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ON SPACE OF OPEN MAPS OF THE UNIT SEGMENT

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We consider the space of the equivalence classes of continuous open maps defined on a unit segment induced by the Hausdorff metric. The main result provides a description of topology of the components of this space.

Key words: hyperspace, open map, segment, Hilbert space.

Recall that a map of topological spaces is said to be open if the image of every open set is open. It turns out that, in the case of compact Hausdorff spaces, the set of all open maps can be endowed with the natural topology generated by the Vietoris topology on the corresponding hyperspace.

The aim of this note is to investigate the space of open maps of the unit segment. Our main result states that every non-degenerated component of this space is homeomorphic to the separable Hilbert space l^2.

All maps are assumed to be continuous.

1. Space $\Phi(X)$. Let X be a compact Hausdorff space. Following [1] we say that two continuous onto maps $f_i\colon X \to Y_i$, $i = 1, 2$, are equivalent if there exists a homeomorphism $h\colon Y_1 \to Y_2$; here Y_1, Y_2 are compact Hausdorff spaces such that $f_2 = hf_1$. Let $\Phi(X)$ denote the set of equivalence classes and $\Psi(X)$ denote the subset of $\Phi(X)$ consisting of classes of open maps.

Recall that, given a compact Hausdorff space X, we denote by $\exp X$ the set of all nonempty closed subsets of X. The Vietoris topology is the topology whose base consists of the sets of the form

$$\langle V_1, \dots, V_n\rangle = \{A \in \exp X \mid A \subset \bigcup_{i=1}^{n} V_i \text{ and } A \cap V_i \neq \varnothing, \text{ for all } i \in \{1, 2, \dots, n\}\},$$

where $V_1, \dots, V_n$ are open subsets in X, $n \in \mathbb{N}$.

In the case of a compact metric space (X,d), the Vietoris topology can be generated by the Hausdorff metric d_H,

$$d_H(E,F)=\inf\{\varepsilon>0 \mid E\subseteq O_\varepsilon(F) \text{ and } F\subseteq O_\varepsilon(E)\}.$$

Suppose now that all spaces under consideration are compact Hausdorff. We identify every equivalence class $[f]$ of a map $f\colon X\to Y$ with the family $\langle f\rangle=\{f^{-1}(y)\mid y\in Y\}$. The latter is a closed subset of $\exp X$, i.e. an element of $\exp^2 X=\exp(\exp X)$. We endow $\Psi(X)$ with the topology induced from $\exp^2 X$ by this identification [3].

2. The set $\Psi(I)$. By I we denote the unit segment. Recall that a map $f\colon I\to X$ is called a piecewise homeomorphism if there exists a partition $0=a_0<a_1<\dots<a_n=1$ of I such that the embedding $f|_{[a_{i-1},a_i]}\colon [a_{i-1},a_i]\to X$ is a homeomorphism, for all $i=1,2,\dots,n$.

We will use the following result (see [2] for its proof).

Theorem 1. *Every continuous open map from I onto a non-degenerated Hausdorff space X is necessarily a piecewise homeomorphism onto X.*

We denote by K_n, $n\in\mathbb{N}$, the set of all $\langle f\rangle\in\Psi(I)$ such that the following condition holds: there exists a partition $0=t_0<t_1<t_2<\dots<t_n=1$ for which every restriction $f|_{[t_{i-1},t_i]}\colon [t_{i-1},t_i]\to X$ is a homeomorphism, $i=1,2,\dots,n$. Evidently, $K_m\cap K_n=\emptyset$, if $m\neq n$.

One can conclude that $\Psi(I)=\bigcup\limits_{i=1}^{\infty}K_i\cup\{\langle c\rangle\}$, where $c\colon I\to\{*\}$ is the constant map onto a singleton.

Proposition 1. *Every K_n, $n\in\mathbb{N}$, is an open and closed subset of the space $\Phi(I)$.*

Proof. Let $\langle g\rangle\in\Psi(I)\setminus K_n$, then there exists $m\neq n$ such that $\langle g\rangle\in K_m$.

There exists $x\in X$ such that $|g^{-1}(x)|=m>n$. Let $g^{-1}(x)=\{y_1,y_2,\dots y_m\}$, where $y_1<y_2<\dots<y_m$ and $y_i\in I$ for all $i=1,2,\dots m$. Let us choose open disjoint subsets $U_i\subset I$ such, as $y_i\in U_i$ for all $i=1,2,\dots m$. Then $\langle U_1,\dots,U_m\rangle$ is an open subset in $\exp I$ and $\langle g\rangle\cap\langle U_1,\dots,U_m\rangle\neq\emptyset$. From this it follows that $\langle g\rangle\in\langle\exp I,\langle U_1,\dots,U_m\rangle\rangle=W$. The set W is an open subset of the space $\exp^2 I$, i.e., is a neighborhood of an element $\langle g\rangle\in\exp^2 I$.

Let $\langle\widetilde{g}\rangle\in W$. Then $\langle\widetilde{g}\rangle\cap\langle U_1,\dots,U_m\rangle\neq\emptyset$, and there exists $y\in Y$ such that $\widetilde{g}^{-1}(y)\in\langle U_1,\dots,U_m\rangle$. From this it follows that $\widetilde{g}^{-1}(y)\cap U_i\neq\emptyset$, for every $i=1,2,\dots,m$. Inasmuch, as the sets $U_1,U_2\dots,U_m$ are disjoint, we can draw a conclusion that $|\widetilde{g}^{-1}(y)|\geqslant m>n$. Therefore $\langle\widetilde{g}\rangle$ does not belong to the set K_n.

We are going to show that the set $M=\bigcup\limits_{i=1}^{\infty}K_i\cup\{\langle c\rangle\}$ is closed in $\Psi(I)$. Assume the contrary and let $(\langle f_i\rangle)_{i=1}^{\infty}$ be a sequence from M converging to $\langle f\rangle\in K_m$.

Without loss of generality, one may assume that $\langle f_i\rangle\in M\setminus\{\langle c\rangle\}$. Consider $f\colon I\to X$. By Theorem 1, X is homeomorphic to I and let $x_0\in X$ be an interior point of X. Let $f^{-1}(x_0)=\{y_1,y_2,\dots,y_m\}$, where $0<y_1<y_2<\dots<y_m<1$. There exists $a>0$ such that a-neighborhoods of y_i and y_j are disjoint if $i\neq j$. Consider the sequence of neighborhoods $U_i=O_{a/i}(f^{-1}(x_0))$ of the set $f^{-1}(x_0)$.

By the pigeon-hole principle, for every i there exist $f_{k(i)}$ and $x_{k(i)}\in f_{k(i)}(I)$ such that $|f_{k(i)}^{-1}(x_{k(i)})\cap O_{a/i}(y_j)|\geqslant 2$, for some $j\in\{1,2,\dots,m\}$.

Passing, if necessary, to a subsequence, one can assume, without loss of generality, that $|f_{k(i)}^{-1}(x_{k(i)}) \cap O_{a/i}(y_1)| \geqslant 2$ for all $i \in \mathbb{N}$. Denote by w_i and z_i the endpoints of $f_i(I)$. Theorem 1 it follows that $f^{-1}(\{w_i, z_i\}) \cap O_{a/i}(y_i) \neq \emptyset$. Passing to the limit, we obtain

$$f^{-1}(\{w, z\}) \cap \{y_i\} \neq \emptyset$$

where w, z are the endpoints of $X_0 = f(I)$. Note that the limits of the preimages of the endpoints are also endpoints. This follows from the fact that the preimages of the endpoints are precisely the sets that contain 0 or 1. We obtain that $f(y_1)$ is an endpoint of X and this is a contradiction.

3. We say that f preserves segments, if, for every $a, b \in I$, $f([a, b]) = [f(a), f(b)]$. It is easy to observe that in this case the image of every convex subset of I is convex and $f^{-1}\colon f(I) \to I$ also preserves segments.

As usual, $C(I)$ denotes the set of all continuous real functions on I endowed with the uniform convergence topology. We denote by $C_0(I)$ the set of all $f \in C(I)$ such that f preserves segments.

Let $\langle f\rangle \in \Psi(I)$, then $f\colon I \to X$ is a piecewise homeomorphism. Suppose now that $[0, t_1] \subset I$ is a subsegment in I such that $f|_{[0,t_1]}\colon [0, t_1] \to X$ is a homeomorphism. Then there exists a partition $0 = t_0 < t_1 < t_2 < \dots < t_{n-1} = 1$ for which every restriction $f|_{[t_{i-1},t_i]}\colon [t_{i-1}, t_i] \to X$ is a homeomorphism, $i = 1, 2, \dots, n$.

Denote by

$$\mathrm{L} = f^{-1}(f(0)) \cup f^{-1}(f(t_1)) = \{t_0, t_1, t_2, \dots t_{n-1}\},$$

the partition on I, thus $t_0 = 0$ and $t_{n-1} = 1$, $n \in \mathbb{N}$. Consider a map $\phi_i\colon [0, t_1] \to [t_{i-1}, t_i]$, defined by the formula

$$\phi_i(t) = f^{-1}(f(t)) \cap [t_{i-1}, t_i].$$

Let $f(0) = 0$ and consider a map $\varphi_i\colon I \to I$, defined by the formula

$$\varphi_i(x) = \begin{cases} \frac{\phi_i(x\cdot t_1) - t_{i-1}}{t_i - t_{i-1}}, & \text{when } i \text{ is an odd number } i = 2k+1, \\ \frac{t_i - \phi_i(x\cdot t_1)}{t_i - t_{i-1}}, & \text{when } i \text{ is an even number } i = 2k, \end{cases}$$

where $k = \{0, 1, 2, \dots, [n/2]\}$.

So, every $\langle f\rangle \in \Psi(I)$ can be identified with the element

$$(t_0, t_1, t_2, \dots t_{n-1}, \varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)) \in I^n \times C_0^{n-1}(I)$$

where $n \in \mathbb{N}$.

And, vice versa, let $(t_0, t_1, t_2, \dots t_{n-1}; \varphi_1, \varphi_2, \dots \varphi_n)$ be some element of the space $I^n \times C_0(I)^{n-1}$ such that $0 < t_1 < \dots < t_{n-2} < 1$ and $\varphi_i\colon I \to \mathbb{R}$ is continuous maps which preserves segments, such that $\varphi_1(I) = \varphi_2(I) = \dots \varphi_{n-1}(I) = Y \subset \mathbb{R}$. Consider maps $\xi_i\colon [t_{i-1}, t_i] \to Y$ defined by the formula:

$$\xi_i(x) = \begin{cases} \varphi_i\left(\frac{x - t_{i-1}}{t_i - t_{i-1}}\right) & \text{if } i \text{ is an odd number,} \\ \varphi_i\left(\frac{x - t_i}{t_{i-1} - t_i}\right) & \text{if } i \text{ is an even number.} \end{cases}$$

Consider a map $f\colon I \to Y$ defined by the formula

$$f(x) = \begin{cases} \xi_1(x) \text{ if } x \in [0, t_1], \\ \xi_2(x) \text{ if } x \in [t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots \\ \xi_{n-1}(x) \text{ if } x \in [t_{n-1}, 1]. \end{cases}$$

Then f is a continuous open map from I onto $Y \subset \mathbb{R}$ and $\langle f \rangle = \{f^{-1}(f((x)) \mid x \in I\} \in \Psi(I)$.

In this way, we obtain a one-to-one map between the elements of subset K_n in the space $\Psi(I)$ and the elements of some subset $\mathcal{K}_n$ in the space $I^n \times C_0^{n-1}(I)$, for each $n \in \mathbb{N}$. The set K_n is a closed subset in $\Psi(I)$, therefore $\mathcal{K}_n$ is a closed subset in $I^{n+1} \times (C_0(I))^n$ as well.

Proposition 2. *For every $n \in \mathbb{N}$, the set $\mathcal{K}_n$ is convex in $\mathbb{R}^{n+1} \times (C_0(I))^n$.*

Proof. Let $(t_1, t_2, \dots, t_{n-1}; \varphi_1, \varphi_2, \dots, \varphi_n)$ and $(r_1, r_2, \dots, r_{n-1}; \psi_1, \psi_2, \dots, \psi_n)$ be elements of the set $\mathcal{K}_n \subset I^{n+1} \times (C_0(I))^n$. Let s be some element in I. Let us consider the following transformation:

$$\begin{aligned} s(t_1, t_2, \dots, t_{n-1}; \varphi_1, \varphi_2, \dots, \varphi_n) + (1-s)(r_1, r_2, \dots, r_{n-1}; \psi_1, \psi_2, \dots, \psi_n) = \\ = (st_1 + (1-s)r_1, st_2 + (1-s)r_2, \dots, st_{n-1} + (1-s)r_{n-1}; \\ s\varphi_1 + (1-s)\psi_1, s\varphi_2 + (1-s)\psi_2, \dots, s\varphi_n + (1-s)\psi_n) \end{aligned}$$

For all $i = \{1, 2, \dots n-1\}$, we have $0 < t_i < 1$, $0 < r_i < 1$. Take $s \in I$, then $0 < st_i < s$ and $0 < (1-s)r_i < (1-s)$, whence $st_i + (1-s)r_i < 1$. As φ_i, $\psi_i \in C_0(I)$, we see that $a \leqslant \varphi_i(t) \leqslant b$ and $a \leqslant \psi_i(t) \leqslant b$, where $a, b \in \mathbb{R}$. Then

$$sa + (1-s)a \leqslant s\varphi_i(t) + (1-s)\psi_i(t) \leqslant sb + (1-s)b$$

or $a \leqslant s\varphi_i(t) + (1-s)\psi_i(t) \leqslant b$. Whence $s\varphi_i(t) + (1-s)\psi_i(t) \in C_0(I)$.

Therefore,

$$s(t_1, t_2, \dots, t_{n-1}; \varphi_1, \varphi_2, \dots, \varphi_n) + (1-s)(r_1, r_2, \dots, r_{n-1}; \psi_1, \psi_2, \dots, \psi_n) \in \mathcal{K}_n.$$

Then, for all $n \in \mathbb{N}$, the set $\mathcal{K}_n$ is a closed convex subset in the space $\mathbb{I}^{n+1} \times (C_0(I))^n$.

Let us state the Anderson-Kadec Theorem. Recall that a *Fréchet space,* by definition, is a locally convex linear complete metric space.

Theorem 2. *([5]) Every infinite-dimensional Fréchet space is homeomorphic to l^2.*

The following statements are well known (see [5]):

(1) The countable infinite product of non-degenerated separable Banach spaces is homeomorphic to l^2;
(2) $I^n \times l_2 \cong l_2$, where $n \in \mathbb{N}$;
(3) Each convex bounded subset in l^2 is homeomorphic to l^2.

Then, $C(I) \cong l^2$ according to Theorem 2. The set $C_0(I)$ is dense in $C(I)$, therefore $C_0(I) \cong l^2$. From Statements 1 and 2 it follows that $I^{n+1} \times C_0^n(I) \cong l^2$. From Statement 3 we obtain $\mathcal{K}_n \cong l^2$, for every $n \in \mathbb{N}$.

Corollary 1. *The set $\Psi(I) \setminus \{\langle c \rangle\}$ is homeomorphic to the disjoint sum of the singleton and a countable number of copies of a separable Hilbert space l^2.*

Thus, $\Psi(I) \setminus \{\langle c \rangle\} \cong \bigoplus\limits_{i=1}^{\infty} (l^2)_i \oplus \{\langle c \rangle\}$.

4. By a *compactification* of a completely regular topological space X we mean any pair (Y, r) such that Y is a compact space and $r \colon X \to Y$ is an embedding such that $r(X)$ is dense subset in Y. The *one-point compactification* of X is defined to be a space $Y = X \cup \{\infty\}$, where ∞ is an arbitrary point which does not belong to X.

We will prove that space $\Psi(I)$ is not a one-point compactification of $\Psi(I) \setminus \{\langle c \rangle\}$.

Example 1. Let us consider a sequence $\{\langle f_n \rangle \mid n \in \mathbb{N}\}$ of elements of space $\Psi(I)$ such that:

1) $\langle f_n \rangle \in K_n$ for all $n \in \mathbb{N}$;
2) the limit of the sequence is a constant map, $\lim\limits_{i\to\infty} \langle f_n \rangle = \langle c \rangle$.

Let $0 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = 1$, where $t_i = \frac{i}{n}$ and $i = 0, 1, 2, \dots n$. Define $f_n \colon I \twoheadrightarrow I$ by the conditions:

1) $f_n(t_{2i}) = 0$, for $i = 0, \dots [\frac{n}{2}]$;
2) $f_n(t_{2i+1}) = 1$, for $i = 0, \dots \big([\frac{n}{2}] + ((-1)^{n+1} - 1)\frac{1}{2}\big)$;
3) $f_n \colon [t_{i-1}; t_i] \twoheadrightarrow I$ is a linear map, for all $i = 1, 2, \dots n-1$.

Let us estimate the distance

$$d_{HH}(\langle f_n \rangle, \langle c \rangle) = d_{HH}(\{f_n^{-1}(f_n(x)) \mid x \in I\}, \{c^{-1}(c(x)) \mid x \in I\}).$$

According to condition (3), on each segment $[t_{i-1}; t_i]$, where $i = 1, 2, \dots n-1$, there exists a point from $\{f_n^{-1}(f_n(x))\}$, where $0 \leqslant x \leqslant 1$. Thus,

$$d_H(f_n^{-1}(f_n(x)), c^{-1}(c(x))) = d_H(f_n^{-1}(f_n(x)), I) \leqslant \frac{1}{n}.$$

From this it follows:

$$d_{HH}(\langle f_n \rangle, \langle c \rangle) = d_{HH}(\{f_n^{-1}(f_n(x)) \mid x \in I\}, \{I\}) \leqslant \frac{1}{n}.$$

We conclude that $\lim\limits_{i\to\infty} \langle f_n \rangle = \langle c \rangle$.

Example 2. Let us consider a sequence $\{\langle f_n \rangle \mid n \in \mathbb{N}\}$ of elements of the space $\Psi(I)$ such that:

1) $\langle f_n \rangle \in K_n$ for each $n \in \mathbb{N}$;
2) $\langle f_n \rangle$ does not converge to the constant map $\lim\limits_{i\to\infty} \langle f_n \rangle \neq \langle c \rangle$.

Let $0 = t_0 < t_1 = \frac{1}{2} < t_2 < t_3 < t_4 < \dots < t_{n-1} < t_n = 1$, where $t_i = \frac{1}{2} + \frac{i-1}{2n}$ and $i = 0, 1, 2, \dots n+1$. Define $f_n \colon I \twoheadrightarrow I$ by the conditions:

1) $f_n(x) = 2x$, for $x \in [0, \frac{1}{2}]$;
2) $f_n(t_{2i}) = 1$, for $i = 0, \dots [\frac{n}{2}]$;
3) $f_n(t_{2i+1}) = 0$, for $i = 0, \dots \Big([\frac{n}{2}] + \frac{((-1)^{n+1} - 1)}{2}\Big)$;
4) $f_n \colon [t_{i-1}; t_i] \twoheadrightarrow I$ is a linear map, for all $i = 0, 1, 2, \dots n+1$.

Let us estimate the distance

$$d_{HH}(\langle f_n \rangle, \langle c \rangle) = d_{HH}(\{f_n^{-1}(f_n(x)) \mid x \in I\}, \{c^{-1}(c(x)) \mid x \in I\}).$$

According to condition (4), on each segment $[t_{i-1}; t_i]$, where $i = 1, 2, \dots n-1$, there exists a point from $\{f_n^{-1}(f_n(x))\}$, where $\frac{1}{2} \leqslant x \leqslant 1$. Therefore,

$$d_H(f_n^{-1}(f_n(x)), c^{-1}(c(x))) = d_H(f_n^{-1}(f_n(x)), I) \leqslant \frac{1}{n}.$$

However, for $x \in \left[0, \frac{1}{2}\right]$, we obtain $f_n^{-1}(f_n(x)) = f_n^{-1}(2x) = \{x\}$ and then

$$d_H(f_n^{-1}(f_n(x)), c^{-1}(c(x))) = d_H(f_n^{-1}(f_n(x)), I) = \max\{x, x - \frac{1}{2}\} \geqslant \frac{1}{4}.$$

Thus,

$$d_{HH}(\langle f_n \rangle, \langle c \rangle) = d_{HH}(\{f_n^{-1}(f_n(x)) \mid x \in I\}, \{c^{-1}(c(x)) \mid x \in I\}) \geqslant \frac{1}{4},$$

which means that the sequence is not convergent.

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ПРО ПРОСТІР ВІДКРИТИХ ВІДОБРАЖЕНЬ ОДИНИЧНОГО СЕГМЕНТА

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Розглянуто простір класів еквівалентності неперервних відкритих відображень одиничного сегмента у топології, індукованій метрикою Гаусдорфа. Основний результат дає опис топології компонент цього простору.

Ключові слова: гіперпростір, відкрите відображення, сегмент, гільбертів простір.

О ПРОСТРАНСТВЕ ОТКРЫТЫХ ОТОБРАЖЕНИЙ ЕДИНИЧНОГО СЕГМЕНТА

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Рассмотрено пространство классов эквивалентности открытых отображений единичного сегмента в топологии, индуцированной метрикой Хаусдорфа. Основной результат дает описание топологии компонент этого пространства.

Ключевые слова: гипперпространство, единичное отображение, сегмент, гільбертово пространство.

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