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## RESPONSE LOCALIZATION IN DISORDERED STRUCTURES GOVERNED BY THE STURM-LIOUVILLE DIFFERENTIAL EQUATION. (REVIEW)

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### ЛОКАЛИЗАЦИЯ ОТВЕТА В НЕУПОРЯДОЧЕННЫХ КОНСТРУКЦИЯХ, ОПИСЫВАЕМЫХ ДИФФЕРЕНЦИАЛЬНЫМ УРАВНЕНИЕМ ШТУРМА-ЛИУВИЛЛЯ. (ОБЗОР)

*The review is dedicated to the relatively new problem in structural engineering: localization of the response by structural irregularities. This review is aimed to outline all relevant discoveries in the response localization in mechanical problems (vibration, buckling) from the perspective of the common mathematical representation through Sturm-Liouville problem. Two possible approaches to analyze the influence of the disorder are discussed: exact dynamic stiffness formulation of the mistuned structure and the perturbation of the eigen solution of the tuned structure. Both approaches shown to lead to the same localization phenomena and exponential decay of the eigenvector from the source of disorder. In the section dedicated to the buckling mode localization the approach to analyze localization of the randomly disordered multi-span beam based on the Furstenberg's theorem is presented. The examples of the localization phenomena in the real engineering structures are given.*

*Keywords:* response localization, wave propagation, buckling, irregularities, disorder, review.

#### Introduction

The problem of structural response localization can be dated up to as early as the famous article by Anderson [1] where he studied the conductivity in disordered solids. His research attracted a considerable interest in the solid-state physics [2, 3, 4]. The similar phenomena of localization was discovered in the structural mechanics by Hodges and Woodhouse [5, 6]. Their pioneering study in the field of vibration localization of mistuned structures started a new field in structural dynamics [7, 8, 9]. The logical continuation of the research led to the analysis of the localization phenomena in elastic stability where several approaches to analysis of the buckling modes localization are known [10, 11]. Despite the fact that these two problems have a completely different physical backgrounds they all have one common mathematical representation. Both problems of structural vibrations and elastic stability are generally governed by the same Sturm-Liouville differential equation. Therefore, all these problems can be studied either by the analysis of the transcendental eigenvalue problem arising from the straightforward implementation of the Sturm-Liouville equation to each structural member of the mistuned system and assembling the total dynamic stiffness equation [12] or perturbing the eigen solutions of the tuned problem [13].

This article gives outlines of the field of response localization in disordered structures from the perspective of the generally common mathematical formulation. It is shown that all structures corresponding to the Sturm-Liouville problem exhibit the same kind of the response localization defined through the fundamental coupling to disorder ratio. It is further shown that the eigenvalue pass and stop bands exist both in the vibration and buckling problems in disordered structures. Vibration and buckling modes both decay from the disorder following the Lyapunov exponent. Exact eigenvalue analysis and the asymptotic analysis methods further shown to solve the governing Sturm-Liouville equations.

#### Vibration localization

This part of the review is mainly focused on the localization phenomena in the dynamic response of the multispan beams. They have a wide range of implementations in engineering from railway rails to multi-span columns and above-ground pipelines. Even aircraft parts can be modeled to some level of accuracy with multi-span beams (fuselage, wings). Therefore, the problem of predicting the dynamic response of such structures naturally arises.

Hodges and Woodhouse [6] were the first to design a simple experiment to show the localization of vibration and to predict the results of the experiment using a simple coupled pendulum model. Their experiment consisted of the excitation of the string with concentrated masses shown in Figure 1. The idea was to excite the string and to measure frequencies and vibrational modes of the string with masses evenly spaced and with one span between masses being

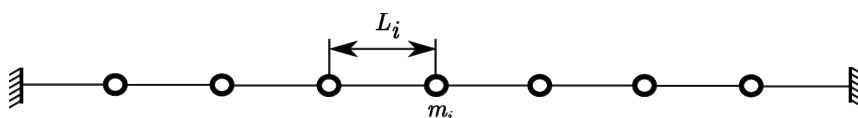


Figure 1. String with discrete masses [6]

slightly different from the others. In such a way the comparison of the dynamic response of the perfectly periodic string and the string with a disorder in the single span was made. Then these results were predicted using the  $n$ -coupled pendulum model shown in Figure 2.

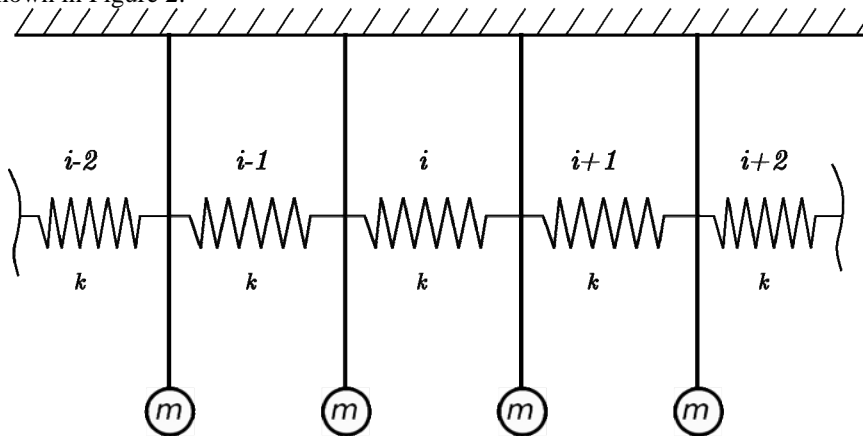


Figure 2.  $n$ -coupled pendulum model [6]

The authors showed that the frequencies of the periodic ordered string appear in groups called *pass bands*. It means that in the frequency space there are special regions where frequencies do appear in high density and there are regions where frequencies do not appear at all. The schematic example of the band nature of frequencies in the ordered string is shown in Figure 3.

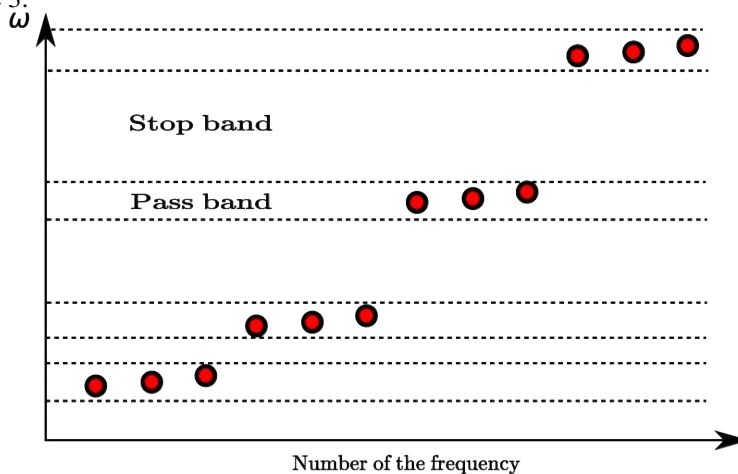


Figure 3. Band structure of the frequency space

It is apparent from the figure that indeed frequencies appear in groups. It should be emphasized that the number of frequencies within a single band is equal to the number of masses on the string, or in general case, to the number of spans in the multi-span beam. To explain such behavior one should consider  $n$  pendulums coupled to each other by  $n - 1$  springs with stiffness  $k$  each as shown in Figure 2. Each pendulum has its natural frequency  $\omega_n$ . If  $\Delta_i$  is a displacement of  $i$ -th pendulum, then its motion will be governed by

$$\Delta_i (\omega_k^2 - \Omega^2 + 2k) = k\Delta_{i+1} + k\Delta_{i-1}. \tag{1}$$

Here  $\Omega$  represents frequency of the whole model.

If pendulums are assumed to be identical then all natural frequencies are identical as well  $\omega_i = \Omega$ . Each pendulum displacement will be written as  $\Delta_i = \Delta = e^{in\phi}$  falls in the range  $[0, \pi]$  and stands for different modes of vibration.

After substitution of  $\Delta_i$  in equation (1) and extracting  $\omega^2$ :

$$\omega_k^2 = \Omega^2 + 2k(1 - \cos \phi). \tag{2}$$

It is apparent from equation (2) that squares of the natural frequencies of the structure lie in bounds:

$$\Omega^2 < \omega^2 < \Omega^2 + 4k. \tag{3}$$

These bounds are known as a *pass bands* of the periodic structure.

Profound analysis of the pendulum equation of motion (1) gives the insight on the basic properties of the pass and stop bands. Indeed, the number of frequencies within a single band is equal to the number of pendulums, the band width is defined by the coupling stiffness  $k$ . If  $k \rightarrow 0$  than the structure degenerates into  $n$  independent pendulums with their own natural frequencies and the band degenerates into a single point  $\omega^2 = \Omega^2$ . It is apparent from equation (3) that the lowest frequency within the band corresponds to such a mode of vibration of the whole system when all pendulums go in the same direction, while higher frequencies correspond to the modes with opposite signs.

After the analysis of the ordered periodic system was performed the authors introduced a small disorder by assuming the length of the  $i$ -th span to be  $L_i + \Delta L_i$  and studied the mode localization in the string and explained this phenomena with the disordered pendulum model. They showed that the mode shapes and hence the degree of localization depend only on the ratio of coupling to disorder. They also managed to use the coupled pendulum model to predict the exponential decay of the vibrational amplitude from the disordered span or pendulum.

Another approach to study the localization phenomena in vibrations was described by Pierre et al. [14]. They considered a simple two-span beam with a torsional spring in between the spans. The torsional spring stiffness  $c$  was used to model the coupling between spans. One span was assumed slightly different in length from the other, where the difference is some small non-dimensional number  $\Delta l$ . The proposed structure is governed by two coupled Sturm-Liouville equations of the kind:

$$\frac{d}{dx} \left[ p(x) \frac{dW}{dx} \right] + q(x)W = -\lambda r(x)W. \quad (4)$$

In such model it was advantageous to use a modified perturbation method. According to this method the disorder is firstly introduced in the unperturbed system and the interface between the two beams is assumed rigid (clamped) in the unperturbed configuration  $1/c = 0$ . The perturbation was then introduced by replacing the clamped support with a spring with a very high stiffness resulting in  $1/c \rightarrow 0$ . According to such formulation the perturbation is assumed to be small. Therefore the first order expansion of the frequency is:

$$\omega = \omega_0 + \delta\omega + O(1/c^2). \quad (5)$$

Where  $\delta\omega$  is the first order perturbation. Substituting equation (5) into the governing the equation of movement the authors calculated the eigenmodes of the perturbed system which showed to be localized at one of two spans. The perturbed modes are shown in Figure 4.

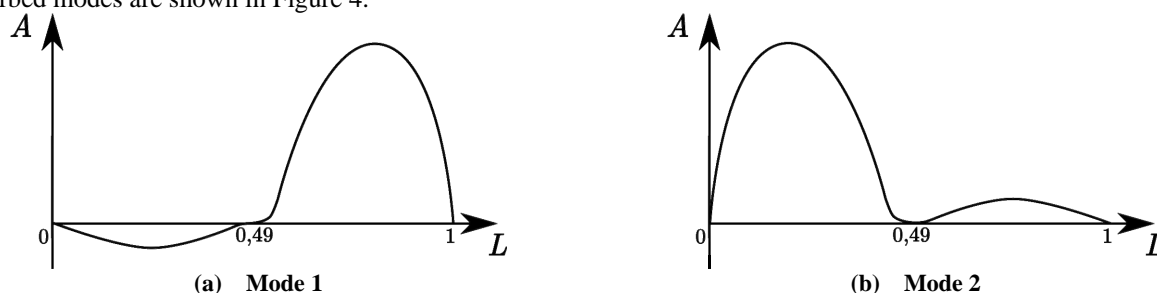


Figure 4. First two eigenmodes of the perturbed two-span beam [14]

Hence, the asymptotic analysis showed strongly localized mode shapes that form a linear combination of the mode shapes of each span.

Only the general outline of the research dedicated to wave localization in disordered structures is presented here. More profound analysis of the localization in the real dynamic structure is kept for another occasion. For instance, it can be shown that if an above-ground pipeline with multiple supports has some level of mistuning introduced through irregularities in the support placement is dynamically excited, its modes of vibration will be highly localized about the disordered span. It can be further shown that the mode amplitudes decay following the Lyapunov exponent. The similar behavior takes place even when all spans are randomly disordered and the disorder is evenly distributed along the length. Such phenomena is also observed in the elastic stability problem discussed below.

### Localization of the buckling modes

The localization phenomena in buckling problems were first discovered by Pierre and Plaut [15]. They studied a 2-span beam by means of perturbation analysis on the eigenvalues and the eigenvectors of the structure. Later, Li et al. [10] considered a general  $n$ -span beam with a disorder localized in a single arbitrary span. They applied finite difference calculus to form a transitional operators that governed the transition of the bending moment from span to span. Xie [11] studied a general  $n$ -span beam in which disorder was assumed distributed randomly in all spans.

Generally, the elastic stability of the disordered nearly periodic structure is performed in two different ways. It is either analyzed in the straightforward way by constructing appropriate transition operators [10, 11] or through small

perturbations on eigenvalues and eigenvectors of the governing structural matrix [15]. Both approaches are further discussed.

Consider a multi-span beam in Figure 5. All spans are identical and have length  $L$  and bending stiffness  $EI$ . Each support contains a torsional spring with stiffness  $k_r$ , which models coupling between spans. The beam is subjected to the axial compressing force  $P$ . Angles of rotation of the beam at each support are given by  $\theta_i$ .

The governing differential equation that accounts for the interaction between spans can now be derived from a Sturm-Liouville equation

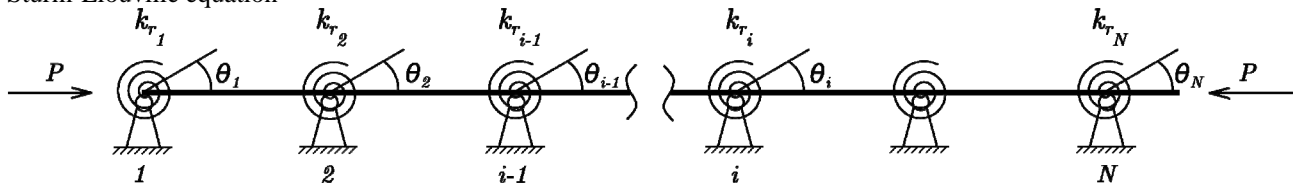


Figure 5. Multi-span beam [11]

(Look at equation (4)) replacing  $p(x)$  with a beam-bending stiffness  $EI$ ,  $q(x)$  with axial compressing force  $P$  and the non-zero right side is expressed in terms of the bending moment that is translated between spans:

$$EI \frac{d^2 W_i(x)}{dx^2} + P \frac{dW_i(x)}{dx} = M_i, \tag{6}$$

where  $M_i$  the bending moment on the  $i$ -th span yet to be determined,  $W_i(x)$  is the lateral deflection of the  $i$ -th span. In order to determine the moment  $M_i$  consider the moment equilibrium at the  $i$ -th support. An arbitrary span is cut out of the system and drawn in Figure 6 with acting forces and corresponding displacements. Indexes  $R$  and  $L$  denote that the bending moments and lateral forces are taken from the right and from the left side of the support.

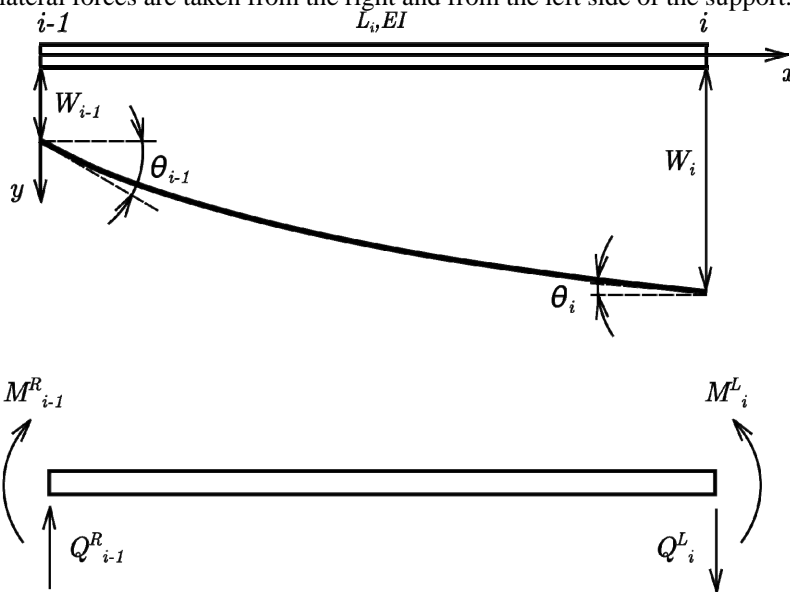


Figure 6. Equilibrium of the  $i$ -th span [10]

It is apparent from the figure that moment equilibrium is defined as:

$$\frac{M_{i-1}^R}{EI} \left( \frac{x_i}{L_i} - 1 \right) = \frac{M_i^L}{EI} \frac{x_i}{L_i}. \tag{7}$$

Therefore, the governing differential equation can be written in the following form:

$$W'' + \lambda^2 W = \frac{M_{i-1}^R}{EI} \left( \frac{x_i}{L_i} - 1 \right) - \frac{M_i^L}{EI} \frac{x_i}{L_i}. \tag{8}$$

This is a non-homogeneous differential equation with constant coefficients and its solution is known:

$$W(x) = C_i \sin(\lambda x_i) + D_i \cos(\lambda x_i) + \frac{M_{i-1}^R}{EI} \left( \frac{x_i}{L_i} - 1 \right) - \frac{M_i^L}{EI} \frac{x_i}{L_i}. \tag{9}$$

$C_i$  and  $D_i$  – are yet unknown coefficients. They can be found from the boundary conditions:

$$W|_{x=0} = 0; \quad W|_{x=L_i} = 0. \tag{10}$$

Therefore,  $C_i$  and  $D_i$  are defined as:

$$C_i = -\frac{M_{i-1}^R \cos(\lambda L_i)}{P \sin(\lambda L_i)} + \frac{M_i^L}{P \sin(\lambda L_i)}; \quad D_i = \frac{M_{i-1}^R}{P}. \quad (11)$$

After substitution of equation (11) in solution (9) the deflected shape of  $i$ -th span  $W_i(x_i)$  can be written. Then, after taking first derivative and substituting  $x_i = 0$  and  $x_i = L_i$ , the equations for rotation in each support is formulated:

$$\theta_{i-1} = \frac{M_{i-1}^R}{\lambda^2 L_i EI} \left[ 1 - \frac{\lambda L_i \cos(\lambda L_i)}{\sin(\lambda L_i)} \right] - \frac{M_i^L}{\lambda^2 L_i EI} \left[ 1 - \frac{\lambda L_i}{\sin(\lambda L_i)} \right]; \quad (12)$$

$$\theta_i = \frac{M_{i-1}^R}{\lambda^2 L_i EI} \left[ 1 - \frac{\lambda L_i}{\sin(\lambda L_i)} \right] - \frac{M_i^L}{\lambda^2 L_i EI} \left[ 1 - \frac{\lambda L_i \cos(\lambda L_i)}{\sin(\lambda L_i)} \right]. \quad (13)$$

Bending moments at supports can now be expressed in terms of rotational angles:

$$M_{i-1}^R = \frac{2EI}{L_i} [2s\theta_{i-1} + c\theta_i]; \quad M_i^L = -\frac{2EI}{L_i} [2s\theta_i + c\theta_{i-1}] \quad (14)$$

Here  $s$  and  $c$  are stability functions, which are derived from equations (12) and (13) in a straightforward way:

$$s = \frac{\lambda L_i [\sin(\lambda L_i) - \lambda L_i \cos(\lambda L_i)]}{4[2 - 2\cos(\lambda L_i) - \lambda L_i \sin(\lambda L_i)]}; \quad c = \frac{\lambda L_i [\lambda L_i - \sin(\lambda L_i)]}{2[2 - 2\cos(\lambda L_i) - \lambda L_i \sin(\lambda L_i)]} \quad (15)$$

Thus, the general stability functions that express moment-rotation dependence for any multi-span beam are developed. They can be used to formulate transitional operators or to define a characteristic equation for the buckling load parameter.

The simplest structure consisting of a 2-span beam analyzed by Pierre and Plaut [15] is considered first. In this model one span is assumed to have slightly different length from the other one. This difference  $\Delta L$  is considered as a measure of disorder. Such a system is presented in Figure 7.

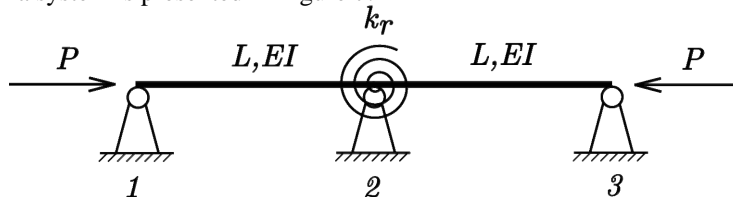


Figure 7. Two-span-beam model [15]

After introducing the disorder, span lengths are defined as  $L_{12} = L - \Delta L$ ,  $L_{23} = L + \Delta L$ . And non-dimensional spring stiffness is  $\gamma = k_r L / EI$ . Substituting these values in equations (14) and considering moment equilibrium at the  $i$ -th support  $M_i^L - M_i^R = k_r \theta_i$  the characteristic equation for the buckling load parameter  $\lambda$  is stated to take form:

$$\begin{aligned} & (\lambda^2 L_{12} + \gamma) \lambda L_{23} \sin(\lambda L_{12}) \cos(\lambda L_{23}) - (\lambda^2 + \gamma) \sin(\lambda L_{12}) (\lambda L_{12}) \sin(\lambda L_{23}) + \\ & + (\lambda^2 L_{23} + \gamma) \lambda L_{12} \cos(\lambda L_{12}) \sin(\lambda L_{23}) - \lambda^2 L_{12} L_{23} \gamma \cos(\lambda L_{12}) \sin(\lambda L_{23}) = 0. \end{aligned} \quad (16)$$

The corresponding buckling modes for each span are:

$$W_{12}(x) = A_1 \left[ \sin(\lambda x) - \frac{x}{L_{12}} \sin(\lambda L_{12}) \right]; \quad W_{23}(x) = A_2 \left[ x - L_{12} - L_{23} (1 - \cos(\lambda L_{23}) + \cot(\lambda L_{23}) \sin(\lambda L_{23})) \right] \quad (17)$$

In equations (17)  $A_1$  and  $A_2$  are undetermined scaling constants for modes. The relation between these constants in terms of system parameters and arbitrary constant  $A$  is given as:

$$A_1 = A \left[ \gamma \sin(\lambda L_{23}) - \gamma \lambda L_{23} \cos(\lambda L_{23}) + L_{23} \lambda^2 \sin(\lambda L_{23}) \right]; \quad A_2 = A \lambda^2 \sin(\lambda L_{12}) \sin(\lambda L_{23}). \quad (18)$$

Now the influence of a disorder on the mode localization in the two-span beam can be analyzed. It should be pointed out that the response of this mechanical system is fully determined by two parameters: disorder magnitude  $\Delta L$  and the strength of coupling  $\gamma$ .

Pierre and Plaut discussed two limiting cases in their article [15]: the case of weak and strong coupling. For each of these cases they numerically solved the equation (16) to define critical buckling load parameters for two lowest modes  $\lambda_1$  and  $\lambda_2$  for different values of disorder. Substituting these values in equations (17) corresponding buckling modes were derived. The  $\lambda_1$  and  $\lambda_2$  curves as well as corresponding buckling modes for weak and strong coupling cases are plotted in Figure 8.

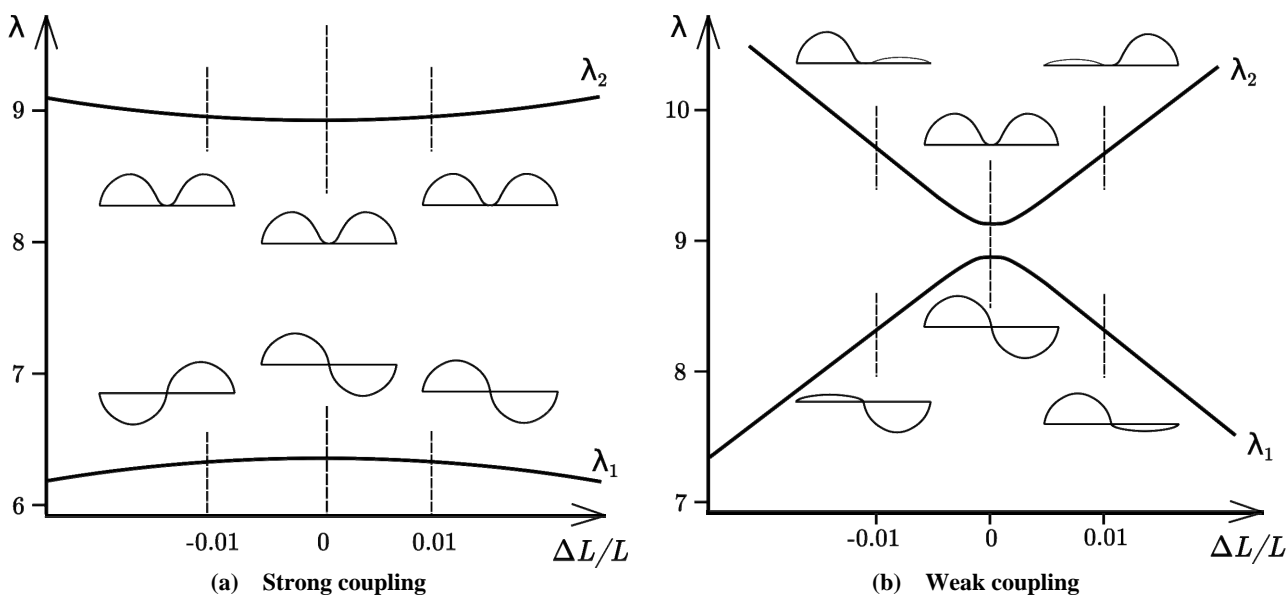


Figure 8. Influence of the disorder on critical buckling force for strong and weak coupling [15]

Strong *inter-span* coupling corresponds to small values of  $\gamma \rightarrow 0$  and means that each span response separately from another. It is apparent from figure 8(a) that  $\lambda_1$  and  $\lambda_2$  give two well-separated buckling modes and disorder doesn't influence the buckling behavior. The eigenvalue loci do not approach to each other and no localization of the modes occurs.

Weak *inter-span* coupling corresponds to high values of  $\gamma$ . The authors obtained eigenvalue loci curves in figure 8(b) for  $\gamma = 600$ . Here the eigenvalue loci approach each other and the smallest distance between them is reached for  $\Delta L = 0$ . This phenomena is known as *eigenvalue loci veering*. In the veering point, buckling modes become degenerate and are no longer linearly independent. Even small disorder of the system dramatically changes buckling modes in the veering point and mode localization occurs.

Their analysis was extended to cover the response of the multi-span beam in Figure 5 by Li et al. [10]. They assumed that the disorder in the system is introduced through any single arbitrary span being slightly longer or shorter than others. This difference in length was called *length imperfection*  $\Delta L$ . The authors used finite difference calculus approach to study the transition of the response of the system from span to span.

Their approach involved a so-called transition operator which was defined from the moment equilibrium at  $i$ -th support and hence operates stability functions (equation (15) and equation (15)):

$$M_i^L - M_i^R = k_r \theta_i. \tag{19}$$

Then, substituting equations (14) they derived the angle transition equation:

$$c(\theta_{i+1} + \theta_{i-1}) + 4(s + \gamma)\theta_i, \tag{20}$$

where  $\gamma$  is a non-dimensional coupling and  $s$  and  $c$  are stability functions.

In such a way, the transition operator  $\mathbb{T}$ , which defines continuous transition of parameters between spans, was written as:  $\theta_{i+1} \mathbb{T} \theta_i$ . As a result, equation (20) can be rewritten using the transition operator

$$\left[ c(\mathbb{T} + \mathbb{T}^{-1}) + 4(s + \gamma) \right] \theta_i = 0. \tag{21}$$

Equation (21) is a finite difference equation. Its solution is written in the form:

$$\theta_i = A e^{i\phi}. \tag{22}$$

After substituting equation (22) in equation (21) one obtains:

$$\cosh \phi = -\frac{2(s + \gamma)}{c}. \tag{23}$$

Three cases of  $\cosh \phi$  depending on values of  $\gamma$  were considered:  $\cosh \phi < -1$ ,  $\cosh \phi \in [-1, 1]$  and  $\cosh \phi > 1$ .

For  $\cosh \phi > 1$ :  $\phi_{1,2} = \pm \phi$ ,  $\phi = \cosh^{-1} \left( -\frac{2(s + \gamma)}{c} \right)$  and the solution is written as:

$$\theta_i = A e^{i\phi} + B e^{-i\phi}. \tag{24}$$

Here  $A$  and  $B$  are arbitrary constants. When  $\cosh \phi < -1$ :  $\phi_{1,2} = \pm(\alpha + j\pi)$ ,  $\alpha = \cosh^{-1}\left(-\frac{2(s+\gamma)}{c}\right)$  and the solution is:

$$\theta_i = (A \cosh(\alpha i) + B \sinh(\alpha i)) \cosh(\pi i). \quad (25)$$

If  $\cosh \phi \in [-1, 1]$ :  $\phi_{1,2} = \pm\beta$ ,  $\beta = \cosh^{-1}\left(-\frac{2(s+\gamma)}{c}\right)$  and the solution is:

$$\theta_i = A \cosh(\beta i) + B \sinh(\beta i). \quad (26)$$

Now consider periodic  $n$ -span beam in which  $(i+1)$ th span has a length imperfection  $\Delta L$ . The beam is assumed to be divided into three parts: two parts contain a large number of equal spans (name the first and the third parts) and the second part contains a single disordered span. The problem is solved for the first and the third parts in a straightforward way substituting solutions (24), (25), (26) into equation (21), boundary and continuity conditions. The second part is analyzed substituting solutions (24), (25), (26) into equation (21) and shape functions as well as (28) boundary and continuity conditions with original length  $L$  replaced by a length with imperfection  $L + \Delta L$ . The algebraic details of the substitution can be found in [10] and are omitted here to make the argument more transparent.

After performing all operations for each part of the beam, six homogeneous algebraic equations were derived for each case. Writing them in a matrix form gives:

$$\mathbf{K}(\lambda) \mathbf{W} = 0. \quad (27)$$

This is an exact dynamic stiffness formulation of the current problem. In equation (27),  $\mathbf{K}$  is a transcendental matrix, which has an intrinsic dependence on critical buckling load parameter  $\lambda$  through the stability functions,  $\mathbf{W}$  is a vector of displacements of all nodes in the system (currently  $\mathbf{W}$  contains only rotational degrees of freedom, which is not the case in general formulation).

The equation (27) is valid when either  $\mathbf{W} = 0$  or  $\det[\mathbf{K}(\lambda)] = 0$ . Because  $\mathbf{W} = 0$  stands for the trivial case, when no buckling occurs

$$\det[\mathbf{K}(\lambda)] = 0. \quad (28)$$

is the criteria from which critical buckling load factors  $\lambda$  are calculated. Then, substituting  $\lambda$  in the solution of governing differential equation (9) corresponding buckling modes for each span were calculated one by one.

The authors ran this analysis for two types of coupling: weak coupling  $\gamma = 0.3$  and strong coupling  $\gamma = 10$ . For each type of coupling 11-span beam with different cases of disorder placement was analyzed. For the strong coupling case, 100-span and 400-span beams were also considered in order to observe strong buckling mode localization near the disordered span. For the further details, the reader is referenced to the original paper [10].

In figure 9(a) the buckling modes for a beam with 100 spans are shown. It is shown in figure 9(b) that the envelope of buckling modes is indeed an exponential function. Hence, it is concluded that buckling mode amplitudes of a disordered longitudinally periodic structure exponentially decay from the disordered span.

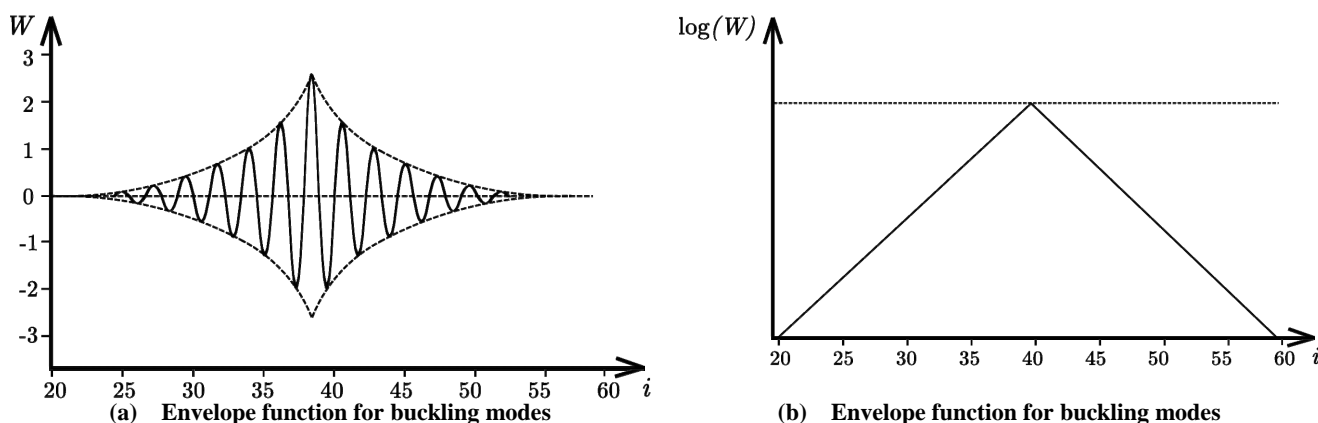


Figure 9. Buckling mode localization in multi-span beam [10]

Now we are in a position to define the strict mathematical formulation of the exponential decay of the buckling response. Consider the beam in figure 5 to be randomly disordered. It means that each span has a small but random length imperfection. This case is the closest to the real structure because spans of the multi-span beam are never perfectly identical. Usually, each span slightly varies in length and this variation is random in nature. The model of the randomly disordered multi-span beam was studied by Xie [11].

Adopting his notation each span length is  $L_i$  and each flexural stiffness is  $k_i = EI / L_i$  while the averaged quantities for the whole structure are  $L$  and  $k$ . Spring stiffness are defined by  $k_r$ . Stability functions vary from span to span along with  $L_i$ . However, it is possible to define them in invariant form. This form is derived from equations (12) and (13) separating the random quantities  $L_i$  and simplifying:

$$\bar{s}_i = \frac{\lambda_i(1-2\lambda_i \cot(\lambda_i))}{\tan(\lambda_i) - \lambda_i}; \quad \bar{c}_i = \frac{(1-2\lambda_i \cot(\lambda_i))(2\lambda_i - \sin(2\lambda_i))}{(\tan(\lambda_i) - \lambda_i)(\sin(2\lambda_i) - 2\lambda_i \cos(2\lambda_i))} \quad (29)$$

Then, transition equation is:

$$\bar{s}_{i-1}\bar{c}_{i-1}k_{i-1}\theta_{i-1} + (\bar{s}_{i-1}k_{i-1} + \bar{s}_i k_i + k_r)\theta_i + \bar{s}_i\bar{c}_i k_i \theta_{i+1} = 0. \quad (30)$$

Xie introduced non-dimensional quantities  $\bar{k}_i = EI / EI L_i$ ,  $\bar{k}_r = k_r / k$  and defined transition operator in a matrix form:

$$\mathbf{x}_i = \mathbf{T}_i \mathbf{x}_{i-1}, \quad \mathbf{x}_i = \begin{Bmatrix} \theta_{i-1} \\ \theta_i \end{Bmatrix}; \quad \mathbf{T}_i = \begin{bmatrix} -\frac{s_{i-1}\bar{k}_{i-1}\theta_{i-1} + s_i\bar{k}_i + k_r}{s_i c_i \bar{k}_i} & -\frac{s_{i-1}c_{i-1}\bar{k}_{i-1}}{s_i c_i \bar{k}_i} \\ 1 & 0 \end{bmatrix}. \quad (31)$$

Hence, the transition equation for the whole beam is written as:

$$\mathbf{x}_n = \mathbf{T}_n \mathbf{T}_{n-1} \dots \mathbf{T}_2 \mathbf{x}_1. \quad (32)$$

The disorder was assumed to be random, independent of other spans and have a common probability distribution. Such assumptions result in matrices  $\mathbf{T}_i$  being independent and identically distributed. Then, the response of the system  $\mathbf{x}_i$  is defined as a product of random matrices  $\mathbf{T}_i$ . Products of the random independent matrices follow the rule defined by Furstenberg's theorem [16]. According to this theorem if:

1.  $\mathbf{T}_i$  are non-singular, independent and identically distributed,
2. each matrix  $\mathbf{T}_i = \mathbf{T}_i(\varepsilon)$  where  $\varepsilon$  is a random vector with probability density  $p(\varepsilon)$ ,
3.  $x_0 \neq 0$ ,
4.  $\lim_{n \rightarrow \infty} \frac{1}{n} \ln[\det(\mathbf{T}_1) \cdot \det(\mathbf{T}_2) \dots \det(\mathbf{T}_n)] = 0$ ,

then:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\mathbf{T}_n \mathbf{T}_{n-1} \dots \mathbf{T}_1 \mathbf{x}_0\| = \lambda > 0, \quad (33)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\mathbf{T}_n \mathbf{T}_{n-1} \dots \mathbf{T}_1\| = \lambda. \quad (34)$$

Therefore, it is easy to show that:

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} \ln \|\mathbf{x}_n\| = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\mathbf{T}_n \mathbf{T}_{n-1} \dots \mathbf{T}_2 \mathbf{x}_1\| = \lambda. \quad (35)$$

Hence, for large  $n$ :

$$\|\mathbf{x}_n\| = e^{\lambda(n-1)} \|\mathbf{x}_1\|. \quad (36)$$

Equation 36 is a mathematical form of defining the rate of buckling mode amplitudes decay from the disorder and can be treated as a quantitative description of localization. Indeed constant  $\lambda$  defines the average exponential rate of change of the norm of the state vector, which in this case is the vector of angles of rotation. This constant is known as Lyapunov's exponent.

Lyapunov's exponent can be found considering the product of matrices  $\mathbf{T}_i$ . Let  $\mathbf{B}_n = \mathbf{T}_n \mathbf{T}_{n-1} \dots \mathbf{T}_2$ , hence

$$\mathbf{x}_n = \mathbf{B}_n \mathbf{x}_1. \quad (37)$$

Take the Euclidean norm of the vector  $\mathbf{x}_n$ :

$$\|\mathbf{x}_n\|_{\varepsilon}^2 = \mathbf{x}_n \mathbf{x}_n = \mathbf{x}_1^T \mathbf{B}_n^T \mathbf{B}_n \mathbf{x}_1. \quad (38)$$

Let the eigenvalues of the 2x2 matrix  $\mathbf{B}_n^T \mathbf{B}_n$  to be  $\sigma_1$  and,  $\sigma_2$  where  $\sigma_1^2 = \sigma_{\max}^2$ ,  $\sigma_2^2 = \sigma_{\min}^2$  and  $\sigma_1^2 \geq \sigma_2^2$ . Each eigenvalue corresponds to its eigenvector:  $\sigma_1^2 \rightarrow e_1$ ,  $\sigma_2^2 \rightarrow e_2$ .

Compose Rayleigh quotient:

$$R(x) = \frac{\mathbf{x}^T \mathbf{B}_n^T \mathbf{B}_n \mathbf{x}}{\mathbf{x}^T \mathbf{x}}, \quad (39)$$

therefore,



$$\sigma_{\min}^2 = \frac{\mathbf{e}_2^T \mathbf{B}_n^T \mathbf{e}_2}{\|\mathbf{e}_2\|_{\varepsilon}^2}, \quad \sigma_{\max}^2 = \frac{\mathbf{e}_1^T \mathbf{B}_n^T \mathbf{e}_1}{\|\mathbf{e}_1\|_{\varepsilon}^2}; \quad (40)$$

according to the properties of the Rayleigh quotient:

$$\sigma_{\min}^2 \leq R(x) \leq \sigma_{\max}^2 \quad (41)$$

substituting equation (39) and equation (40) into equation (41) on obtains:

$$\sigma_{\min} \|\mathbf{x}_1\| \leq \|\mathbf{x}_n\| \leq \sigma_{\max} \|\mathbf{x}_2\|. \quad (42)$$

Taking a limit of the inequality (42) one obtains inequality similar to equation (35):

$$\lim_{x \rightarrow \infty} \frac{1}{n-1} \ln \sigma_{\max} \geq \lim_{x \rightarrow \infty} \frac{1}{n-1} \ln \|\mathbf{x}_n\| \geq \lim_{x \rightarrow \infty} \frac{1}{n-1} \ln \sigma_{\min}. \quad (43)$$

From which the Lyapunov's exponents are defined as:

$$\lambda_1 = \lambda_{\max} = \lim_{x \rightarrow \infty} \frac{1}{n-1} \ln \sigma_{\max} \quad (44)$$

and

$$\lambda_2 = \lambda_{\min} = \lim_{x \rightarrow \infty} \frac{1}{n-1} \ln \sigma_{\min}. \quad (45)$$

## Conclusions

The results have shown that firstly the nearly periodic structures in vibration and buckling are capable of respond qualitatively same fashion. The localization of the response is governed by the disorder to coupling strength ratio. Hence to possible types of localization are possible: strong and weak localization. Both of them can be analyzed using either a straightforward eigenvalue approach or the perturbation techniques.

In case of the weak localization disorder dramatically influences the behavior of the structure resulting in the vibration or buckling modes being highly localized in the area of disorder. If the disordered is randomly distributed in the structure the localization still occurs and can be analyzed from the properties of the product of random transition matrices.

All longitudinally nearly periodic structures in buckling exhibit strong localization of modes near the disordered span. Which means that the buckling mode amplitudes will be the largest near the disordered span and then exponentially decay from this span. Such behavior is very easy to observe in the railway rails. If assembled without gaps that account for the thermal expansion, the rails are under high axial compressing load and naturally should lose its stability and buckle along the whole length. However, this is not what is happening in the real structure. Actually, rails buckle in a much localized fashion having high lateral displacements near the disordered span and almost not deforming far from the localization area. This behavior was observed all multi-span beam models with a single disordered span or if the disorder is randomly distributed.

**Аннотация.** Данный обзор посвящен относительно новой в инженерной практике задаче, а именно феномену локализации ответа за счет неупорядоченности конструкции. Целью обзора является дать общее представление об исследованиях в области локализации ответа в механических системах (колебания и потеря устойчивости) с точки зрения единого математического представления через задачу Штурма-Лиувилля. В статье рассматриваются два возможных подхода к описанию влияния неупорядоченности: аналитический подход с использованием динамической матрицы жесткости для анализа неупорядоченной конструкции и метод возбуждения собственного решения упорядоченной конструкции. Показано, что оба подхода адекватно описывают феномен локализации, и с их помощью возможно получить экспоненциальное затухание собственного вектора при удалении от источника неупорядоченности. В части статьи, посвященной локализации форм потери устойчивости, описано подход для анализа локализации ответа в конструкции со случайно распределенной неупорядоченностью. Данный подход базируется на известной теореме Фюрстенберга. Также приведены примеры проявления эффекта локализации ответа в реальных инженерных конструкциях.

**Ключевые слова:** локализация ответа, распространение волн, потеря устойчивости, неупорядоченность, неоднородность, обзор.

**Анотація.** Даний огляд присвячено відносно новій проблемі в інженерній практиці, а саме феномену локалізації відгуку за рахунок невпорядкованості конструкції. Огляд направлений на те, щоб дати загальне уявлення про дослідження в області локалізації відгуку в механічних системах (коливання, втрата стійкості) з точки зору єдиного математичного представлення через задачу Штурма-Ліувалля. В статті розглянуто два можливих підходи до описання впливу невпорядкованості: аналітичний підхід з використанням динамічної матриці жорсткості для аналізу конструкції з

неоднорідностями і метод збудження власного розв'язку впорядкованої конструкції. Показано, що обидва підходи здатні адекватно описати феномен локалізації відгуку, і з їх допомогою можна отримати експоненційне затухання власного вектора при віддаленні від джерела неоднорідності. В частині статті, що присвячена локалізації форм втрати стійкості, описано підхід для аналізу конструкції з випадково розподіленими неоднорідностями. Даний підхід базується на теоремі Фюрстенберга. Також наведені приклади виникнення ефекту локалізації відгуку в реальних інженерних конструкціях.

**Ключові слова:** локалізація відгуку, розповсюдження хвиль, втрата стійкості, невпорядкованість, неоднорідність, огляд.

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