УДК 519.713.2

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Ріст ініціальних оборотних автоматів з двома станами над бінарним алфавітом

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Growth of initial invertible automata with two states over binary alphabet

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Обчислено функції росту для усіх ініціальних оборотних автоматів з двома станами над бінарним алфавітом. Як наслідок показано, що автомат нескінченного порядку з цього класу має раціональну (алгебраїчну) функцію росту тоді і лише тоді, коли множина його станів породжує групу блимаючих лампочок, яка є вінцевим добутком циклічної групи порядку 2 та групи цілих чисел.

Ключові слова: ініціальний автомат, функція росту.

Automata were introduced in the middle of the 20th century to investigate the properties of different computational schemes. They appeared as a part of applied mathematics but now automata are widely used in such abstract areas as group theory and dynamical systems. In particular, automata were used to solve the Milnor problem on group growth.

An initial automaton is a model which allows us to realize certain transformations of a set of words over some finite alphabet. The growth function of such an automaton counts the minimal number of states in the automata that implement different powers of corresponding action on the given set of words. In this article we consider the initial invertible automata with two states that act on the set of binary words. We compute the growth function for each automaton from this automata class. As a corollary we prove that the only infinite order automata in this class with rational (algebraic) growth function are the lamplighter automata.

Keywords: initial automaton, growth function.
Communicated by Prof. Kirichenko V. V.

1 Introduction

Modern computers translate any type of information to the binary sequences to store and process it. Operations on such a binary presentation of data are implemented by different types of abstract automata. If a given function over the data can be realized by an automaton then this realization is not unique. It is natural to find the automaton with the minimal number of states which realizes a given function.

One of the simplest type of automata are initial Mealy automata that implement certain functions on the space X^* of words over an alphabet X. The composition of such functions is realized by the consecutive connection of the corresponding Mealy automata. Moreover, if we define the n-th iteration of an automaton as a composition with itself n times then we may introduce the growth function $\gamma(n)$ which counts the number of states in n-th iteration. In other words, it calculates the minimal number of states that could be used to implement the corresponding iteration of function given by the automaton.

The set of all initial invertible automata over a fixed alphabet forms a group with respect to the automata composition called the group of finite automata. Every finite invertible automaton A generate a finitely generated subgroup of this group: one can take an initial automaton A_s for each state s of A and generate a subgroup. There were many investigations during the last three decades in order to understand which groups can be generated by finite automata. The full list of groups generated by two-state automata over a two-letter alphabet is given in [6]. In particular, it was proved that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is generated by such an automaton. Also groups generated by three-state automata over a two-letter alphabet were deeply investigated in [3].

The growth function of non-initial automaton has the same type as the growth function of generated group. This connection allows one to construct examples of groups and automata with intermediate growth functions between polynomial and exponential. The first such an example was constructed by Grigorchuk in [5] and provided an answer to the question of Milnor on group growth. Interesting growth functions of non-initial automata were computed Bartholdi and Reznikov in [1] and [2]. The growth function of the generalized adding machine is computed in [7].

In this paper we compute the growth functions of initial invertible Mealy automata with two states over a binary alphabet. Note that we call a function $f : \mathbb{N} \to \mathbb{R}$ rational (algebraic) if the corresponding generating function of the sequence $\{f(n)\}_{n=0}^{\infty}$ is rational (algebraic). Our main result can be formulated as follows.

Theorem 1. *Let $\mathcal{A}_{(2,2)}$ be the set of invertible initial two-state automata over a two-letter alphabet and $\gamma_s(n)$ be the growth function of an automaton s. Then:*

1) *If an automaton $s \in \mathcal{A}_{(2,2)}$ has finite order then it has rational growth function. In this case $\gamma_s(n) = 1$ or $\gamma_s(n) = 1 + (n \bmod 2)$.*

2) *The only initial automata from $\mathcal{A}_{(2,2)}$ that have both infinite order and rational growth function are the lamplighter automata, which have growth function $\gamma_s(n) = 2^n$.*

3) *Any other automaton $s \in \mathcal{A}_{(2,2)}$ has non-algebraic growth function. More precisely, the following inequalities hold:*

$$2[\log_2 n] - p - 1 \leqslant \gamma_s(n) \leqslant 2[\log_2 n] - p + 2,$$

where $n = 2^p m$ with odd m.

The precise values of the growth functions $\gamma_s(n)$ for every $s \in \mathcal{A}_{(2,2)}$ are given in the proof.

The growth function in item 3) is non-algebraic, because it is bounded by two logarithmic functions $\log_2(x) - 2$ and $\log_2(x) + 3$, while the coefficients of an algebraic function cannot have logarithmic behavior (see Theorem 8.2 in [4]).

2 Automata and their growth functions

Let X be a finite set of elements $\{x_1, x_2, ..., x_n\}$, $n \geqslant 2$. We call this set the alphabet. Also let us denote by X^* the set of words over X including the empty word.

Definition 2.1. An *automaton* A is a triple (X, S, λ), where S is the set of states, and

$$\lambda : S \times X \to X \times S$$

is the output-transition map.

Definition 2.2. An automaton A is called *finite* if its set of states is finite.

Definition 2.3. An automaton $A = (X, S, \lambda)$ is *invertible* if for every its state $s \in S$ the restricted map $\lambda_s = \lambda(s, \cdot)|_X$ acts on X as some permutation $\pi_s \in \mathrm{Sym}(X)$.

In this article we consider only finite invertible automata. Note that every state s of such an automaton can be defined using the following wreath recursion notation:

$$s = (s_1, s_2, ..., s_n)\pi_s,$$

where $s_i = \lambda(x_i, s)|_S$.

Definition 2.4. An *initial* automaton A_s is an automaton $A = (X, S, \lambda)$ with a fixed initial state $s \in S$. We will use notation s for initial automaton A_s for better readability.

Each initial automaton s acts on the set of words over X. This action is constructed as follows. We pass the first letter y_1 of an input word to the initial automaton s. As the output we get a pair $\lambda(s, y_1) = (y_1', s')$ where y_1' is the output letter and s' is the next state. On the next step, we pass the second letter y_2 to the state s' and so on. Repeating of this procedure gives us an output word $y_1' y_2' \ldots$.

Since different automata can produce the same action on X^*, we must consider the notion of *minimal* automata.

Definition 2.5. An initial automaton s is called *minimal* if it has a minimal number of states among the automata that act on X^* as s.

Now let we have two initial invertible automata s_1 and s_2 over the same alphabet X. We can consider their composition $s_1 \cdot s_2$ as the minimal automaton which acts on each word $\omega \in X^*$ by the rule $s_1 \cdot s_2(\omega) = s_1(s_2(\omega))$.

Definition 2.6. The *growth function* of an initial automaton s is the function

$$\gamma_s(n) = \#States(s^n).$$

Definition 2.7. An initial automaton s has *finite order* if s^n acts trivially on X^* for some $n \in \mathbb{N}$.

Definition 2.8. Let A be an invertible non-initial automaton. The group generated by the transformations A_s for all states s of A is called the *automaton group* of A or *group generated by the automaton* A.

Definition 2.9. The automata given by recursions $b = (a, b)$, $a = (b, a)\sigma$ and $b = (a, b)$, $a = (a, b)\sigma$ are called the lamplighter automata.

These automata get their name because they generate the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ (see [6]).

3 Proof of Theorem 1

Every automaton with two states a and b over the binary alphabet $X = \{0, 1\}$ can be written in terms of wreath recursions as

$$a = (q_{11}, q_{12})\pi_a \quad \text{and} \quad b = (q_{21}, q_{22})\pi_b,$$

where $q_{ij} \in \{a, b\}$ and $\pi_a, \pi_b \in \mathrm{Sym}(X) = \{(), \sigma = (0\,1)\}$. If $\pi_a = \pi_b = ()$, then $\gamma_a(n) = \gamma_b(n) = 1$. If $\pi_a = \pi_b = \sigma$, then $\gamma_a(n) = \gamma_b(n) = 1 + (n \; mod \; 2)$. Further, without loss of generality, we suppose that $\pi_a = \sigma$ and $\pi_b = ()$.

Proof of Item 1. We list the automata such that a and b have finite order:

- $b = (b, b)$, $a = (a, a)\sigma$ (generates C_2);
- $b = (b, b)$, $a = (b, b)\sigma$ (generates C_2);
- $b = (a, a)$, $a = (a, a)\sigma$ (generates $C_2 \times C_2$);
- $b = (a, a)$, $a = (b, b)\sigma$ (generates $C_2 \times C_2$);
- $b = (a, b)$, $a = (a, a)\sigma$ (generates D_∞);
- $b = (a, b)$, $a = (b, b)\sigma$ (generates D_∞).

For these cases we can directly get that $\gamma_s(n) = 1$ or $\gamma_s(n) = 1 + (n \; mod \; 2)$. $\square$

Remark 1. *The two states of every automaton representation of the dihedral group D_∞ listed above have order two and they have rational growth function. However, the product of these two states has infinite order and its growth function is of logarithmic type thus it is not algebraic.*

Now let us consider the lamplighter automata.

Proof of Item 2. There are two automata in our family that generate the group $\mathbb{Z}_2 \wr \mathbb{Z}$:

- $b = (a, b)$, $a = (b, a)\sigma$;
- $b = (a, b)$, $a = (a, b)\sigma$.

The states of the second automaton are inverse to the states of the first automaton. That is why the corresponding growth functions are equal. So we can consider only the automaton $b = (a, b)$, $a = (b, a)\sigma$. We claim that each of the automata a^n and b^n consists of 2^n states which correspond to all words of length n over $\{a, b\}$. This follows from two well known facts about the action of a and b (see [3]). First, the semigroup generated by a and b is free; therefore, all words of length n over $\{a, b\}$ represent different initial automata. Second, the so-called dual automaton is isomorphic to the lamplighter automaton and acts transitively on all words of length n over $\{a, b\}$. Therefore, the automata a^n and b^n contain a state for every word of length n over $\{a, b\}$. Hence $\gamma_a(n) = \gamma_b(n) = 2^n$. $\square$

Proof of Item 3. The other automata, according to the classification from [6], generate the group $\mathbb{Z}$. There are two such automata. The first one is the so-called adding machine $b = (b, b)$, $a = (b, a)\sigma$. Its growth function was computed in [7]:

$\gamma_b(n) = 1$ for all $n \in \mathbb{N}$ and

$$\gamma_a(n) = \begin{cases} 2m - p + 2, & \text{if } \varepsilon_{m-1} = 1 \text{ or } p = m; \\ 2m - p + 1, & \text{otherwise.} \end{cases}$$

where $n = \varepsilon_0 + 2\varepsilon_1 + \ldots + 2^m\varepsilon_m$ is the expansion of n in base 2, $m = [\log_2 n]$ and p is the first non-zero position.

The second automaton is defined by recursions $b = (a, a)$, $a = (a, b)\sigma$. It is easy to check that $\gamma_a(1) = \gamma_b(1) = 2$ and for any $n > 2$ we have $\gamma_b(n) = \gamma_a(n) + 1$. That is why it is enough to consider only the growth function of a.

Let us notice that $ab = ba$ and $a^2b = 1$. It means that $b = a^{-2}$ and $a = (a, a^{-2})\sigma$. As a has infinite order we have that any two different powers of a define different actions on X^*.

It is easy to check the following formula for the state a^n:

$$a^n = \begin{cases} (a^{-\frac{n}{2}}, a^{-\frac{n}{2}}), & n = 0 \; mod \; 2; \\ (a^{-[\frac{n}{2}]+1}, a^{-[\frac{n}{2}]-2})\sigma, & \text{otherwise,} \end{cases}$$

which implies $\gamma_a(2^k n) = k + \gamma_a(n)$ for $n > 1$ and $\gamma_a(2^k) = k+1$. Therefore, it is enough to consider only odd values of n. Let $t_1 = -[\frac{n}{2}] - 2$. Then

$$a^n = (a^{t_1+3}, a^{t_1})\sigma.$$

To construct the set A of all states of the given finite state automaton we can consider the following procedure. First of all we add this automaton to the set A. Using the corresponding wreath recursion we add all coordinates to the set A. On the next step we consider only new elements that appeared in A during the previous step. We add to A all their coordinates. We can repeat this procedure again and again. As the given automaton is finite state then on some step there will be no new elements in A from the previous step. This means that A is constructed.

Let us use the procedure from above to find all the states of an automaton a^n, where n is a big enough odd number. First of all we add the state a^n and on the first step we also add states a^{t_1+3} and a^{t_1} with

$$t_1 = -[\frac{n}{2}] - 2.$$

On the first step both states are new because of proposition that n is big enough number. It is easy to check that for any parity of t_1 we will get on the next step exactly 2 new elements a^{t_2} and a^{t_2+3} for some integer number t_2. Note, that every next step gives us only 2 new states a^{t_k} and a^{t_k+3} for some t_k. Moreover, the following inequality holds:

$$|t_1| > |t_2| > \cdots > |t_k|,$$

until we get the first value of $|t_k|$ or $|t_k + 3|$ that is less then 20.

So the idea is to calculate how many steps we made until we get a value t_l such that $|t_l| \leqslant 20$. In addition we must calculate all other states of a^n that can be obtained from the states a^{t_l} and a^{t_l+3}. Based on this we get that the growth function satisfies the following equality

$$\gamma_a(n) = 2l + 1 + \delta(t_l)$$

where $\delta(t_l)$ counts the number of powers in the union of states a^{t_l}, a^{t_l+3} except themselves.

The table 1 gives us the full list of all possible disjoint cases that cover all the odd values of $n > 20$. In other words, we describe such a set of pairs $x, x+3$ (x is an integer number) that for any such a number n there is exactly one step of the algorithm which gives us a pair from this set. Notice that we must take into account that sign of t_k is changed from step to step. Nevertheless, it is enough to consider only absolute values of t_k.

Using the algorithm for each a^n we will reach only one of the listed pairs t_l, t_l+3. So it is enough to define what numbers can give us a pair t and $t+3$ on some. To describe such numbers we may consider only absolute value of t. We can construct the inverse transformations of powers from the wreath recursion :

$$x \mapsto 2x,$$

$$x \mapsto 2x+3, x \mapsto 2x-3.$$

Moreover, as we consider only odd numbers n then it is enough to use only maps $x \mapsto 2x+3$ and $x \mapsto 2x-3$. So on the first level we can construct only one such a number $n = 2t+3$ and then we can use both maps. As a result all numbers can be represented as follows:

$$n = 2(2(2(\dots 2(2t+3) \pm 3) \cdots \pm 3) \pm 3) \pm 3 =$$
$$= 2^s t + 3(2^{s-1} \pm 2^{s-2} \pm \cdots \pm 1).$$

This can be rewritten as $n = 2^s t + 3x$, where $x \in \{1, 3, 5, \dots, 2^s - 1\}$, $s \geqslant 1$. Also using this formula we can compute the number of steps

$$l = [\log_2 \frac{n}{t+3}] + 1.$$

Now we can introduce a boolean function $\psi(n,t)$ which equals to 1 if and only if there exist two numbers s and $x \in \{1, 3, 5, \dots, 2^s - 1\}$ such that $n = 2^s t + 3x$. Then the growth function is

$$\gamma_a(n) = 2[\log_2 n] + \sum_{8 \leqslant t \leqslant 17, t \neq 16} \psi(n,t) f(n,t), \quad (1)$$

where $f(n,t) = 2[\{\log_2 n\} - \log_2 (t+3)] + \delta(t) + 3$.

Further we simplify this expression. We need the following simple observation.

Lemma 1. *Let $\alpha \in (1,2)$ and $n \in \mathbb{N}$. Then $\{\log_2 n\} \geqslant \log_2 \alpha$ if and only if $\frac{n}{2^{[\log_2 n]}} \in [\alpha; 2)$.*

Proof. $\{\log_2 n\} = \log_2 \alpha$ if and only if the difference $\log_2 n - [\log_2 n]$ is equal to $\log_2 \alpha$.

Based on this we have

$$\log_2 n = \log_2 \alpha + [\log_2 n] \implies n = 2^{\log_2 \alpha + [\log_2 n]},$$

$$\frac{n}{2^{[\log_2 n]}} = 2^{\log_2 \alpha} = \alpha.$$

This means that $\log_2 n \geqslant \log_2 \alpha$ if and only if $\frac{n}{2^{[\log_2 n]}} \in [\alpha; 2)$. □

t_l	t_l+3	Powers	Powers for $-t_l-3$, $-t_l$	$\delta(t_l)$
8	11	$\{-4,-7,2,5,-1\}$	$\{4,7,-2,-5,1\}$	5
9	12	$\{-3,-6,3,0\}$	$\{3,6,-3,0\}$	4
10	13	$\{-5,-8,1,4,-2\}$	$\{5,8,-1,-4,2\}$	5
11	14	$\{-4,-7,2,5,-1\}$	$\{4,7,-2,-5,1\}$	5
12	15	$\{-6,-9,3,6,-3,0\}$	$\{6,9,-3,-6,3,0\}$	6
13	16	$\{-5,-8,1,4,-2\}$	$\{5,8,-1,-4,2\}$	5
14	17	$\{-7,-10,2,5,-1,-4\}$	$\{7,10,-2,-5,1,4\}$	6
15	18	$\{-6,-9,3,6,-3,0\}$	$\{6,9,-3,-6,3,0\}$	6
17	20	$\{-7,-10,2,5,-1,-4\}$	$\{7,10,-2,-5,1,4\}$	6

Table 1: Some values of the function $\delta(t)$

Since only one summand in (1) is nonzero, we consider each possibility for t separately to hold the condition $\psi(n,t)=1$.

1) If $t=9$ then

$$f(n,t)=2[\{\log_2 n\}-\log_2 1.5]+1.$$

On the other hand $n=9\cdot 2^s+3x=2^{s+3}+2^s+3x$, $x\in\{1,3,\ldots,2^s-1\}$, $s\geqslant 1$. From Lemma 1 for $\alpha=1.5$ we get that $\{\log_2 n\}<\log_2 1.5$. Therefore

$$\gamma_a(n)=2[\log_2 n]-1.$$

2) For $t\in\{12,14,15,17\}$ we have $\delta(t)=6$. Therefore

$$f(n,t)=2([\{\log_2 n\}-\log_2(t+3)]+4)+1=$$
$$=2[\{\log_2 n\}+\log_2\frac{16}{t+3}]+1.$$

a) For $t\in\{14,15,17\}$ we have $\log_2\frac{16}{t+3}<0$ and therefore

$$\gamma_a(n)=2[\log_2 n]+2[\{\log_2 n\}-\log_2\frac{t+3}{16}]+1=$$
$$=\begin{cases}2[\log_2 n]+1, & \{\log_2 n\}\geqslant\log_2\frac{t+3}{16},\\ 2[\log_2 n]-1, & \text{otherwise.}\end{cases}$$

b) For $t=12$ we have

$$\gamma_a(n)=2[\log_2 n]+2[\{\log_2 n\}+\log_2\frac{16}{15}]+1.$$

By construction we have

$$n=12\cdot 2^s+3x=2^{s+3}+2^{s+2}+3x$$

for some $s\geqslant 1$ and $x\in\{1,3,\ldots,2^s-1\}$. Therefore $[\log_2 n]=s+3$ and we get

$$\frac{n}{2^{[\log_2 n]}}=\frac{n}{2^{s+3}}=\frac{2^{s+3}+2^{s+2}+3x}{2^{s+3}}=$$
$$=1.5+\frac{3x}{2^{s+3}}<1.5+\frac{3\cdot 2^s}{2^{s+3}}=\frac{15}{8}.$$

Then $\{\log_2 n\}<\log_2\frac{15}{8}$ by Lemma 1 and we get that $\gamma_a(n)=2[\log_2 n]+1$.

3) For $t\in\{8,10,11,13\}$ we have the following equality:

$$f(n,t)=2([\{\log_2 n\}-\log_2(t+3)]+4)=$$
$$=2[\{\log_2 n\}+\log_2\frac{16}{t+3}].$$

a) If $t=13$ then $\log_2\frac{16}{t+3}=0$ and we get that $\gamma_a(n)=2[\log_2 n]$.

b) If $t\in\{8,10,11\}$ then we can repeat the same steps that was used for $t=12$. As a result for these cases $\{\log_2 n\}<\log_2\frac{t+3}{8}$ and $\gamma_a(n)=2[\log_2 n]$.

The final result for any odd number $n>20$ is defined by the formula $\gamma_a(n)=2[\log_2 n]+\delta(n)$,

$$\delta(n)=\begin{cases}1, & \text{if } \psi(n,12)=1,\\ -1, & \text{if } \psi(n,9)=1,\\ 1, & \text{if } \psi(n,t)=1,\ t\in\{14,15,17\}\\ & \text{and } \{\log_2 n\}\geqslant\log_2\frac{t+3}{16},\ ,\\ -1, & \text{if } \psi(n,t)=1,\ t\in\{14,15,17\}\\ & \text{and } \{\log_2 n\}<\log_2\frac{t+3}{16},\\ 0, & \text{otherwise.}\end{cases}$$

where $\psi(n,t)$ is a boolean function which equals to 1 if and only if there are exist numbers s and $x\in\{1,3,5,\ldots,2^s-1\}$ such that $n=2^st+3x$.

It is easy to see that the statement of *Item 3* holds for both a and b. Moreover, this proof gives us a possibility to construct any iteration of the automata directly without calculation of automata composition.

□

Remark 2. *We can use the similar procedure as was used in the proof to calculate the growth function of any automaton a which is defined by the following wreath recursion*

$$a=(a^s,a^{1-s})\sigma,s\in\mathbb{Z}.$$

Note, that in this case the adding machine corresponds to $s=0$.

Список використаних джерел

1. *Bartholdi L.* The smallest Mealy automaton of intermediate growth / L. Bartholdi , I.I. Reznykov, V.I. Sushchansky // J. Algebra. – 2006 – №2, **295** – pp. 387–414.

2. *Bartholdi L.* A Mealy machine with polynomial growth of irrational degree / L. Bartholdi, I.I. Reznykov // Internat. J. Algebra Comput. – 2008 – №1, **18** – pp. 59–82.

3. *Bondarenko I.* On classification of groups generated by 3-state automata over a 2-letter alphabet / I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, V. Nekrashevych, D. Savchuk, Z. Sunik // Algebra and Discrete Mathematics – 2008 – №1 – pp. 1-163.

4. *Flajolet Ph.* Analytic combinatorics / Ph. Flajolet, R. Sedgewick // Cambridge University Press – 2009.

5. *Grigorchuk R.I.* On the Milnor problem of group growth / R.I. Grigorchuk // Dokl. Akad. Nauk SSSR – 1983 – №1, **271** – pp. 30–33.

6. *Grigorchuk R. I.* Automata, dynamical systems and groups / R.I. Grigorchuk, V.V. Nekrashevych, V.I. Sushchansky // Proceedings of the Steklov Institute of Mathematics – 2000 – **231** – pp. 128–203.

7. *Skochko V.* The growth function of the adding machine / V. Skochko // accepted for publishing in Algebra and Discrete Mathematics – 2018.

References

1. BARTHOLDI, L., REZNYKOV, I.I. and SUSHCHANSKY V.I., (2006) "The smallest Mealy automaton of intermediate growth", J. Algebra, №2, **295**, pp. 387–414.

2. BARTHOLDI, L. and REZNYKOV, I.I. (2008) "A Mealy machine with polynomial growth of irrational degree", Internat. J. Algebra Comput., №1, **18**, pp. 59–82.

3. BONDARENKO, I., GRIGORCHUK, R., KRAVCHENKO, R., MUNTYAN, Y., NEKRASHEVYCH, V., SAVCHUK, D. and SUNIK Z.(2008) "On classification of groups generated by 3-state automata over a 2-letter alphabet", Algebra and Discrete Mathematics, №1, pp. 1-163.

4. FLAJOLET, Ph. and SEDGEWICK, R. (2009) "Analytic combinatorics", Cambridge University Press.

5. GRIGORCHUK, R. I. (1983) "On the Milnor problem of group growth", Dokl. Akad. Nauk SSSR, №1, **271**, pp. 30-33.

6. GRIGORCHUK, R.I., NEKRASHEVYCH, V.V. and SUSHCHANSKY, V.I. (2000) "Automata, dynamical systems and groups", Proceedings of the Steklov Institute of Mathematics, **231**, pp. 128–203.

7. SKOCHKO V. (2018) "The growth function of the adding machine", accepted for publishing in Algebra and Discrete Mathematics.

Received: 11.08.2017