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M. Kuzmynykh,
Lviv Polytechnic National University,
Department of Computer-Aided Design

SIMULATION OF EXPERIMENTAL DATA BY STATISTICAL DISTRIBUTIONS OF CAUCHY, MAXWELL AND BOLTZMANN

The paper is simulation and statistical analysis of random data, distributed by the laws is executed of Cauchy or the Cauchy-Lorentz, Gibbs, Maxwell and Boltzmann and mixed on their basis distributions. Computer simulation of statistical mean and dispersion was carried out.

Key words: probability distribution, statistical simulation, statistical mean and dispersion.

МОДЕЛЮВАННЯ ДАНИХ ЕКСПЕРИМЕНТУ СТАТИСТИЧНИМИ РОЗПОДІЛАМИ КОШІ, МАКСВЕЛЛА І БОЛЬЦМАНА

Виконано моделювання і статистичний аналіз випадкових даних, розподілених за законами Коші або Коші–Лоренца, Гіббса, Максвелла і Больцмана, та змішаних розподілів на їх основі. Здійснено комп'ютерне моделювання статистичних середніх і дисперсії.

Ключові слова: ймовірнісний розподіл, статистичне моделювання, статистичні середні та дисперсія.

Introduction

In the physical simulation, the statistics of the ratio of normally distributed random variables $\frac{X}{Y}$ obtained wide application. It is known as distributions of Cauchy*) [1–3]. It is a partial case of important in physics distribution of Breit-Wigner [4–5] with the parameters γ and x_0, which describes the section of resonance scattering with probability density

$$f_{Ca}(x) = \frac{1}{\pi}\frac{\gamma/2}{\gamma^2/4+(x-x_0)^2}, \quad X \in (-\infty,+\infty), \tag{1}$$

that is why it is used to simulate the thermal effects in the coolant channels of the nuclear reactor [6].

Recently, the distribution of Cauchy has been used in the tasks of assessing the consequences of natural disasters [7], accidents of complex technical systems [8], signal processing [9]. This is explained by the fact that the distribution of Cauchy diminishes more slowly than the Gaussian distribution. Therefore, the process or its fragment associated with the peaks of the sharp increase of probability in the random field of Cauchy appears more often than in the Gaussian field [10]. In addition, the processing of data of the systems with Cauchy distributions by classical for normal distribution method of the least squares can lead to serious errors. Therefore, in order to minimize gross errors in the processing of results, so-called robust methods are developed [11].

Cauchy's distribution was opened by Poisson to construct a counterexample for the central limit theorem

The maximum of the density function (1) corresponds to the coordinate $x|_{f=\max} = x_0$, which is determined as a solution of a differential equation $\dfrac{\partial f_{Ca}}{\partial x} = 0$.

The maximum value of the function (1) is:

$$\max_{f_{Ca}} = \left(\sqrt{\frac{2}{\pi}}\right)^2 \frac{1}{\gamma}. \tag{2}$$

The height of the Cauchy's contour is equal to half the maximum at the coordinate points:

$$x|_{\frac{1}{2}\max f_{Ca}} = x_0 \pm \frac{1}{2}\gamma, \tag{2a}$$

consequently

$$\begin{gathered} x|_{\frac{1}{2}\max f_{Ca}} = x_0 \pm \frac{1}{\pi}\frac{\pi}{2}\gamma = x_0 \pm \frac{1}{\pi \max_{f_{Ca}}} \quad \Rightarrow \quad \frac{1}{\pi \max_{f_{Ca}}} = \left(x_0 - x|_{\frac{1}{2}\max f_{Ca}}\right) \quad \Rightarrow \\ \Rightarrow \quad \pi \max_{f_{Ca}} \left(x_0 - x|_{\frac{1}{2}\max f_{Ca}}\right) = 1. \end{gathered} \tag{2b}$$

If we solve the differential equation $\dfrac{\partial^2 f_{Ca}}{\partial x^2} = 0$, then we obtain the coordinates of the points of inflection of curve (1):

$$x_1 = x_0 + \frac{1}{2\sqrt{3}}\gamma \quad \Rightarrow \quad (\pm x_1 \mp x_0) = \frac{1}{\sqrt{3}}\gamma \tag{2c}$$

at the height of the contour

$$f|_{x=x_1} = \frac{3}{2\pi}\frac{1}{\gamma}. \tag{3}$$

Since the Fourier transform for the Cauchy function has a compact form:

$$F_X\left[\frac{1}{\pi}\frac{\gamma/2}{(x-x_0)^2+\gamma^2/4}\right](k) = \exp\left(-2\pi i k x_0 - \pi\gamma|k|\right) \tag{3a}$$

then the Lorentz-Cauchy model is widely used in physics to simulate resonances.

The Cauchy distribution is two-parameter. Normalized ($\sigma_{Ca} = 1$) and centred ($\mu_{Ca} = 0$) function of density has the form:

$$f_{Ca}(x) = \frac{1}{\pi}\frac{d}{dt}\left(acr\tan(x)\right) = \frac{1}{\pi\left[x^2+1\right]} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(1)}{\Gamma(1/2)(x^2+1)}, \quad X \in (-\infty,+\infty) \tag{4}$$

In the unnormalized and non-centered representation, function (4) has the form

$$f_{Ca}(x) = \frac{1}{\pi}\frac{\sigma_{Ca}}{(x-\mu_{Ca})^2+\sigma_{Ca}^2}, \quad X \in (-\infty,+\infty), \tag{4a}$$

where μ_{Ca} and $\sigma_{Ca} > 0$ are the random values. Here μ_{Ca} is local parameter and is not the mean, because Cauchy distributions do not have means. The value $\sigma_{Ca} > 0$ is the scale parameter. Neither is ¾ the standard deviation, because Cauchy distributions do not have variances. Thus, the parameters μ_{Ca}, σ_{Ca} in the distribution (4a) are the parameters of location (mode) and scale (the half-width at half the height of the density curve). The Cauchy distribution is symmetric relative to $x = 0$ (or x_0), so all its odd moments are zero. The integral distribution function has the form:

$$\begin{gathered} F_{Ca}(x) = \int_{-\infty}^{x} f_{Ca}(x)dx = \frac{1}{\pi}\int_{-\infty}^{x}\frac{\sigma_{Ca}}{(x-\mu_{Ca})^2+\sigma_{Ca}^2}dx = \frac{1}{\pi}\int_{-\infty}^{x}\frac{\sigma_{Ca}}{(x-\mu_{Ca})^2+\sigma_{Ca}^2}dx \\ = \frac{1}{2} + \frac{1}{\pi}\arctan(x-\mu_{Ca}). \end{gathered} \tag{4b}$$

Not having the first and higher order of moments, the distribution of Cauchy does not obey the law of large numbers. For the Cauchy distribution, the position parameter m_{Ca} coincides with the mode and the median of distribution, and the scale parameter σ_{Ca} with the width of the contour at half of its height. In the Cauchy distribution, the tails approach the horizontal axis much more slowly than in normal distribution, therefore, it is much more probable for the large and small values and as a consequence - the average does not exist.

The Cauchy distribution is infinitely divided, so independent random variables in the right-hand side of the sum $X = x_1 + x_2 + ... + x_N$ are also distributed according to the Cauchy's law, but with the parameters $\frac{m_{Ca}}{N}$ i $\frac{\sigma_{Ca}}{N}$. It means that the average $\frac{1}{N}\left(x_1 + x_2 + ... + x_N\right)$ has the same parameters of distribution, as each value X_i. For the Cauchy distribution, the Lebesgue integral

$$\int_{-\infty}^{\infty} x^{\nu} f_X(x)dx \qquad (5)$$

is not determined for $\nu \geq 1$, neither mathematical expectation (although the integral of the first moment in the sense of the main value is equal to

$$\lim_{\Theta\to\infty} \int_{-\Theta}^{\Theta} x \frac{1}{\pi} \frac{\sigma_{Ca}}{(x - m_{Ca})^2 + \sigma_{Ca}^2} dx = m_{Ca} , \qquad (5a)$$

nor dispersion, nor the moments of older orders of Cauchy's distribution are not defined. Sometimes it is said that mathematical expectation is not defined, and the variance overlaps. It means that existing sampling values for average meaning

$$\overline{X_{Ca}} = m_{Ca} = \frac{1}{N}\left(x_1 + x_2 + ... + x_N\right) \qquad (5b)$$

of random variables and mean square deviation σ_{Ca} do not coincide and do not have functional connection with the parameters of distribution, because the points are selected irregularly to the point $x = 0$ of the standard distribution (1). Since on the infinity the sub-integral function of the integral (3) changes as $\frac{1}{x}$, then for the Cauchy distribution the integral of the moment of the first and higher orders does not have a finite value [12]. For the Cauchy distribution, the mathematical expectation equals

$$\begin{aligned} m_X &= \lim_{\Delta\to\infty} \frac{1}{\pi} \int_{-\Delta}^{+\Delta} \frac{x \cdot \gamma/2}{\gamma^2/4 + (x - x_0)^2} dx = x_0 + \lim_{\Delta\to\infty} \frac{\gamma}{4\pi} \ln\left(\frac{\gamma^2}{4} + \left(x - x_0\right)^2\right)\Bigg|_{-\Delta}^{+\Delta} = \\ &= x_0 + \lim_{\Delta\to\infty} \frac{\gamma}{4\pi} \ln\left(\frac{\gamma^2/4 + \Delta^2}{\gamma^2/4 + (-\Delta^2)}\right)\Bigg|_{-\Delta}^{+\Delta} = x_0 + 0 \end{aligned} \qquad (6)$$

to the mode x_0, and the dispersion

$$\sigma_X^2 = \frac{\gamma/2}{\pi} \int_{-\infty}^{+\infty} \frac{(x - x_0)^2}{\gamma^2/4 + (x - x_0)^2} dx = \frac{\gamma/2}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\gamma^2}{4} - \frac{1}{\gamma^2/4 + (x - x_0)^2}\right) dx = \infty - 1$$

is unlimited. Therefore parameter γ in fact characterizes the width of the density distribution curve, as a width of a peak at half of the height of FWHM.

For the Cauchy distribution the rule of convolution is realized

$$f_{Ca}(x) \cdot f_{Ca}(y) = f_{Ca}(x + y). \qquad (7)$$

and for independent random variables $X_1, X_2, ..., X_N$, the value $\frac{1}{N}\left(X_1 + X_2 + ... + X_N\right)$ is distributed with the probability density of that X_i. Therefore, in fact, in the experiment there is no accumulation of the mean in the vicinity of zero, which could take place in terms of the law of large numbers and the number of measurements does not affect the accuracy of its measurement. Since the distribution characteristics m_{Ca}, σ_{Ca} for the Cauchy distribution do not have a functional relationship with the parameters of the distribution itself, we will analyze the statistical mean and dispersion of the distribution intermediate between the Gaussian and Cauchy distributions [13–15]. The intermediate distribution represents practical interest in terms of modeling the statistics of signal detectors [16].

The distribution of Cauchy, as the Gaussian distribution, is symmetric with respect to the mode. Therefore, we substantiate the formal rule «3σ »[*)]for the distribution of Cauchy:

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{\sigma_{Ca}}{x^2 + \sigma_{Ca}^2} dx,\ X \in (-\infty, +\infty) \Rightarrow \frac{\pi}{2} erf\left(\frac{x}{\sqrt{2}\sigma}\right) = \tan^{-1}\left(\frac{x}{\sigma_{Ca}}\right) \Rightarrow$$

$$\Rightarrow \begin{cases} x = 1\sigma : \dfrac{\pi}{2} erf\left(\dfrac{1}{\sqrt{2}}\right) = \tan^{-1}\left(\dfrac{\sigma}{\sigma_{Ca}}\right) \Rightarrow 1\sigma_{Ca} = \dfrac{\sigma}{\tan\left[\dfrac{\pi}{2} erf\left(\dfrac{1}{\sqrt{2}}\right)\right]}, \approx 68.2\%, \\ x = 2\sigma : \dfrac{\pi}{2} erf\left(\dfrac{2}{\sqrt{2}}\right) = \tan^{-1}\left(\dfrac{2\sigma}{\sigma_{Ca}}\right) \Rightarrow 2\sigma_{Ca} = \dfrac{\sigma}{\tan\left[\dfrac{\pi}{2} erf\left(\dfrac{2}{\sqrt{2}}\right)\right]}, \approx 94.4\%, \\ x = 3\sigma : \dfrac{\pi}{2} erf\left(\dfrac{3}{\sqrt{2}}\right) = \tan^{-1}\left(\dfrac{3\sigma}{\sigma_{Ca}}\right) \Rightarrow 3\sigma_{Ca} = \dfrac{\sigma}{\tan\left[\dfrac{\pi}{2} erf\left(\dfrac{3}{\sqrt{2}}\right)\right]} \approx 99.8\%. \end{cases} \tag{8}$$

Modelling of the Cauchy-Lorentz optical resonance

In the theory of probabilities, complex random numbers have been developed [17–18]:

$$Z = X + iY \tag{9}$$

The representation of the complex scalar $z = x + i\,y$ is as the real 2×2 matrix $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. It is the analogy between $Z = X + i\,Y$ and $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ leads to a proper definition of the variance of a complex random vector. A complex random variable is similar to a joint distribution of two random variables. A usual probability distribution function of random variable is characterized by the moment functions. The complex random variable is also characterized by moment functions and the first order moment or the mean of complex random variable Z is given by:

$$\overline{Z} = \overline{X} + i\overline{Y} \tag{9a}$$

and dispersion

$$D_Z = \sigma_Z^2 = \overline{\left|Z - \overline{Z}\right|^2} = D_{\mathrm{Re}(Z)} + D_{\mathrm{Im}(Z)}\,. \tag{9b}$$

Besides that

$$\begin{aligned} &E[z \cdot z^*] = E[x^2 + y^2] = E[x^2] + E[y^2] \\ &aber \\ &E[z^2] = E[x^2 + 2ixy - y^2] = E[x^2] - E[y^2] + 2iE[xy] \end{aligned} \tag{9c}$$

The rule "3σ " is actually developed and is valid for normal distribution

The method of complex variables is convenient to use for modelling optical resonance of Cauchy-Lorenz, which in optics is called optical dispersion. Optical dispersion is associated with electronic processes that are accompanied by the propagation of electromagnetic wave in crystals. In phenomenological form, the phenomenon of dielectric resonance is modelled using Maxwell's dispersion equations [19–22], which for non-magnetic substance ($\mu \approx 1$*) in the complex form is written as follows:*

$$\tilde{n}^2(\omega) = \tilde{\varepsilon}(\omega) = \tilde{\varepsilon}_1(\omega) + i\tilde{\varepsilon}_2(\omega), \tag{10}$$

where $\tilde{n}(\omega)$ – is a complex function that binds to one another real refraction $n(\omega)$ and absorption $\chi(\omega)$ rates.

From the thermodynamic considerations [24], in the optics the connection between the indices $n(\omega)$ and $\chi(\omega)$ is justified in the form:

$$\tilde{n}(\omega) = n(\omega) - i\chi(\omega). \tag{10a}$$

then

$$\tilde{\varepsilon}(\omega) = (n - i\chi)^2 = \varepsilon_1 + i\varepsilon_2 \Rightarrow \begin{cases} \varepsilon_1 = n^2 - \chi^2 \\ \varepsilon_2 = -2n\chi \end{cases}, \tag{10b}$$

from there

$$\begin{cases} n = \sqrt{\dfrac{1}{2}\left(\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)}, \\ \chi = \sqrt{\dfrac{1}{2}\left(-\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)}. \end{cases} \tag{10c}$$

Here we restrict ourselves to the law of dispersion $\varepsilon(\omega)$ in a one-electron approximation. This approach is widely used in literature to simulate resonance transitions [23]. The essence of one-electron approximation is that in view of the interaction of the electric field of a light wave with the electron system of a crystal, into account is taken the so-called optical electron, which is weakly connected with the nucleus.

In the one-electron approximation, the dielectric permittivity function is formulated in the complex form [23]:

$$(\varepsilon(\omega) - \varepsilon_0) = \frac{4\pi\alpha\omega_0^2}{(\omega_0^2 - \omega^2) - i\omega\gamma}. \tag{11}$$

where its real and imaginary parts are:

$$\begin{aligned} \mathrm{Re}\,\tilde{\varepsilon} &= \varepsilon_1 = \varepsilon_0 + \frac{4\pi\alpha\omega_0^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}, \\ \mathrm{Im}\,\tilde{\varepsilon} &= \varepsilon_2 = \frac{4\pi\alpha\omega_0^2\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}. \end{aligned} \tag{12}$$

Here ε_0 is the dielectric permittivity of the medium at low frequencies or the so-called background dielectric constant, $4\pi\alpha$ is the force of the oscillator of the transition to the electronic state with the resonant frequency ω_0, γ - the fade parameter [24].

Figure 1 shows graphs of functions (12). In the region of the resonance frequency ω_0 *of the function* $n(\omega,\gamma)$ *and* $\chi(\omega,\gamma)$*, there are sharply expressed nonlinear dispersions. The laws of the Cauchy-Lorentz dielectric resonance lead to the appearance in the region of optical dispersion of wavelength intervals with normal* $\dfrac{dn}{d\omega} > 0$ *and abnormal* $\dfrac{dn}{d\omega} < 0$ *process velocity. The area with an abnormal* *dispersion combines branches* $\frac{dn}{d\omega}>0$ *and is concentrated between the positions of the boundary frequencies of the transverse* ω_0 *and longitudinal* ω_L *resonance excitations of the crystal:*

$$\omega_L^2 = \omega_0^2 \left(1 + \frac{4\pi\alpha}{\varepsilon_0}\right). \tag{13}$$

This is a well-known Sachs-Ladyne-Taylor formula [23].

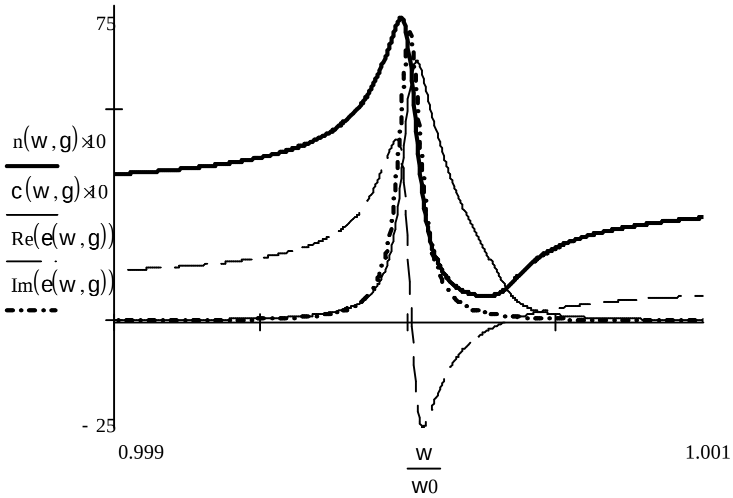
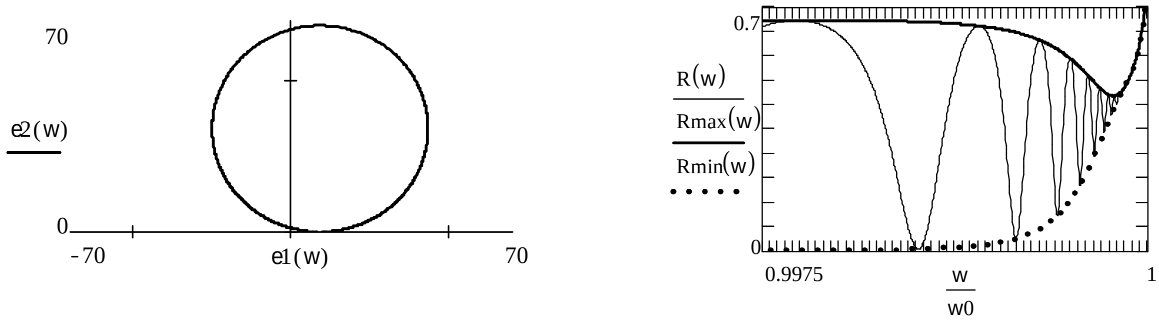


Fig. 1. Graphs of functions

The width of the longitudinal-transverse splitting $\omega_{LT} = \omega_L - \omega_T$ *depends on the amount of fading in the system of resonant excitations* γ *and in this area the absolute refractive index may be less than one, and the phase velocity of the light exceeds the corresponding for the vacuum* c_0. *With disappearing fading* $\gamma \to 0$ *the outline* $\chi(\omega)$ *obtains a delta-like kind.*

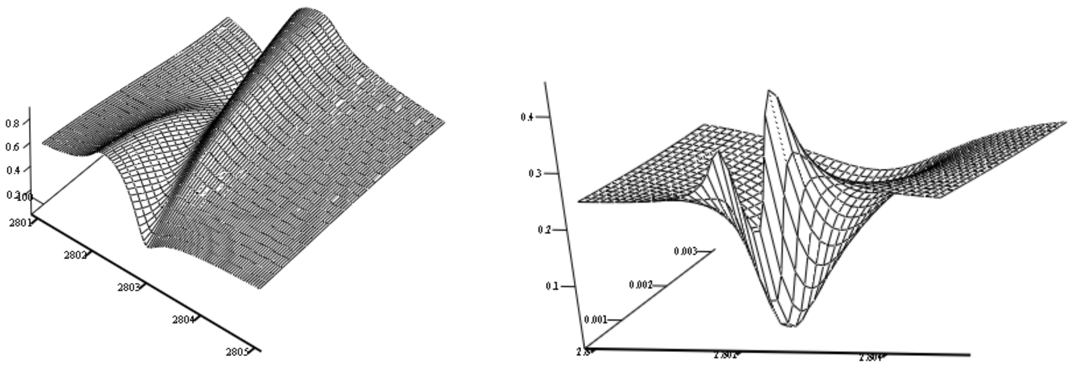
The dispersions of functions (11) are shown in Figure 1, and the hodograph of the complex function $\tilde{\varepsilon}(\omega)$ *has the form of a circle (Fig. 2 (a)), which collapses with increasing* γ. *This kind of hodograph reflects the fact that in the model of optical transitions, one mechanism of relaxation of energy (motion resistance of an electron is proportional to its velocity) is established. If they are more, then the form of the contour of the hodograph is complicated and converted into polymodal one.*



a *b*

Fig. 2. a) the hodograph of the complex function; b) the resonance region the oscillatory characteristics of the interference processes of light waves substantially decay

The Cauchy or Cauchy-Lorentz resonance greatly affects the dispersion of electronic processes in this spectral region. As it is evident from Fig. 2 (b), in this resonance region the oscillatory characteristics *of the interference processes of light waves substantially decay, since in this area the density of dissipative losses by the light wave of energy is significantly increased. The modulation of the spectrum by the Cauchy-Lorentz resonance is manifested in various regions of the spectrum of electron excitations of the crystal, including those associated with the peculiarities of its atomic-molecular structure. In Fig. 3 (a), it is depicted in the region of resonant excitation of optical phonons and excitons [25–29].*



a *b*

Fig. 3. The modulation of the spectrum by the Cauchy-Lorentz resonance in the region of resonant excitation of optical phonons and excitons

The similarity of surfaces for various resonant excitations is due to the fact that the basis of the justification of the model of the optical oscillator is the same mechanism of dissipation of its energy, namely, the proportionality of the energy loss with the light wave of the velocity of the optical electron in the resonance flaring by its own electric field. Other aspects of simulation of electromagnetic processes in the material environment are fairly well-covered in literature, for example, in [30–31].

Statistical divisions of Gibbs, Maxwell and Boltzmann

Micro and macro systems and their statistical implementation. In physics, for the simulation of the energy of a macrosystem or microsystem, depending on whether it is investigated in terms of the behavior of individual degrees of freedom, or the effects that are caused by their significant amount are studied, the Gibbs distribution is used. In the first case, the object of the research is the microsystems of individual molecules, atoms, microparticles, and in the second one – macrosystems in the form of physical bodies, consisting of a large number of them. The statistical method of studying physical objects is based on the concepts that are characteristic of the microcosm and proceed from the causal relationship between macro and micro phenomena; therefore, one can explain the laws of thermodynamics based on the laws of mechanics (classical or quantum) that regulate the micro phenomena and indicate the possibility of deviations (fluctuations) from the laws of thermodynamics.

The state of the physical system is given if all of its physical characteristics are determined. There is a great number of such characteristics (physical quantities), but there is always a minimum set of independent parameters, having set which it is possible to determine all other quantities. These are the so-called state parameters. In classical mechanics, such parameters are, for example, coordinates and impulses of all degrees of freedom of the physical system, that is, it is postulated that all physical quantities are functions of coordinates and impulses. It is this energy that is in Hamilton's function.

In quantum mechanics, the state parameters are quantum numbers – one for each degree of freedom. For example, it can be the main quantum number n, magnetic m and spin s for one dot particle that determine the wave function of the particle system. Therefore, in order to obtain macroscopic conclusions about the system from its microscopic properties, it is necessary to have a connection between microscopic and macroscopic states. Since one macro-state of the system corresponds to many of its micro-states, then by specifying a macro-state, we outline the set of available micro-states, the quantity of which is called statistical weight. Moreover, the realization of this or that micro-state can only be judged probabilistic, if this probabilistic information is possible to obtain from the data on micro-state. If such information about the micro-states cannot be obtained, this means that they are all equally probable, as is the case in isolated systems under fixed conditions. Consequently, there is a probabilistic relationship between the macro- and micro- states of the system, which precisely determines the dependence of its macroscopic properties on microscopic ones.

The probability of a micro-state is one of the main concepts of the statistical method of describing the system. In connection with the objective necessity of this concept, we introduce the notion of a statistical ensemble as an analogue of the concept of an elementary event in the mathematical probability theory. Then, by definition, the probability of finding a system in this state will be equal to the ratio of the number of systems in this micro-state, to the total number of micro-states.

The statistical ensemble is just selected as a set of objects for the study of physical systems. The statistical ensemble is a sufficiently large set of macroscopically accurate copies of the physical system, having all the macro parameters set equally. The probability of all available micro-states determines the distribution of the system by microstate-statistical distribution. Moreover, as the experience proves, the macro process in the system is irreversible and it is due to the fact that it is the transition of the system from less probable states to more probable. Macro process stops when, under given external conditions, the system has taken the most probable state, that is, has thus acquired an equilibrium state. The equilibrium state is described by the smallest number of system parameters. The transition of the system to a non-equilibrium state is unlikely.

If the system is isolated from external conditions (the so-called thermostat), then all its micro-states are equally probable and the macro-state determines one statistical ensemble – micro-canonical, as established in physics, by the intrinsic energy and volume. If the system is not isolated, that is, it interacts with the thermostat (temperature and volume are given), then the characteristic of the probability of the system being in a certain state is the so-called statistical weight.

Gibbs distribution. The classical micro-state of the system is determined by the generalized impulses P and particle coordinates Q. Moreover, if in the quantum approach the set of values of the parameters of micro-states is discrete, then in the classical case it is continuous. In classical physics, the universal distribution of Gibbs is known as a function of the values of generalized coordinates P, Q:

$$H = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + U(x,y,z)\,, \tag{14}$$

which is used to model the energy of the system. Since the component parts (bricks) of the system - the particles move and can interact with each other. The total energy of the system is given as a function of the impulses p_i and the coordinates q_i of the particles in the form of the Hamilton function[1*)]. For one particle, the Hamilton function reduces to the sum of kinetic and potential energies, therefore, in the Cartesian system, the Gibbs distribution is written in the form

$$f(p,q) = C\exp\left(-\frac{1}{2m\sigma_M^2}\left(p_X^2 + p_Y^2 + p_Z^2\right) - \frac{1}{kT}U(x,y,z)\right). \tag{15}$$

If we are interested only in the distribution of velocities, then by integrating the Gibbs distribution in all coordinates, we get the well-known Maxwell distribution:

$$f(p_X, p_Y, p_Z) = C_p\exp\left(-\frac{1}{2m\sigma_M^2}\left(p_X^2 + p_Y^2 + p_Z^2\right)\right). \tag{16}$$

It is possible to introduce the surface of a stable, complete kinetic energy of n particles in a three-dimensional Euclidean space:

$$\frac{1}{2m}\left(p_X^2 + p_Y^2 + p_Z^2\right) = \frac{3}{2}nkT\,. \tag{17}$$

In quantum mechanics, Hamilton's operator $\hat{H}$ corresponds to Hamilton's function.

Then, if each velocity projection has the same energy $\frac{1}{2}kT$, then at $n \to \infty$ the distribution of each of the components goes to the Maxwellian.

Distribution of Boltzmann. If we are interested in the particle distribution by coordinates, then by integrating the Gibbs distribution at all velocities, we obtain the well-known Boltzmann distribution:

$$f(x,y,z) = C_q \exp\left(-\frac{U(x,y,z)}{2\sigma_M^2}\right). \tag{18}$$

The formula (1.13.4) includes both the potential energy of the particles in the external field, and the potential energy of the interaction between the particles. From (18) it follows that relative value X, Y, Z if the supply of potential energy is true:

$$U(x,y,z) = U(x) + U(y) + U(z). \tag{19}$$

Then, for the Boltzmann distribution function, the factorization condition is fulfilled

$$f(x,y,z) = C_q \exp\left(-\frac{U(x,y,z)}{2\sigma_M^2}\right) = \sqrt[3]{C_q}\, e^{-\frac{U(x)}{2\sigma_M^2}} \cdot \sqrt[3]{C_q}\, e^{-\frac{U(y)}{2\sigma_M^2}} \cdot \sqrt[3]{C_q}\, e^{-\frac{U(z)}{2\sigma_M^2}}. \tag{20}$$

Maxwell distribution. Let $u_1, u_2, ..., u_k$ are independent standard (normalized and centred) distributed relative value. The density of probability of this system is determined by the formula $f_u\left(\vec{u}\right) = \frac{1}{(2\pi)^{\frac{k}{2}}} \exp\left(-\frac{1}{2}\sum_{i=1}^{k} u_i^2\right)$. If we create a new relative value in the form of $\chi = \sqrt{\sum_{i=1}^{k} u_i^2}$, where the number of items k is the number of degrees of freedom, then we can construct a density function with a distribution $f(\chi) = C \cdot \chi^{k-1} \exp\left(-\frac{1}{2}\chi^2\right)$, where the constant coefficient C is determined from the condition of rate setting

$$C\int_0^\infty f(\chi)d\chi = 1 \quad \Rightarrow \quad C = \frac{1}{\int_0^\infty f(\chi)d\chi} = \frac{1}{\int_0^\infty \chi^{k-1}\exp\left(-\frac{1}{2}\chi^2\right)d\chi} = \frac{2}{2^{k/2}\Gamma(k/2)}. \tag{21}$$

At $k = 2$ the well-known Rayleigh distribution law is obtained $f(\chi) = \chi \exp\left(-\chi^2/2\right)$, and at $k = 3$ we obtain Maxwell's law of distribution $f(\chi) = \frac{2}{\sqrt{\pi}}\chi^2 \exp\left(-\chi^2/2\right)$. Thus, Maxwell's distribution is directly related to χ^2 distribution of the relation

$$f_X(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} \exp\left(-x/2\right)U(x), \tag{21a}$$

if to put in formula (20) $n = 3$.

Maxwell's distribution is used to calculate the probability of a random point falling into the sphere and a layer between two concentric spheres in geometry, where the scattering center coincides with the center of the sphere and the center of the spherical layer. Other generalized laws of Maxwell's distribution can be found in [32].

The density of the Maxwell distribution is described by the function:

$$f(x) = \begin{cases} \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} x^2 \exp\left(-x^2/2\sigma^2\right), & x > 0, \\ 0, & x \le 0, \end{cases} \tag{22}$$

where $\sigma > 0$. For it, the central moments are:

$$M(X)=2\sqrt{\frac{2}{\pi}}\sigma_X,\ D(X)=(3-8/\pi)\sigma_X^2,\ \mu_3=2\sigma_X^2\left(\frac{16}{\pi}-5\right)\sqrt{\frac{2}{\pi}},\ \mu_4=\sigma_X^4\left(15-\frac{8}{\pi}\right). \tag{23}$$

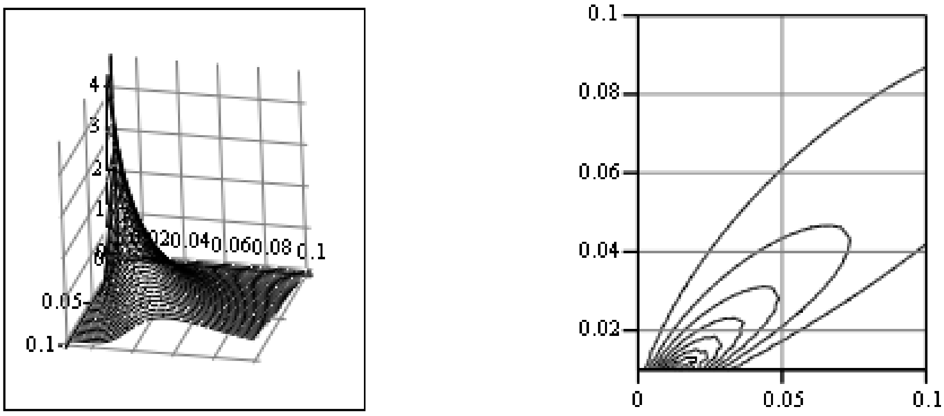
The cumulative distribution is described by the function

$$F(x)=\frac{\sqrt{2}}{\sigma_X^3\sqrt{\pi}}\int_0^x x^2\exp\left(-x^2/2\sigma^2\right)dx=\frac{2}{\sqrt{\pi}}\int_0^{z^2/2\sigma_X^2}\sqrt{z}\exp\left(-z\right)dz. \tag{24}$$

If each of the independent Cartesian projections of the velocity vector is distributed by the Gaussian law

$$f(\vartheta_{X_i})=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\vartheta_{X_i}^2}{2\sigma^2}\right),\quad i=1,2,3;\ X_1=X,\ X_2=Y,\ X_3=Z, \tag{25}$$

then the function of the density distribution of the speed module has the form



fV fV

Fig. 4. The function of the density distribution of the speed module

$$f(\vartheta)=4\pi\sqrt{\left(\frac{m_0}{2\pi kT}\right)^3}\vartheta^2\exp\left(-\frac{m_0\vartheta^2}{2kT}\right), \tag{26}$$

and its surface and corresponding lines of the level are depicted in fig. 4. The basis of the justification of function (26) is the equation of probability balance

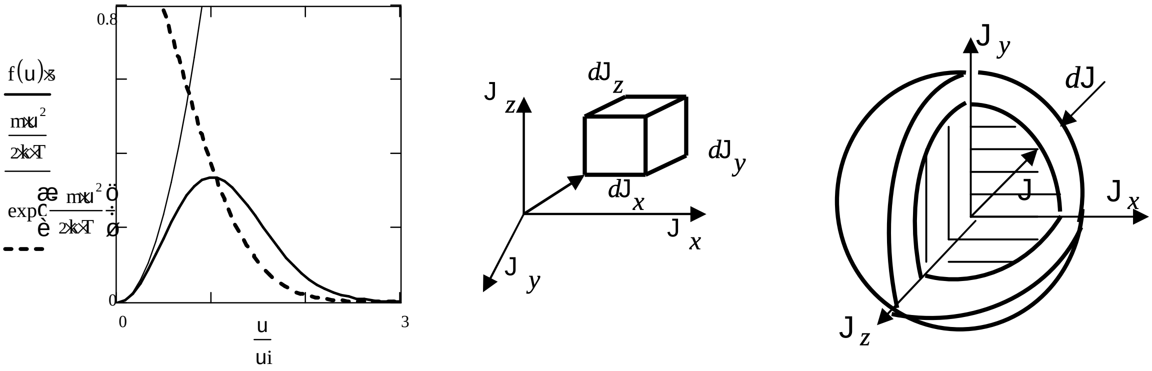
$$f(\vartheta)d\vartheta = f(\vartheta_X,\vartheta_Y,\vartheta_Z)d\vartheta_X d\vartheta_Y d\vartheta_Z \quad \Rightarrow \quad f(\vartheta)=f(\vartheta_X,\vartheta_Y,\vartheta_Z)\frac{d\vartheta_X\cdot d\vartheta_Y\cdot d\vartheta_Z}{d\vartheta} \tag{27}$$

of the equation for the density of the joint distribution of three independent relative values

$$\begin{aligned} f(\vartheta_X,\vartheta_Y,\vartheta_Z) &= f(\vartheta_X)\cdot f(\vartheta_Y)\cdot f(\vartheta_Z)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\vartheta_X^2}{2\sigma^2}}\cdot\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\vartheta_Y^2}{2\sigma^2}}\cdot\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\vartheta_Z^2}{2\sigma^2}}= \\ &=\frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^3}\exp\left(-\frac{1}{2\sigma^2}(\vartheta_X^2+\vartheta_Y^2+\vartheta_Z^2)\right)=\frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^3}\exp\left(-\frac{\vartheta^2}{2\sigma^2}\right)\end{aligned} \tag{28}$$

and the equality of the product $d\vartheta_X\cdot d\vartheta_Y\cdot d\vartheta_Z$ as the volume of the elementary cube (Fig. 5 (b)), of the volume of the elementary spherical ring in Fig. 5 (c) $4\pi d\vartheta^2 d\vartheta$, from which we obtain that

$$\frac{d\vartheta_X\cdot d\vartheta_Y\cdot d\vartheta_Z}{d\vartheta}=4\pi\vartheta^2. \tag{29}$$



a *b* *c*

Fig. 5. Maxwell's functions (a); the volume of the elementary cube (b); the volume of the elementary spherical ring (c)

Maxwell's functions represent the product of two functions of a quadratic ϑ^2 and exponents $\exp\left(-\frac{\vartheta^2}{2\sigma^2}\right)$, the graphs of which are depicted in Fig. 5 (a). By comparing the functions (26) and (28) to each other, we obtain that the mean square deviation equals

$$\sigma = \sqrt{\frac{kT}{m_0}}\,. \tag{30}$$

To the extremum of the function (26) (mode of distribution) corresponds the most probable value of speed ϑ_i. Having solved the differential equation for an extremum

$$\frac{d}{d\vartheta} f(\vartheta)\Big|_{\vartheta=\vartheta_i} = 2\vartheta_i - \frac{\vartheta_i^3}{\sigma^2} = 0 \tag{31}$$

we get that to the Maxwell's distribution mode corresponds the most probable value of speed

$$\vartheta_i = \sqrt{2}\sigma\,. \tag{32}$$

We substantiate the standard valuation for the function (32):

$$C_M \frac{4\pi}{\sqrt{(2\pi\sigma^2)^3}} \int_0^\infty \vartheta^2 \exp\left(-\frac{\vartheta^2}{2\sigma^2}\right) d\vartheta = 1 \quad \Rightarrow \quad C_M = \frac{\sigma^3\sqrt{2\pi}}{2\int_0^\infty \vartheta^2 \exp\left(-\frac{\vartheta^2}{2\sigma^2}\right) d\vartheta} = \tag{33}$$

$$= \frac{2\sigma^3\sqrt{2\pi}}{\sqrt{\pi}\left(2\sigma^3\right)\sqrt{2}} = 1.$$

Then the average velocity $\bar{\vartheta}$ will be equal

$$\bar{V} = \frac{4\pi C_M}{\sqrt{(2\pi\sigma^2)^3}} \int_0^\infty \vartheta^3 \exp\left(-\frac{\vartheta^2}{2\sigma^2}\right) d\vartheta = \frac{4\pi}{2\pi\sigma^3\sqrt{2\pi}} \frac{1}{2}\left(4\sigma^4\right) = 2\sqrt{\frac{2}{\pi}}\sigma = \frac{2}{\sqrt{\pi}}\vartheta_i = 1.128\vartheta_i\,, \tag{34}$$

which is in agreement with (23) taking into account (32). The average of the square of velocity is equal to

$$\sqrt{\overline{V^2}} = \frac{4\pi C_M}{\sqrt{(2\pi\sigma^2)^3}} \int_0^\infty \vartheta^4 \exp\left(-\frac{\vartheta^2}{2\sigma^2}\right) d\vartheta = \frac{4\pi}{2\pi\sigma^3\sqrt{2\pi}} \frac{1}{2} \Gamma\left(\frac{5}{2}\right)\left(2\sigma^2\right)^{5/2} = 3\sigma^2 = \frac{3}{2}\vartheta_i^2\,. \tag{35}$$

Therefore, the dispersion of the velocity of independent components $\bar{\vartheta}$ i $\sqrt{\vartheta^2}$:

$$\Delta_V = \overline{V^2} - \bar{V}^2 = \vartheta_i^2\left(3/2 - 2/\pi\right) = 0.227\vartheta_i^2\,, \tag{36}$$

from where the mean square deviation is

$$\sigma_V = \sqrt{D_V} = 0.48\vartheta_i\,. \tag{37}$$

After integrating the function (24), the finite expression for the Maxwell integral distribution function acquires the form [1.23]

$$F(\vartheta) = erf\left(\frac{\vartheta}{\sqrt{2}\sigma}\right) - \sqrt{\frac{2}{\pi}}\frac{\vartheta}{\sigma}\exp\left(-\frac{\vartheta^2}{2\sigma^2}\right). \quad (38)$$

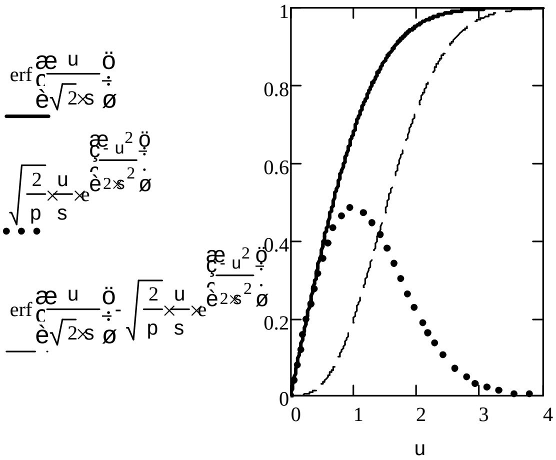


Fig. 6. The graph of the function (38)

The graph of the function (38) is shown in Fig. 6. The first item in (38) is a function of errors, and the second item is a differential Rayleigh function, reduced to a dimensionless form.

Summary

The paper is simulation and statistical analysis of random data, distributed by the laws is executed of Cauchy or the Cauchy-Lorentz, Gibbs, Maxwell and Boltzmann and mixed on their basis distributions. Computer simulation of statistical mean and dispersion was carried out.

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