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# DEVISING AN APPROACH TO THE GEOMETRIC MODELING OF RAILROAD TRACKS ALONG CURVILINEAR SECTIONS

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***This paper proposes an approach to arranging curvilinear sections of the railroad track, which involves the replacement of two transitional ones and a circular curve with one curvilinear section. Modeling this section implies the construction of a curve in the plan and profile, the joint consideration of which forms the spatial appearance of the curve, which ensures a smooth transition between the rectilinear rails. The need for such transitions is due to the terrain, the need to bypass settlements, and the presence of geological, geographical, and other obstacles that occur when laying railroads. A curvilinear section is modeled in the plan using a curve that is represented in natural parameterization under the law of curvature distribution in the form of a fifth-power polynomial. At the same time, at the start and end points of the section, the curvature and its derivative accept zero values. The outer rail elevation retraction (modeling in the profile) is performed using the curve built, whose sections are also represented in natural parameterization with the dependences of the curvature distribution on the arc length in the form of fourth-power polynomials. At the docking point of the sections, the third order of smoothness is ensured, which implies the equality of values of the functions, their derivatives, the curvature, and a derivative of the curvature from the length of its natural arc. Measures have been proposed to ensure the predefined track gauge retraction. The application of the proposed approach to model a railroad track along the curvilinear section with a variable-radius curve could make it possible to achieve a favorable curvature distribution, a smooth elevation retraction of the outer rail and the track gauge. That would consequently improve the safety of rolling stock running along the curvilinear section of the track, reduce the lateral and vertical efforts that predetermine the wear of rails and wheelsets***

***Keywords: railroad track, curvilinear section, outer rail elevation retraction, track gauge***

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## 1. Introduction

Rail transport is an integral part of the material production of any country. It provides passenger, freight, and mixed cargo and passenger transportation. One of the most important issues related to the operation of railroad transport is the introduction of measures to improve traffic safety. Special requirements are put forward to the geometry of a railroad track along curvilinear sections, which are used to change the direction of a train.

Arranging a rail track along the railroad path on curvilinear sections has a series of features that are associated with the specificity in the interaction between the track and rolling stock. The existence of curvilinear sections along a railroad track is determined by the terrain, the need to bypass settlements, various geological and geographical obstacles.

At first glance, it seems that the issue of geometric modeling of the railroad track on curvilinear sections has been mainly resolved. However, there are a series of factors that require devising new approaches to the geometric modeling of a railroad track along curvilinear sections. One of such key factors is a significant increase in the speed of trains and an increase in their mass, which causes an increase in the force interaction between the rails and the rolling stock of locomotives and cars. This leads to increased wear of rails and wheelsets, which can be the cause of emergencies and even environmental and man-made disasters.

A significant increase in the speed of trains puts forward stricter requirements for the design of curvilinear sections of the track, which should ensure the smoothness of change in the curvature at those points where sections of the track connect with rails of different curvatures, primarily rectilinear ones. This is achieved by the equality of the angles of inclination of the tangent, curvature, its derivative at the docking points of the rectilinear and curvilinear sections, as well as the equality to zero of the derivative from the curvature at the start and end points of a curvilinear section.

The above thus predetermines the relevance of research into devising new approaches to the geometric modeling of curvilinear sections of a railroad track. The results from addressing this issue are important for the transport industry of any country.

## 2. Literature review and problem statement

The issue of improving curvilinear sections along a railroad track has challenged railroad specialists, perhaps, from the first years of the railroad's existence. According to [1], back in 1854, problems were identified in Austria when the train switched from a circular section to a straight line. That led to the need to place between these sections of the track of the so-called transition curve. The purpose of those activities was to reduce the effects of abrupt curvature change. However, at that time, relatively low speeds of trains and their small mass made it possible to put up with certain difficulties, which were caused by a jump-like change in the curvature at places of transition from rectilinear rails to circular ones. Gradually, with the increase in the speed of trains, the arrangement of transition curves was paid more attention.

One of the first equations used in the description of transition curves was the cubic parabola, which was explained by the simplicity of the mathematical notation. Later, as indicated in work [1], to describe the curvilinear sections of a railroad track, spirals, clothoids (Cornu spirals), bisinusoids, sinusoidal spirals, radioids, the lemniscates of Bernoulli, as well as other curves, were applied.

The issue of the arrangement of curvilinear sections of a railroad track is still relevant for railroad engineers. It was addressed in numerous papers, monographs, theses, conference materials, etc. The efforts of scientists from different countries were aimed at improving the properties of transition curves. Thus, in work [2], a transition curve is proposed, which is part of the curvilinear section of a railroad track, whose mathematical base is the spirals. The spiral section is arranged between two straight rails providing, alas, only for the second order of smoothness. In [3], it is noted that the spiral curves are not characterized by mathematical features and extremes of curvature. They are used to smooth paths in places of abrupt change in curvature where the speed of a moving object increases. The use of spiral curves is more effective when ensuring a smooth transition between the straight line and the circle, or between two circles.

The authors of [4] evaluated all available types of transition curves of railroad tracks and proposed a variant of a transition curve based on the properties of the clothoid and the simplicity of the description of the cubic parabola. It should be added that the clothoid is still the most common curve used in the modeling of transitional curves of railroad tracks. However, clothoids obey only the linear law of curvature distribution. As regards the cubic parabola, it can be noted that the analytical expressions used in the modeling of transition curves are based on the expansion of the original expressions into a Taylor series, containing, as a rule, two terms and rejecting the terms of higher powers. That does not contribute to the high accuracy of results obtained. Cubic parabola was also used in [5] to simulate transition curves, although these curves are advisable to use on short sections of the track.

Work [6] notes the drawback of a transition curve in the form of a clothoid, due to the linear law of curvature distribution. That leads to bends on the curvature sections that occur in the initial and final regions of the curvilinear section of the track. Therefore, the authors of the cited work propose using sinusoidal curves to solve the problem.

Paper [7] proposes transition curves that are described by polynomials of the 9th and 11th powers. It is shown that the conventional modeling of transition curves by parabolas of the third power does not make it possible, from the point of view of the dynamics of the movement of bogies, to obtain optimal conditions for the passage of trains. Solving the optimization problem determines the polynomials that simulate transition curves, which provide comfortable conditions for passengers. Polynomial curves are also used in [8] to construct a curvilinear section applying two transition curves and one circular curve. The same curves are used to construct vertical arches of transition curves [9]. Paper [10] shows that polynomial curves can be used to model *S*-shaped transition curves. *S*-similarity is achieved by applying polynomial curves with the powers of 5 and 7. However, on the plots of the distribution of curvature, zones with a sharp increase in curvature are marked.

Paper [11] proposes modeling of transitional curves between the rectilinear and circular sections of the track using curves that obey the nonlinear law of curvature distribution. It should be noted that the author of the cited work adheres to the traditional approach to form a curvilinear section of the track. To simulate transition curves, work [12] employs the smoothing *B*-splines, which have certain advantages, in terms of smoothness, at the final points of the curvilinear sections of the track over the clothoid curves. However, in the cited work, *B*-splines are adapted to a smooth transition between two circular curves of an actual section of the railroad track.

Rational Bezier curves of the second order made it possible, by using only one such curve, to construct a transition curve with acceptable characteristics [13]. However, given all the attractiveness of Bezier curves, their use is characterized by certain issues associated with the location of the vertices of the characteristic Bezier broken line. When applying rational Bezier curves, the problem is the so-called weights, which significantly affect the nature of the passage of simulated curves.

An example of the use of sinusoidal hyperbolic functions in the construction of transition curves is given in [14]. It is noted that the sinusoidal hyperbolic functions lead to shortened transition curves, which, in some cases, is undesirable, especially at high speeds of trains.

Our review of the scientific literature reveals that the issue of the movement of trains along the curvilinear sections of a railroad track is challenging for railroad engineers and requires devising new approaches to their geometric representation.

## 3. The aim and objectives of the study

The purpose of this work is to devise an approach to the geometric modeling of the curvilinear sections of a railroad track, which are arranged between rectilinear rails located at some angle. This would improve the conditions of train movement, in particular, reduce the impact of a jump-like change in curvature on the movement of locomotives and cars, decrease the wear of rails and wheelsets, as well as improve comfortable conditions for passengers.

To accomplish the aim, the following tasks have been set:

– to build an outer rail of the curvilinear section of a track in the plan;

– to build an outer rail elevation retraction;

– to build an inner rail of the track while ensuring the predefined track gauge at outer rail elevation retraction.

## 4. The study materials and methods

We have devised a method for the geometric modeling of a curvilinear section of the railroad track using a curve represented in natural parameterization and obeying the predefined law of curvature distribution. Conventionally, a curvilinear section of the track contains two transition curves and one circular curve. It is proposed, instead of the traditional three curves to describe the curvilinear section of the track in the plan, that is, in the horizontal plane of projections, to apply one curve. Fig. 1 shows the proposed curvilinear section of track 1, arranged between two straight rails 2 and 3, whose boundary points 4 and 5 are highlighted by dashes.

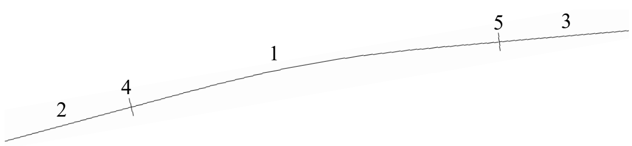


Fig. 1. The proposed structure of the curvilinear section of a railroad track: 1 — curvilinear section; 2, 3 — rectilinear rails; 4, 5 — boundary points

Curve 1 is represented in natural parameterization with the distribution of curvature from the length of its natural arc in the form of a fifth-power polynomial:

$$k(s) = as^5 + bs^4 + cs^3 + ds^2 + es + f, \quad (1)$$

and under the following boundary conditions:

$$k\big|_{\substack{s=0\\s=S}} = 0; \ \frac{dk}{ds}\bigg|_{\substack{s=0\\s=S}} = 0, \quad (2)$$

where $s$ is a parameter in the form of the natural curve length; $S$ is the length of the arc of the modeled curve.

The derivative from the curvature is determined by the following dependence:

$$k' = \frac{dk(s)}{ds} = 5as^4 + 4bs^3 + 3cs^2 + ds + e. \quad (3)$$

The angle of inclination of the tangent to the modeled curve is derived from the expression:

$$\varphi_1 = \varphi_0 + \frac{aS^6}{6} + \frac{bS^5}{5} + \frac{cS^4}{4} + \frac{dS^3}{3} + \frac{eS^2}{2} + fS, \quad (4)$$

where $\varphi_0$ and $\varphi_1$ are the angles of inclination of the tangents to the modeled line at the start and end points of the curvilinear section, respectively.

The boundary conditions (2), applied to the law of curvature distribution (1) and derivative (3), predetermine the zero values for coefficients $f$ and $e$.

Knowing the angles $\varphi_0$ and $\varphi_1$, we use expression (4) to find dependences for the coefficients $b$, $c$, and $d$:

$$b = 30\frac{(\varphi_1 - \varphi_0)}{S^5} - 2.5aS;$$

$$c = -3aS^2 - 2bS;$$

$$d = 2aS^3 + bS^2.$$

The coefficient $a$ and the length of the arc $S$ are determined in the process of solving the optimization problem associated with the coordination of the intermediately obtained endpoint of the transition curve at the specified endpoint of the curvilinear section.

The objective function accepted in the optimization problem is the following expression:

$$\delta = \sqrt{(\bar{x} - x)^2 + (\bar{y} - y)^2},$$

where $\bar{x}$, $\bar{y}$ are the coordinates of the intermediately obtained point with the current values for the desired unknown parameters, and $x$, $y$ are the coordinates of the specified endpoint of the simulated curve.

To solve the optimization problem, a highly effective algorithm proposed in [15] is employed.

The coordinates of the curvilinear section of a railroad track, which is described by the curve represented in natural parameterization, are determined from the following equations:

$$x = x_0 + \int_0^S \cos\varphi(s)\,ds;$$

$$y = y_0 + \int_0^S \sin\varphi(s)\,ds.$$

Since the coordinates of the endpoint of the transition curve are known, these equations are sufficient to determine the two unknowns necessary to model the desired curve.

When rolling stock runs along the curvilinear sections of a railroad track, there are centrifugal forces that seek to move the bogie outside the curve. This can only happen in exceptional cases. Centrifugal forces lead to the redistribution of pressure on the rails of both threads and the overload of the outer thread, which contributes to increased lateral wear of the rails and ridges of the wheels. To avoid these phenomena, an outer rail is elevated over the inner one. This ensures the same wheel pressure on the outer and inner rail threads, and, therefore, the same vertical wear of both rails.

The outer rail elevation retraction is arranged in curvilinear sections where the radius of the curvature takes a value of 4,000 m or less. The size of an outer rail elevation retraction is defined by the regulatory documents of the countries where the railroad is laid. Once at the high speed of the train the estimated value of an outer rail elevation retraction exceeds the normative value, the rated value is still accepted while the speed of the train is limited along a given curvilinear section. Usually, an outer rail elevation retraction is arranged by raising it by increasing the thickness of the ballast under the outer rail thread.

The outer rail elevation retraction shall be simulated in the *xoz* vertical plane using a curve composed of two sections. The curvature of each section is subject to the dependence on the length of its natural arc in the form of a fourth-power polynomial. At the docking point of the sections, the third-order smoothness would be ensured, which implies the equality of values of the functions, their derivatives, the curvature, and derivatives from the curvature along the length of the arc.

The starting, middle, and end points of the outer rail elevation retraction curve are denoted by numbers 0, 1, 2, respectively.

To describe the curvature of the left-hand half of the section of the outer rail of a railroad track, a fourth-power polynomial is used:

$$k(s) = a_1s^4 + b_1s^3 + c_1s^2 + d_1s + e_1, \quad (5)$$

where $a_1$, $b_1$, $c_1$, $d_1$ and $e_1$ are the unknown coefficients to be determined in the process of modeling a curve.

The curvature distribution (5) matches the following dependences of the derivative and the angle of inclination of the tangent to the curve on the length of the arc of the circle:

$$k_1' = 4a_1s^3 + 3b_1s^2 + 2c_1s + d_1; \tag{6}$$

$$\psi(s) = \psi_0 + \frac{a_1s^5}{5} + \frac{b_1s^4}{4} + \frac{c_1s^3}{3} + \frac{d_1s^2}{2} + e_1s, \tag{7}$$

where $\psi_0$ is the angle of inclination of the tangent to the curve at the starting point.

Taking into consideration the boundary conditions (2), at $s=0$, we obtain $d_1=0$, $e_1=0$.

At the starting point of this curve, the angle of $\psi$ is zero. It accepts the same value at the middle point of the curve since it is the point of the maximum elevation of an outer rail.

To construct a section of the outer rail elevation retraction curve, it is necessary to determine the coefficients $a_1$ and $b_1$ and the length of the arc of the curved section $S_1$.

Dependence (7) is used to find an expression for the coefficient $a_1$, which takes the form:

$$a_1 = \Phi_1 - \frac{5b_1}{4S_1} - \frac{5c_1}{3S_1^2}, \tag{8}$$

where

$$\Phi_1 = \frac{5(\psi_1 - \psi_0)}{S_1^5}.$$

In the considered case, $\Phi_1=0$, since the angles $\psi$ at the starting and middle points are zero. However, due to certain geographical features of the track, the angles $\psi$ at the start and end points can take different values.

An expression for the coefficient $a_1$ can be found in another way, taking advantage of the fact that, at $s=S_1$, the derivative from the curve curvature is zero. Then:

$$a_1 = -\frac{3b_1}{4S_1} - \frac{c_1}{2S_1^2}. \tag{9}$$

Equating (8) and (9), we find the expression for coefficient $b_1$:

$$b_1 = 2S_1\Phi_1 - \frac{7c_1}{3S_1}.$$

Thus, as a result of the transformations performed, the number of unknowns is reduced to two. These are the coefficient $c_1$ and the arc length $S_1$.

The parametric equations written for point 1 of the outer rail elevation retraction arc, taking into consideration the law of curvature distribution (5), take the form:

$$x_1 = x_0 + \int_0^{S_1} \cos\left[\psi_0 + \frac{a_1s^5}{5} + \frac{b_1s^4}{4} + \frac{c_1s^3}{3}\right] \mathrm{d}s;$$

$$z_1 = z_0 + \int_0^{S} \sin\left[\psi_0 + \frac{a_1s^4}{4} + \frac{b_1s^3}{3} + \frac{c_1s^3}{3}\right] \mathrm{d}s,$$

where $x_0$, $z_0$ are the coordinates of the starting point of the outer rail elevation retraction arc.

The available data are sufficient to organize the computational process of bringing the intermediately obtained point to point 1. At the values of $c_1$ and $S_1$, selected by the algorithm proposed in [15], we calculate the coefficient $b_1$, and then the coefficient $a_1$.

Upon completion of the simulation of the left-hand part of the curvilinear section of the track, the values of the curvature and the derivative from it are calculated at $s=S_1$, which are hereafter denoted as $K$ and $K'$. These differential characteristics of the curve are necessary to ensure the third order of smoothness when docking the left and right sections of the constructed curves under consideration.

The right-hand sections of the curves are also modeled using the law of curvature distribution in the form of a fourth-power polynomial:

$$k(s) = a_2s^4 + b_2s^3 + c_2s^2 + d_2s + e_2,$$

where $a_2$, $b_2$, $c_2$, $d_2$, and $e_2$ are the unknown coefficients to be determined in the process of modeling a curve.

The dependences of the derivative and the angle of inclination of the tangent on the length of the arc are similar to expressions (6) and (7) with the corresponding change in the indices at unknown coefficients, and, instead of the angle $\psi_0$, $\psi_1$ is used.

When modeling the right section of the curve, it is assumed that at the docking point there is a new countdown of the length of the curve arc. Under these circumstances, we obtain $e_2=K$, $d_2=K'$. Thus, the problem of constructing the right section is reduced to finding the three coefficients $a_2$, $b_2$, $c_2$, and the length of the arc $S_2$. The coefficient $c_2$ and the length of the arc $S_2$ are derived by minimizing the deviation of the intermediately obtained endpoint of the curve from the specified point 2. In this case, the coefficients $b_2$ and $a_2$ are calculated from the following expressions:

$$b_2 = 2S_2\Phi_2 - \frac{7c_2}{3S_2} - 4{,}5\frac{K'}{S_2^2} - 10\frac{K}{S_2^3};$$

$$a_2 = -\frac{b_2}{S_2} - \frac{c_2}{S_2^2} - \frac{K'}{S_2^3} - \frac{K}{S_2^4},$$

where

$$\Phi_2 = \frac{5(\psi_2 - \psi_1)}{S_2^5}.$$

The railroad track gauge is determined by fitting the bogies of rolling stock into the curves of the predefined radius. Preventing a bogie's jammed fitting causes the minimum permissible track gauge.

The maximum track gauge is determined subject to reliable prevention of failure of rolling stock wheels inside the track.

The rated size of the track gauge between the inner edges of the rail heads is established by the relevant regulatory documents of the country of construction of a railroad track.

It was determined which track gauge can be obtained if the inner rail is modeled by the initial values of the angles of inclination of the tangents at the start and end points of the curvilinear section of the track used in the modeling of the outer rail. Due to the fact that within the curves the radius of the inner rail thread is slightly smaller than the radius of the outer rail thread, the length of the inner thread is less than the length of the outer rail thread. Taking into consideration the values of the track gauge along the rectilinear section of the track, the coordinates for the starting and end points of the curvilinear section of the inner rail were determined. Thus, the initial data are sufficient to simulate the inner rail under the law of curvature distribution in the form of a fifth-power polynomial as it was done when simulating the outer rail of the track in the plan.

Next, the distance between the outer and inner rails was found, which was measured along the perpendicular drawn to the outer rail at the point under consideration, and the intersection point of this perpendicular with the inner rail. To construct the perpendicular equation at the considered point, the orthogonal coordinates $x$ and $y$ at this point and the curvilinear coordinate $s$ were used. Expression (4) is used to find the angle of inclination of the tangent. The coordinates of the intersection point of the perpendicular with the inner rail are determined by the numerical method, by solving a nonlinear equation with one variable, which is the length $s$ of the arc of the inner rail. Knowing the length $s$ of the arc, we calculated the orthogonal coordinates for the intersection point of the perpendicular with the inner rail, and then the length of the perpendicular located between the outer and inner rails.

Once the designer of a railroad track is not happy with the results obtained, then the adjustment of the track gauge can be performed by modeling the inner rail with a curve consisting of two sections. That is, to apply the proposed approach to describe the outer rail elevation retraction.

To do this, one needs to additionally determine the coordinates of the point at which the docking of the sections of the curve is carried out. This approach to modeling the inner rail of the railroad track should be considered more promising since it provides for a possibility to ensure the predetermined railroad track gauge.

## 5. Results of studying the geometric modeling of the curvilinear sections of a railroad track

### 5. 1. Modeling the outer rail of the curvilinear section of a track in the plan

Test examples of the curvilinear sections of a track in the plan are shown in Fig. 2. The curves were modeled with the same values of the angles of inclination of tangents at the starting points of curvilinear sections; at the endpoints, the angles of inclination of the tangents accepted the same, but negative, values. For curve 1, the angle of inclination of the tangent at the starting point was 30°, for curve 2 – 25°, and, finally, for curve 3 – 20°. The angles of the inclination of tangents at the starting and end points may take different values by modulo. The digit-based designation of the curves in the figures below is meant to further harmonize them with the plots of the distribution of the curvature and its derivatives.

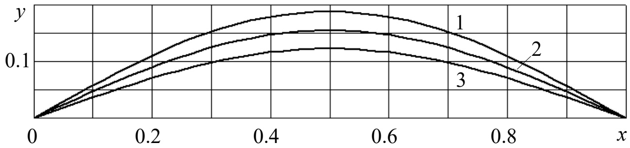


Fig. 2. Railroad track curvilinear sections: 1 – $\varphi_0=30°$; 2 – $\varphi_0=25°$; 3 – $\varphi_0=20°$

Fig. 3 shows the plots of the distribution of the curvature of the curve and its derivative from the relative length of the natural arc.

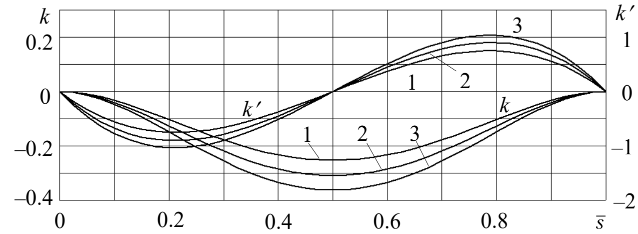


Fig. 3. Distribution plots of the curvature $k$ and its derivative $k'$: 1 – $\varphi_0=30°$; 2 – $\varphi_0=25°$; 3 – $\varphi_0=20°$

The curvature and derivative at the start and end points of the curvilinear section accept zero values corresponding to the boundary conditions (2) of its modeling.

### 5. 2. Modeling the outer rail elevation retraction

Fig. 4 shows the plots of the outer rail elevation retraction curves. These data are illustrative in nature since they are constructed with a significant excess of the values of lifting curves at the middle point. The aim was to separate the built lines from each other since, at the real height of the outer rail elevation, the graphic dependences would be barely higher than the $x$ axis.

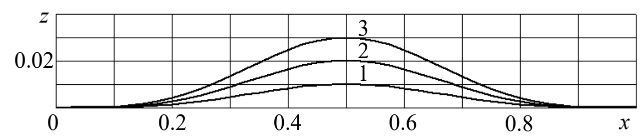


Fig. 4. Railroad track's outer rail elevation curves: 1 – $\varphi_0=30°$; 2 – $\varphi_0=25°$; 3 – $\varphi_0=20°$

The distribution plots of the curvature and its derivative depending on the arc length for the test curves shown above are demonstrated in Fig. 5, 6, respectively.

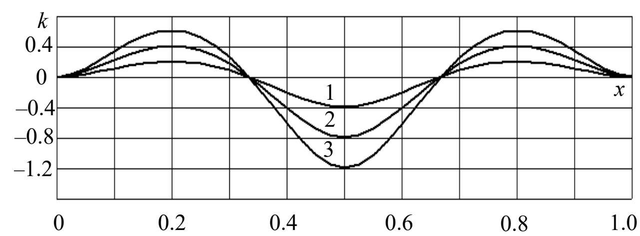


Fig. 5. Outer rails' elevation curvature distribution curves: 1 – $\varphi_0=30°$; 2 – $\varphi_0=25°$; 3 – $\varphi_0=20°$

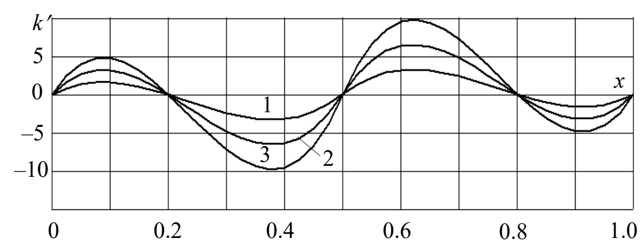


Fig. 6. Distribution curves of the derivative depending on the outer rails' elevation curvature: 1 – $\varphi_0=30°$; 2 – $\varphi_0=25°$; 3 – $\varphi_0=20°$

All the curves shown in Fig. 4–6 demonstrate a smooth character; at the starting and end points of the curvilinear section – zero values for both the curvature and derivative corresponding to the boundary conditions imposed on them (2). In addition, curvature derivative plots cross the abscissa axis at points where the curvature has extremes.

### 5. 3. Modeling the inner rail of the track while ensuring the predefined track gauge at an outer rail elevation retraction

Fig. 7 shows at a significantly increased scale the graphic dependences that correspond to the relative increase in the railroad track gauge. At the start and end points of the curvilinear section, the deviations δ of the track gauge accept zero values. When approaching the middle of the section, the relative distance between the outer and inner rails increases. The numbering of curves in Fig. 7 coincides with the designations of the curves shown in Fig. 2.

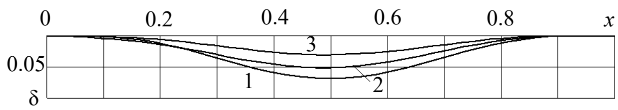


Fig. 7. Plots of a relative increase in the width of a railroad track: 1 – $\varphi_0=30°$; 2 – $\varphi_0=25°$; 3 – $\varphi_0=20°$

The results of calculating an increase in the track gauge by representing the inner rail of the track using the constructed curve are shown in Fig. 8.

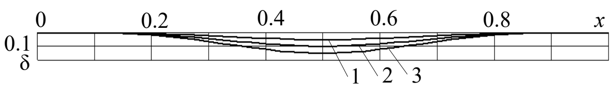


Fig. 8. Plots of the relative increase in the railroad track gauge when representing the inner rail by the constructed curve: 1 – δ=0.05; 2 – δ=0.10; 3 – δ=0.15

When simulating the inner rail, the following relative values of increasing the track gauge were used: 0.05; 0.10; 0.15. These values, of course, exceed the actual values of increasing the track gauge. The increase in the track gauge is predetermined by the desire to separate the curves from each other and distance them from the *x* axis.

## 6. Discussion of results of studying the geometric modeling of the curvilinear sections of railroad tracks

Our positive results from the geometric modeling of the curvilinear sections of a railroad track are due to the validity of mathematical statements, which are based on the positions of analytical and differential geometry and numerical methods. Algorithmizing the methods that correspond to the tasks of our study has made it possible to develop a workable computer code. All research tasks have been practically implemented, which is confirmed by the above graphic results.

The advantage of the proposed approach to model the curvilinear sections of a railroad track is that the law of the distribution of its curvature was accepted as the basis for constructing the curve. Based on this law, the coordinates of the points of the simulated curve are calculated. This is the difference between the proposed approach and those approaches reported in [1–14] that consider the construction of a railroad track's curvilinear sections. In those papers, a curve is constructed, and then the distribution of the curvature and its derivative are determined. With an unfavorable distribution of these differential characteristics of the applied curve, the track designer is deprived of the degrees of freedom of influence. Namely, the distribution of the curvature and its derivative is the most important indicator of the quality of the curvilinear section of a railroad track.

Examples of the simulated railroad tracks in the plan and profile are shown in Fig. 2, 4. The curves shown in Fig. 2 were modeled using the law of curvature distribution in the form of a fifth-power polynomial (1). The curves shown in Fig. 4 were represented by the constructed curve, each section of which was represented under the law of curvature distribution in the form of a fourth-power polynomial (5). The sections were connected while ensuring the third order of smoothness. The curves obtained in the calculations were accompanied by plots of the distribution of the curvature and the first derivative (Fig. 2, 4). They are smooth in nature and confirm compliance with the applied boundary conditions (2).

The increase in the track gauge along the curvilinear section of a track is clearly confirmed by the corresponding plots shown in Fig. 7, 8.

Thus, it would suffice for a designer of the curvilinear section of a railroad track to set the initial data and obtain the coordinates for the track rail lines.

The current study could be advanced by solving the problem in which the boundary conditions (2) are supplemented with equality to zero of the second derivative from the curvature depending on the length of the natural arc at the initial and end points of the curvilinear section. That is, the following conditions must be met:

$$\left.\frac{d^2k}{ds^2}\right|_{\substack{s=0\\ s=S}}=0.$$

Although, at the same time, it will be necessary to increase the power of the polynomial, which should describe the dependence of the curvature on the length of the arc when simulating the outer rail in the plan.

When using numerical methods, including optimization, it is very important to set the initial values, optimized parameters. In all the variants considered in this work, two parameters are optimized, one of which is an unknown coefficient and the second is the length of the curve. The initial value of the length of the arc can be more or less accurately determined. For the first approximation, one can take the distance between the starting and end points of the modeled line. The initial value of an unknown coefficient is somewhat more difficult to define. However, after two or three trial calculations, it can be determined, albeit approximately, I mean the value of this coefficient. Everything else will be made by the optimization algorithm proposed in [15]. For greater confidence in determining the unknown coefficient, calculations were carried out from two or three initial points. As evidenced by the experience of calculating the values of optimized parameters, they differed by such a small value, which is unattainable with the practical bending of the rails.

Based on the proposed measures for the geometric modeling of the curvilinear sections of railroad tracks, computer code was developed in the Fortran PowerStation programming environment. When accessing this code, calculations are performed related to determining the coordinates of the points of the simulated lines in the plan and profile, as well as the track gauge. The code uses subroutines for determining the curvature of curves and derivatives, a subroutine for solving transcendental equations and minimizing the objective function. The developed code, in addition to numerical results, which are the coordinates of the points of the simulated lines, makes it possible to visualize the lines on the computer monitor screen. Graphic data are a clear confirmation of the feasibility of the proposed approach to the geometric modeling of the curvilinear sections of a railroad track.

## 7. Conclusions

1. We have proposed a method for the geometric modeling of the outer rail of a curvilinear section of the railroad track in the plan (horizontal plane), which is based on the natural representation of curves and the law of curvature distribution in the form of a polynomial of the fifth power. To find five unknown coefficients of this polynomial and the length of the arc of the curve, zero boundary conditions are used that reduce the number of unknown coefficients. Joint consideration of the dependences of the curvature, its derivative, and the angle of inclination of the tangent defines expressions for the three unknown polynomial coefficients. The fifth coefficient and the length of the arc are determined by minimizing the deviation of the intermediately obtained endpoint from the curvilinear section of the railroad track specified with the initial data.

2. Our practical calculations have proven that the devised method for the geometric modeling of the railroad track outer rail elevation retraction can be represented by a curve composed of two sections. Each section is described by a curve in natural parameterization using the law of curvature distribution in the form of a polynomial of the fourth power. At the docking point of the sections, the third order of smoothness is ensured. Using boundary conditions, dependences of the angles of inclination of tangents, and the conditions that provide, at the docking point, for the third order of smoothness, the number of unknowns for each section is reduced to two – one coefficient and the length of the arc of the section. The presence of coordinates of the endpoints of the sections makes it possible to solve the problem.

3. It has been shown that the minimum permissible track gauge, predetermined by the increase in the outer rail, is advisable to provide for by modeling the inner rail with two sections of the parametric curve under the laws of curvature distribution, which are described by the polynomials of the fourth power. Under these conditions, it is possible to directly indicate the desired track gauge.

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