite measure on $(\mathbf{T}, \mathcal{A})$. Let $X = \{X(t),\ t \in \mathbf{T}\}$ be a separable stochastic process such that the sample path of X are measurable with respect to $\mathcal{A}$ and $X \in L_U(\Omega)$, where $U(x)$ -N-function.

Let there exist functions $\sigma_t = \{\sigma_t(h), h > 0\}, t \in \mathbf{T}$ such that:

1) $\sigma_t(h) \geqslant 0;\ \sigma_t(h)$ increases as $h > 0$;

2) $\sigma_t(h) \to 0;\ \sigma_t(h)$ is continuous from the right;

3) $\sup_{\rho(t,s)\leqslant h} \|X(t) - X(s)\|_U \leqslant \sigma_t(h)$.

Note that these properties has the function

$$\sigma_t(h) = \sup_{\rho(t,s)\leqslant h} \|X(t) - X(s)\|_U,$$

if the process $X(t)$ is continuous in the norm $\|\cdot\|_U$.

Denote by $\sigma_t^{(-1)}(v)$ the generalized inverse to $\sigma_t(h)$, that is

$$\sigma_t^{(-1)}(v) = \sup\{h :\ \sigma_t(h) \leqslant v\}.$$

Let

$$d(u, v) = \|X(u) - X(v)\|_U$$

and let $\mathbf{S}$ be a set from $\mathcal{A}$ such that

$$(\mu \times \mu)\{(u, v) \in \mathbf{S} \times \mathbf{S},\ \rho(u, v) \neq 0\} > 0.$$

Let $\varepsilon_k(t) > 0,\ k = 1, 2, \ldots$ be a sequence such that $\varepsilon_k(t) > \varepsilon_{k+1}(t), \varepsilon_k(t) \to 0$ as $k \to \infty$, i $\varepsilon_1(t) = \sup_{s\in\mathbf{S}} \rho(t, s)$. Let

$$B_t(U) = \{s : \rho(t, s) \leqslant u\},$$

$$B_t^k = B_t(\varepsilon_k(t)),\ \mu_k(t) = \mu_k(t, \mathbf{S}) = \mu(B_t^k \cap \mathbf{S}).$$

Definition 2. *N-function $U(x)$ belongs the class Δ^2 if there exists constants $Q \geqslant 1$ and $y_0 \geqslant 0$ such that $U^2(x) \leqslant U(Qx)$, while $x \geqslant y_0$.*

Definition 3. *A stochastic process* $X = \{X(t), t \in \mathbf{T}\}$ *belongs the class* Δ^2 *if* $X \in L_U(\Omega)$, *where* $U \in \Delta^2$. *In this case we will write* $X \in \Delta^2$.

Lemma 1. *[1] Let* $X = \{X(t), t \in \mathbf{T}\}$ *be a separable stochastic process from the class* Δ^2 *and* $\gamma(v)$ *be an increasing function. Let the next condition be satisfied*

$$\sup_{t\in\mathbf{S}} \int_0^{\delta_1(t)} U^{(-1)}\left((\nu_t(u))^{-1}\right) du < \infty, \qquad (1)$$

where

$$\delta_1(t) = z\big(2\sigma_t(\sup_{t\in\mathbf{S}} \rho(t,s))\big),$$

$$\nu_t(u) = \mu\left(B\left(t, \sigma_t^{(-1)}\left(z^{(-1)}(u)/2\right)\right) \cap \mathbf{S}\right).$$

Then for any $0 < p < 1$ *the next inequality holds true with probability one*

$$\sup_{t\in\mathbf{S}} \left| X(t) - \int_{\mathbf{S}} X(u) \frac{d\mu(u)}{\mu(\mathbf{S})} \right| \leqslant \frac{\eta}{p(1-p)} \times$$

$$\times \sup_{t\in\mathbf{S}} \left[Q \int_0^{\min\{\delta_1(t)p, r_0\}} U^{(-1)}\left((\nu_t(u))^{-1}\right) du + \right.$$

$$\left. + Z(t) \right] = \eta a_p, \qquad (2)$$

where

$$\eta = \left\| \frac{(X(u) - X(v))\gamma(d(u,v))}{d(u,v)} \right\|_{U,\mu\times\mu}^{S\times S},$$

r_0 *is a number such that* $(\nu_t(u))^{(-1)} \geqslant U(r_0)$, *if* $u \leqslant r_0$;
$Z(t) = 0$ *if* $\delta_1(t)p \leqslant r_0$ *and*

$$Z(t) = \int_{\min\{\delta_1(t)p, r_0\}}^{\delta_1(t)p} U^{(-1)}\left((\nu_t(u))^{-2}\right) du,$$

if $\delta_1(t)p \geqslant r_0$.

Consider now applications of this lemma for processes from the space $L_U(\Omega)$ with N-function $U(x) = \exp\{|x|\} - |x| - 1$. This N-function $U(x)$ belongs to class Δ^2 with $Q = 2,\ y_0 = 0$.

Theorem 1. *Let all assumptions of Lemma 1 be satisfied for the* N*-function* $U(x) = \exp\{|x|\} - |x| - 1$. *Then for all* $x \geqslant \hat\gamma$ *the following inequality holds true*

$$\mathbf{P}\{\eta > x\} \leqslant 2\exp\left\{ -\ln(1 + \frac{1}{A(\mathbf{S})})\frac{x}{\hat\gamma} \right\}, \qquad (3)$$

where $\hat\gamma = \sup_{u,v\in\mathbf{S}} \gamma(d(u,v))$,

$$\eta = \left\| \frac{(X(u) - X(v))\gamma(d(u,v))}{d(u,v)} \right\|_{U,\mu\times\mu}^{S\times S},$$

$$A(S) = \int_{\mathbf{S}}\int_{\mathbf{S}} \frac{\gamma(d(u,v))}{\hat\gamma} d(\mu(u)\times\mu(v)).$$

Proof. Lets put $d(\mu(u)\times\mu(v)) = d\mu$. It is easy to see that

$$\mathbf{P}\{\eta > x\} \leqslant$$

$$\leqslant \mathbf{P}\left\{ \int_{\mathbf{S}}\int_{\mathbf{S}} \left(\exp\left\{\frac{\eta}{x}\right\} - \frac{\eta}{x} - 1 \right) d\mu > 1 \right\} \leqslant$$

$$\leqslant \mathbf{P}\left\{ \int_{\mathbf{S}}\int_{\mathbf{S}} \left(\exp\left\{\frac{\eta}{x}\right\} - 1 \right) d\mu > 1 \right\}. \qquad (4)$$

Since

$$\exp\left\{ \frac{|(X(u) - X(v))\gamma(d(u,v))|}{x} \right\} - 1 \leqslant$$

$$\leqslant \frac{\gamma(d(u,v))}{\hat\gamma} \left(\exp\left\{ \frac{|(X(u) - X(v))\hat\gamma|}{x} \right\} - 1 \right),$$

then for all $q > 1$ the next relations hold

$$\mathbf{P}\{\eta > x\} \leqslant$$

$$\mathbf{P}\left\{ \int_{\mathbf{S}}\int_{\mathbf{S}} \frac{\gamma(d(u,v))}{\hat\gamma} \exp\left\{ \frac{|(X(u) - X(v))\hat\gamma|}{d(u,v)x} \right\} d\mu \right.$$

$$\left. > 1 + A(\mathbf{S}) \right\} \leqslant \mathbf{E}\left(\int_{\mathbf{S}}\int_{\mathbf{S}} \frac{\gamma(d(u,v))}{\hat\gamma} \times \right.$$

$$\left. \exp\left\{ \frac{|(X(u) - X(v))\hat\gamma|}{d(u,v)x} \right\} d\mu \right)^q (1 + A(\mathbf{S}))^{-q} \leqslant$$

$$\left(\frac{A(\mathbf{S})}{1 + A(\mathbf{S})} \right)^q \mathbf{E} \int_{\mathbf{S}}\int_{\mathbf{S}} \frac{\gamma(d(u,v))}{\hat\gamma} \times$$

$$\times \exp\left\{ \frac{q|(X(u) - X(v))\hat\gamma|}{d(u,v)x} \right\} \frac{d\mu}{A(\mathbf{S})}. \qquad (5)$$

If we take $q = \frac{x}{\hat\gamma}$, then for $x \geqslant \hat\gamma$ we will obtain

$$\mathbf{P}\{\eta > x\} \leqslant \left(\frac{A(\mathbf{S})}{1 + A(\mathbf{S})} \right)^{\frac{x}{\hat\gamma}} \times$$

$$\int_{\mathbf{S}}\int_{\mathbf{S}} \frac{\gamma(d(u,v))}{\hat\gamma} \mathbf{E}\exp\left\{ \frac{|(X(u) - X(v))|}{d(u,v)} \right\} \frac{d\mu}{A(\mathbf{S})}$$

$$\leqslant 2\left(\frac{A(\mathbf{S})}{1 + A(\mathbf{S})} \right)^{\frac{x}{\hat\gamma}} = 2\exp\left\{ -\ln(1 + \frac{1}{A(\mathbf{S})})\frac{x}{\hat\gamma} \right\}.$$

The Theorem 1 is proved.

2 How to test the hypothesis concerning the form of covariance function of Gaussian stochastic process

Assume that $Y = \{Y(t), t \in \mathbf{T}\}$ is a separable real-valued stationary Gaussian process with $\mathbf{E}Y(t) = 0$ and continuous correlation function

$$B(\tau) = \mathbf{E}Y(t+\tau)Y(t).$$

Lets consider the sample correlation function (correlogram)

$$\widehat{B}_T(\tau) = \frac{1}{T}\int_0^T Y(t+\tau)Y(t)dt, \quad 0 \leqslant T \leqslant \infty.$$

Theorem 2. *The next inequality holds true for correlogram $\widehat{B}_T(\tau)$ of correlation function $B(\tau)$*

$$\mathbf{P}\left\{\sup_{0\leqslant\tau\leqslant S} |B(\tau) - \widehat{B}_T(\tau) - \frac{1}{S}\int_0^S B(u)du + \right.$$

$$\left. +\frac{1}{S}\int_0^S \widehat{B}_T(u)du| > x\right\} \leqslant$$

$$\leqslant 2\exp\{-\ln(1+\frac{1}{A(S)})\frac{x}{\hat{\gamma}|a_p|}\},$$

where $x \geqslant 1$.

Proof. This theorem is a consequence of Lemma 1 and Theorem 1 for $X(t) = B(t) - \widehat{B}_T(t)$, where $\mathbf{S} = [0, S]$. Here $X(t) = B(t) - \widehat{B}_T(t)$ is square Gaussian stochastic process and

$$\|\eta\| \leqslant \sqrt{\frac{C}{\ln 2}}\left(\mathbf{E}\eta^2\right)^{1/2},$$

where

$$C = \inf_{r\in(0,1)} r^{-2}|\ln(1-r)| - r^{-1}.$$

The Theorem 2 is proved.

Let H be a hypothesis which for $0 \leqslant \tau \leqslant T$ lie down that correlation function of Gaussian stochastic process is equals to $B(\tau)$. We take $\widehat{B}_T(\tau)$ as an estimate of $B(\tau)$. Let consider next test.

Criterion 1. *Let find x_α for fixed α, $0 \leqslant \alpha \leqslant 1$ that*

$$f(x_\alpha) = \alpha,$$

where $f(x) = 2\exp\left\{-\ln(1+\frac{1}{A(S)})\frac{x}{\hat{\gamma}|a_p|}\right\}$.

Hypothesis that the covariance function of the process $X(t)$ is $B(\tau)$ for $0 \leqslant \tau \leqslant T$ is accepted if

$$\sup_{0\leqslant\tau\leqslant S}\left|B(\tau) - \widehat{B}_T(\tau) - \frac{1}{S}\int_0^S B(u)du + \right.$$

$$\left. +\frac{1}{S}\int_0^S \widehat{B}_T(u)du\right| < x_\alpha$$

and is rejected otherwise.

Example 1. *Let consider next function $B(f) = \exp\{-f^2\}$. Let put $Z_T(f) = \widehat{B}_T(f) - B(f),\ \ f \geqslant 0$. Using the formula (6.1.3) from [2, p.168] we will obtain*

$$\mathbf{E}\left(Z_T(u) - Z_T(v)\right)^2 =$$

$$= \frac{2}{T}\int_0^T \left(1 - \frac{t}{T}\right)[(B(t-u) - B(t-v)) \cdot$$

$$\cdot (B(t+u) - B(t+v)) +$$

$$B(t)\left(2B(t) - B(t+u-v) - B(t-(u-v))\right)]dt$$

Let put $h = u - v \geqslant 0$ and estimate

$$|2B(t) - B(t+u-v) - B(t-(u-v))| \leqslant$$

$$2|\exp\{-t^2\} - \exp\{-(t+h)^2\}| +$$

$$|\exp\{-(t+h)^2\} - \exp\{-(t-h)^2\}| =$$

$$= 2A + B.$$

$$A = \exp\{-t^2\}|1 - \exp\{t^2 - (t+h)^2\}| \leqslant$$

$$\leqslant 2th + h^2$$

$$B = \exp\{-(t-h)^2\}\times$$

$$\times|\exp\{(t-h)^2 - (t+h)^2\} - 1| \leqslant 4th.$$

$$|B(t-u) - B(t-v)| \leqslant 2.$$

$$|B(t+u) - B(t+v)| = \exp\{-(t+v)^2\}\times$$

$$\times|\exp\{(t+v)^2 - (t+u)^2\} - 1| \leqslant$$

$$\leqslant \exp\{-(t+v)^2\}(2th + 2Sh).$$

$$\left|1 - \frac{t}{T}\right| \leqslant 1, \quad t \in [0, T].$$

So,

$$|\mathbf{E}\left(Z_T(u) - Z_T(v)\right)^2| \leqslant$$

$$\frac{4}{T}\int_0^T \exp\{-(t+v)^2\}(2th + 2Sh)dt +$$

$$+\frac{2}{T}\int_0^T \exp\{-t^2\}(4th + 2h^2 + 4th)dt \leqslant$$

$$\leqslant \frac{12h}{T} + \frac{6hS\sqrt{\pi}}{T} = h\frac{12 + 6S\sqrt{\pi}}{T}.$$

This estimate gives us $\sigma_t(h)$:

$$\sigma_t(h) = \sqrt{hC_{ST}}, \quad C_{ST} = \frac{12 + 6S\sqrt{\pi}}{T}\frac{C}{\ln 2}.$$

Then

$$\sigma_t(h)^{(-1)} = \frac{h^2}{C_{ST}}.$$

Functions $\delta_1(t)$ *and* $\nu_t(u)$ *will be next:*

$$\delta_1(t) = z(2\sqrt{tC_{ST}}),$$

$$\nu_t(u) = \mu\left(B(t, \frac{u^2}{C_{ST}}) \cap [0, S]\right).$$

Here we will obtain $\nu_t(u) \geqslant \frac{u^2}{C_{ST}}$. *Let put* $\gamma(y) = 1$. *Then*

$$z(y) = y, \quad z^{(-1)}(y) = y, \quad \hat{\gamma(y)} = 1, \quad A(S) = S^2.$$

Hence

$$a_p \leqslant \frac{1}{p(1-p)} \sup_{0 \leqslant t \leqslant S} \left(2\int_0^{2\sqrt{tC_{ST}}p} \ln\left(2 + \frac{2C_{ST}}{u^2}\right) du\right) = \frac{1}{p(1-p)} \times$$

$$\left(2\int_0^{2\sqrt{SC_{ST}}p} \ln\left(2 + \frac{2C_{ST}}{u^2}\right) du\right) =$$

$$= \frac{4}{p(1-p)}\left(\sqrt{12 + 6S\sqrt{\pi}}p \ln\left(2 + \frac{1}{2T}\right) + \frac{12 + 6S\sqrt{\pi}}{T}\arctan(2\sqrt{T}p)\right).$$

So, the next estimate holds true, while $x > 1$

$$\mathbf{P}\Big\{\sup_{0 \leqslant \tau \leqslant S} |B(\tau) - \widehat{B}_T(\tau) - \frac{1}{S}\int_0^S B(u)du + \frac{1}{S}\int_0^S \widehat{B}_T(u)du| > x\} \leqslant$$

$$\leqslant 2\exp\left\{-\ln(1 + \frac{1}{S^2})x \times \left[\frac{4}{p(1-p)}\left(\sqrt{12 + 6S\sqrt{\pi}}p\ln\left(2 + \frac{1}{2T}\right) + \frac{12 + 6S\sqrt{\pi}}{T}\arctan(2\sqrt{T}p)\right)\right]^{-1}\right\}.$$

for separable square-Gaussian process $X(t), t \in [0, T]$ *when it is true that*

$$\sup_{\rho(t,s) \leqslant h} (\mathbf{E}(X(t) - X(s))^2)^{\frac{1}{2}} \leqslant \sigma(h) = \sqrt{hC_{ST}}$$

Hence, let find x_α *for fixed level* α, $0 \leqslant \alpha \leqslant 1$, *that*

$$f(x_\alpha) = \alpha,$$

where

$$f(x) = 2\exp\left\{-\ln(1 + \frac{1}{S^2})x \times \left[\frac{4}{p(1-p)}\left(\sqrt{12 + 6S\sqrt{\pi}}p\ln\left(2 + \frac{1}{2T}\right) + \frac{12 + 6S\sqrt{\pi}}{T}\arctan(2\sqrt{T}p)\right)\right]^{-1}\right\}.$$

Hypothesis that the covariance function of the process $X(t)$ *is* $B(\tau)$ *for* $0 \leqslant \tau \leqslant T$ *is accepted if*

$$\sup_{0 \leqslant \tau \leqslant S}\left|B(\tau) - \widehat{B}_T(\tau) - \frac{1}{S}\int_0^S B(u)du + \frac{1}{S}\int_0^S \widehat{B}_T(u)du\right| < x_\alpha$$

and is rejected otherwise.

3 Conclusions

In paper the criterion of hypothesis testing on covariance functions of the stochastic processes is obtained. As an example was considered continuous correlation function $B(\tau) = \exp\{-\tau^2\}$.

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