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ON A REDUCTION OF A LINEAR HOMOGENEOUS DIFFERENTIAL SYSTEM WITH OSCILLATING COEFFICIENTS TO A SYSTEM WITH SLOWLY VARYING COEFFICIENTS IN RESONANCE CASE

Щоголев С. А. Про зведення лінійної однорідної диференціальної системи з коливними коефіцієнтами до системи з повільно змінними коефіцієнтами у резонансному випадку. Для лінійноЇ однорідної диференціальної системи, коефіцієнти якої зображувані у вигляді абсолютно та рівномірно збіжних рядів Фур'є з повільно змінними коефіцієнтами та частотою, отримано умови існування лінійного перетворення аналогічної структури, що зводить цю систему до системи з повільно змінними коефіцієнтами у нерезонансному випадку на асимптотично великому проміжку зміни незалежної змінної за умови наявності деяких резонансних співвідношень.
Ключові слова: диференціальний, повільно змінний, ряди Фур'є.

Щёголев С. А. О сведении линейной однородной системы с осциллирующими коэффициентами к системе с медленно меняющимися коэффициентами в резонансном случае. Для линейной однородной дифференциальной системы, коэффициенты которой представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены условия существования линейного преобразования аналогичной структуры, приводящего эту систему к системе с медленно меняющимися коэффициентами в нерезонансном случае на асимптотически большом промежутке изменения независимой переменной при условии наличия некоторых резонансных соотношений.
Ключевые слова: дифференциальный, медленно меняющийся, ряды Фурье.

Shchogolev S. On a reduction of a linear homogeneous differential system with oscillating coefficients to a system with slowly-varying coefficients in resonance case. For the linear homogeneous differential system, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, conditions of existence of the linear transformation with coefficients of similar structure, this system leads to a system with slowly-varying coefficients in a noresonance case in asymptotical large interval of independent variable, are obtained subject to the availability of certain resonance relations.
Key words: differential, slowly varying, Fourier series.

Introduction. This paper continues [1], where considered the next linear system of the differential equations:

$$\frac{dx}{dt} = (\Lambda(t,\varepsilon) + \varepsilon A(t,\varepsilon) + \mu P(t,\varepsilon,\theta))x, \tag{1}$$

$x = \mathrm{colon}(x_1, \ldots, x_N)$, elements of matrix $\Lambda(t,\varepsilon) = \mathrm{diag}(\lambda_1(t,\varepsilon), \ldots, \lambda_N(t,\varepsilon))$ and matrix $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ in some sense are slowly varying functions, and elements of matrix $P(t,\varepsilon,\theta) = (p_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency:

$$p_{jk}(t,\varepsilon,\theta)) = \sum_{n=-\infty}^{\infty} p_{jk,n}(t,\varepsilon)e^{in\theta(t,\varepsilon)},$$

$d\theta/dt = \varphi(t,\varepsilon)$, μ – small parameter. In [1] are obtained the conditions of existence of the transformation of kind

$$x = \Phi(t,\varepsilon,\theta,\mu)z, \tag{2}$$

where structure of matrix Φ is analogous of structure of matrix P, which reducing the system (1) to kind:

$$\frac{dz}{dt} = \widetilde{A}(t,\varepsilon,\mu)z, \tag{3}$$

where elements of matrix $\widetilde{A}$ are slowly varying. It was assumed that $\lambda_j(t,\varepsilon) - \lambda_k(t,\varepsilon) = i\omega_{jk}(t,\varepsilon)$, $\omega_{jk}(t,\varepsilon) \in \mathbf{R}$ and $|\omega_{jk}(t,\varepsilon) - \nu\varphi(t,\varepsilon)| \geqslant \gamma > 0 \ \forall\ \nu \in \mathbf{Z}$ $((j-k)^2 + \nu^2 > 0)$. Means considered the noresonance case. In this paper the analogous problem are solved by the condidtion, that $\lambda_j(t,\varepsilon) = ir_j\varphi(t,\varepsilon)$ $(r_1,\ldots,r_N \in \mathbf{Z})$. Thus we have the case of the resonance between eigenfrequencies $\omega_{jk}(t,\varepsilon)$ and external frequency $\varphi(t,\varepsilon)$.

In what follows we use the denotions and the definitions from [1].

Main Results

1. Statement of the Problem.

Consider the following differential system:

$$\frac{dx}{dt} = (A(t,\varepsilon) + \mu P(t,\varepsilon,\theta(t,\varepsilon)))x, \tag{4}$$

$t,\varepsilon \in G(\varepsilon_0) = \{t,\varepsilon : \ 0 < \varepsilon < \varepsilon_0,\ -L\varepsilon^{-1} \leqslant t \leqslant L\varepsilon^{-1},\ 0 < L < +\infty\ \}$, $x = \operatorname{colon}(x_1,...,x_N)$, $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$, $a_{jk} \in S(m,\varepsilon_0)$, $P(t,\varepsilon,\theta(t,\varepsilon)) = (p_{jk}(t,\varepsilon,\theta(t,\varepsilon)))_{j,k=\overline{1,N}}$, $p_{jk} \in F(m,l,\varepsilon_0,\theta)$, $m,l \in \mathbf{N}\cup\{0\}$, $m \geqslant 2$ (the definitions and basic properties of classes $S(m,\varepsilon_0)$ and $F(m,l,\varepsilon_0,\theta)$ are given in [1]), $\mu \in (0,\mu_0) \subset \mathbf{R}^+$. We assume that eigenvalues of matrix $A(t,\varepsilon)$ have a kind $ir_j\varphi(t,\varepsilon)$, $r_j \in \mathbf{N}$, and $r_j \neq r_k$ if $j \neq k$ $(j,k = \overline{1,N})$, where $\varphi(t,\varepsilon)$ – is a function, which figured in definition of class $F(m,l,\varepsilon,\theta)$.

We study the problem of the conditions of existence of the transformation of kind (2), where $\Phi \in F(m^*,l,\varepsilon^*,\theta)$, which reducing the system (4) to kind (3), where $\widetilde{A}(t,\varepsilon,\mu) \in S(m^*,\varepsilon^*)$ $(m^* \leqslant m,\ \varepsilon^* \leqslant \varepsilon_0)$.

2. Auxiliary results.

Lemma 1. $\exists\ \mu_2 \in (0,\mu_0)$, $\exists\ \varepsilon_1 \in (0,\varepsilon_0)$ *such that* $\forall\ \mu \in (0,\mu_2)$, $\forall\ \varepsilon \in (0,\varepsilon_1)$ *exists the transformation of kind*

$$x = Z(t,\varepsilon,\theta,\mu)y, \tag{5}$$

where $Z \in F(m-1,l,\varepsilon_1,\theta)$, *which reducing the system (4) to the form:*

$$\frac{dy}{dt} = (\varepsilon\Lambda_1(t,\varepsilon) + \varepsilon^2 H_1(t,\varepsilon,\theta) + \mu U(t,\varepsilon,\mu) + \mu\varepsilon V(t,\varepsilon,\theta,\mu))y, \tag{6}$$

where Λ_1 – *diagonal matrix with elements from* $S(m-1,\varepsilon_1)$, H_1 – *square matrix with elements* from $F(m-2,l,\varepsilon_1,\theta)$, U – *square matrix with elements from* $S(m,\varepsilon_1)$, V – *square matrix with elements from* $F(m-1,l,\varepsilon_1,\theta)$.

Proof. Based on the condition $r_j \neq r_k$ $(j \neq k)$ and properties of function $\varphi(t,\varepsilon)$, we can state that exists matrix $L(t,\varepsilon) \in S(m\varepsilon_0)$ such that $\inf\limits_{G(\varepsilon_0)} |\det L(t,\varepsilon)| > 0$, and $L^{-1}AL = i\varphi(t,\varepsilon)\Lambda$, where $\Lambda = \mathrm{diag}(r_1,\ldots,r_N)$. We make in the system (4) the substitution:

$$x = L(t,\varepsilon)x^1, \tag{7}$$

where x^1 – the new unknown N-dimensional vector. We obtain:

$$\frac{dx^1}{dt} = (i\varphi(t,\varepsilon)\Lambda + \varepsilon H(t,\varepsilon) + \mu P_1(t,\varepsilon,\theta))x^1, \tag{8}$$

where $H = -\frac{1}{\varepsilon}L^{-1}\frac{dL}{dt} \in S(m-1,\varepsilon_0)$, $P_1 = L^{-1}PL \in F(m,l,\varepsilon_0,\theta)$. We increase the order of smallness with respect parameter ε of the off-diagonal slowly varying elements in matrix of system (8). For this purpose in system (8) we make the substitution:

$$x^1 = (E - \varepsilon Q(t,\varepsilon))x^2, \tag{9}$$

where $(Q)_{jj} \equiv 0$, $(Q)_{jk} = \frac{(H)_{jk}}{i(r_j - r_k)\varphi}$ $(j \neq k)$. Then we have: $i\varphi(\Lambda Q - Q\Lambda) = H - \Lambda_1$, where $\Lambda_1 = \mathrm{diag}((H)_{11},\ldots,(H)_{NN})$. Obviously that $\exists\ \varepsilon_1 \in (0,\varepsilon_0)$ such that $\forall \varepsilon \in (0,\varepsilon_1)$ the transformation (9) is non-degenerate. As result we obtain:

$$\frac{dx^2}{dt} = (i\varphi(t,\varepsilon)\Lambda + \varepsilon\Lambda_1(t,\varepsilon) + \varepsilon^2 H_0(t,\varepsilon) + \mu P_1(t,\varepsilon,\theta) + \mu\varepsilon P_2(t,\varepsilon,\theta))x^2, \tag{10}$$

where $H_0 = (E-\varepsilon Q)^{-1}(HQ + Q\Lambda_1 + \varepsilon^{-1}dQ/dt) \in S(m-2,\varepsilon_1)$,
$P_2 = (E-\varepsilon Q)^{-1}(QP_1 - P_1Q) \in F(m-1,l,\varepsilon,\theta)$.

We make in the system (10) the substitution:

$$x^2 = \exp(i\Lambda\theta(t,\varepsilon))x^3, \tag{11}$$

and as result we obtain the system:

$$\frac{dx^3}{dt} = (\varepsilon\Lambda_1(t,\varepsilon) + \varepsilon^2 H_1(t,\varepsilon,\theta) + \mu P_3(t,\varepsilon,\theta) + \mu\varepsilon P_4(t,\varepsilon,\theta))x^3, \tag{12}$$

where $H_1 = \exp(-i\Lambda\theta)H_0\exp(i\Lambda\theta) \in F(m-2,l,\varepsilon_1,\theta)$, $P_3 = \exp(-i\Lambda\theta)P_1\exp(i\Lambda\theta) \in F(m,l,\varepsilon_1,\theta)$, $P_4 = \exp(-i\Lambda\theta)P_2\exp(i\Lambda\theta) \in F(m-1,l,\varepsilon_1,\theta)$.

Consider the auxilary matrix equation:

$$\varphi(t,\varepsilon)\frac{\partial\Psi}{\partial\theta} = P_3(t,\varepsilon,\theta) - U(t,\varepsilon,\mu) + \mu(P_3(t,\varepsilon,\theta)\Psi - \Psi U(t,\varepsilon,\mu)). \tag{13}$$

We show, that for the some choice of matrix $U(t,\varepsilon,\mu)$ and for sufficiently small values of parameter μ the equation (13) has a solution $\Psi(t,\varepsilon,\theta,\mu)$, which belongs to class $F(m,l,\varepsilon_1,\theta)$.

Together with equation (13) we consider equation:

$$\varphi(t,\varepsilon)\frac{\partial\Psi_0}{\partial\theta} = P_3(t,\varepsilon,\theta) - U(t,\varepsilon). \tag{14}$$

We set:

$$(U_0)_{jk} = \Gamma_0[(P_3)_{jk}] = \Gamma_{r_j - r_k}[(P_1)_{jk}] \ (j,k = \overline{1,N}),$$

$$(\Psi_0)_{jk} = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \frac{\Gamma_\nu[(P_3)_{jk}]}{i\nu\varphi(t,\varepsilon)} \exp(i\nu\theta(t,\varepsilon)) \ (j,k = \overline{1,N}).$$

Obviously $U_0 \in S(m,\varepsilon_1)$, $\Psi_0 \in F(m, l\varepsilon_1, \theta)$, and $\exists\, K_1 > 0$ such that $\|\Psi_0\|^*_{F(m,l\varepsilon_1,\theta)} \leqslant K_1\|P_3\|^*_{F(m,l\varepsilon_1,\theta)}$, $\|U_0\|^*_{F(m,l\varepsilon_1,\theta)} \leqslant K_1\|P_3\|^*_{F(m,l\varepsilon_1,\theta)}$.

We make in the system (13) the substitution:

$$\Psi = \Psi_0 + \mu\widetilde{\Psi},\ U = U_0 + \mu\widetilde{U}, \tag{15}$$

where $\widetilde{\Psi}$, $\widetilde{U}$ – the new unknown matrices. We obtain:

$$\varphi(t,\varepsilon)\frac{\partial\widetilde{\Psi}}{\partial\theta} = (P_3(t,\varepsilon,\theta)\Psi_0(t,\varepsilon,\theta) - \Psi_0(t,\varepsilon,\theta)U_0(t,\varepsilon)) - \widetilde{U}+$$

$$+\mu(P_3(t,\varepsilon,\theta)\widetilde{\Psi} - \widetilde{\Psi}U_0(t,\varepsilon) - \Psi_0(t,\varepsilon,\theta)\widetilde{U} - \mu\widetilde{\Psi}\widetilde{U}). \tag{16}$$

We construct the next process of successive approximations:

$$(\widetilde{U}_0)_{jk} = \Gamma_0[(P_3\Psi_0 - \Psi_0 U_0)_{jk}], \tag{17}$$

$$(\widetilde{\Psi}_0)_{jk} = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \frac{\Gamma_\nu[(P_3\Psi_0 - \Psi_0 U_0)_{jk}]}{i\nu\varphi} \exp(i\nu\theta), \tag{18}$$

$$(\widetilde{U}_s)_{jk} = (\widetilde{U}_0)_{jk} + \mu\Gamma_0[(P_3\widetilde{\Psi}_{s-1} - \widetilde{\Psi}_{s-1}U_0 - \Psi_0\widetilde{U}_{s-1} - \mu\widetilde{\Psi}_{s-1}\widetilde{U}_{s-1})_{jk}], \tag{19}$$

$$(\widetilde{\Psi}_s)_{jk} = (\widetilde{\Psi}_0)_{jk}+$$

$$+\mu \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \frac{\Gamma_\nu[(P_3\widetilde{\Psi}_{s-1} - \widetilde{\Psi}_{s-1}U_0 - \Psi_0\widetilde{U}_{s-1} - \mu\widetilde{\Psi}_{s-1}\widetilde{U}_{s-1})_{jk}]}{i\nu\varphi} \exp(i\nu\theta), \tag{20}$$

$(j,k = \overline{1,N};\ s = 1,2,3,\ldots)$

Obviously, that $\forall\, s$: $\widetilde{U}_s \in S(m,\varepsilon_1)$, $\widetilde{\Psi}_s \in F(m,l,\varepsilon_1,\theta)$.

We define sets:

$$\Omega_1 = \left\{ \widetilde{\Psi} \in F(m,l,\varepsilon_1,\theta) :\ \|\widetilde{\Psi} - \widetilde{\Psi}_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant d \right\},$$

$$\Omega_2 = \left\{ \widetilde{U} \in S(m,\varepsilon_1) \subset F(m,l,\varepsilon,\theta) :\ \|\widetilde{U} - \widetilde{U}_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant d \right\}.$$

Using a technique known contraction mapping principle [2], it is easy to show that $\exists\ \mu_1 \in (0,\mu_0)$ such that $\forall\ \mu \in (0,\mu_1)$ all approximations $\widetilde{\Psi}_s$ $(s = 1,2,3,\ldots)$ belongs to set Ω_1, and all approximations $\widetilde{U}_s$ belongs to set Ω_2, and process (18)–(20) is converge by the norm $\|\cdot\|^*_{F(m,l,\varepsilon_1,\theta)}$ to solution $\widetilde{\Psi}(t,\varepsilon,\theta,\mu) \in F(m,l,\varepsilon_1,\theta)$ of equation (16).

We make in system (12) the substitution:

$$x^3 = (E + \mu\Psi(t, \varepsilon, \theta, \mu))y, \quad (21)$$

where Ψ are defined by formula (15). We obtain the system (6), where matrix V are defined from equation:

$$(E + \mu\Psi)V = \Lambda_1\Psi - \Psi\Lambda_1 + P_4(E + \mu\Psi) + \varepsilon(H_1\Psi - \Psi H_1) - \frac{1}{\varepsilon}\frac{\partial\Psi}{\partial t}. \quad (22)$$

Obviously $\exists\ \mu_2 \in (0, \mu_1)$ such that $\forall\ \mu \in (0, \mu_2)$ equation (22) has a solution $V(t, \varepsilon, \theta, \mu) \in F(m-1, l, \varepsilon_1, \theta)$.

Lemma 1 are proved.

We write the system (6) in the form:

$$\frac{dy}{dt} = (\varepsilon\Lambda_1(t, \varepsilon) + \mu(U_0(t, \varepsilon) + \mu\widetilde{U}(t, \varepsilon, \mu)) + \varepsilon^2 H_1(t, \varepsilon, \theta) + \mu\varepsilon V(t, \varepsilon, \theta, \mu))y. \quad (23)$$

Lemma 2. *Let the eigenvalues $u_1(t, \varepsilon), \ldots, u_N(t, \varepsilon)$ of matrix $U_0(t, \varepsilon)$ holds:*

$$\inf_{G(\varepsilon_0)} |\mathrm{Re}(u_j(t, \varepsilon) - u_k(t, \varepsilon))| > 0 \quad (j \neq k). \quad (24)$$

Then $\exists\ \mu_3 \in (0, \mu_0)$, $C_1 \in (0, +\infty)$ such that $\forall\ \mu \in (0, \mu_3)$ exists the transformation of kind:

$$y = L(t, \varepsilon, \mu)y^2, \quad (25)$$

where $L \in S(m-1, \varepsilon_2(\mu))$, $\varepsilon_2(\mu) = C_1\mu$, which reducing the system (23) to the form:

$$\frac{dy^2}{dt} = (\mu\Lambda_2(t, \varepsilon, \mu) + \varepsilon\Lambda_3(t, \varepsilon, \mu) + \frac{\varepsilon^2}{\mu}\ K_1(t, \varepsilon, \mu) + \varepsilon^2 H_2(t, \varepsilon, \theta, \mu) +$$

$$+ \mu\varepsilon V_1(t, \varepsilon, \theta, \mu))y^2, \quad (26)$$

where Λ_2 – diagonal matrix with elements from $S(m, \varepsilon_2(\mu))$, Λ_3 – diagonal matrix with elements from $S(m-1, \varepsilon_2(\mu))$, H_2 – square matrix with elements from $F(m-2, l, \varepsilon_2(\mu), \theta)$, V_1 – square matrix with elements from $F(m-1, l, \varepsilon_2(\mu), \theta)$.

Proof. We denote $\widetilde{u}_j(t, \varepsilon, \mu)$ $(j = \overline{1, N})$ – eigenvalues of matrix $U_0(t, \varepsilon) + \mu\widetilde{U}(t, \varepsilon, \mu)$. From the perturbation theory of eigenvalues of matrices [3] and condition (24) implies that $\exists\ \mu_3 \in (0, \mu_0)$ such that $\forall\ \mu \in (0, \mu_3)$ holds:

$$\inf_{G(\varepsilon_1)} |\mathrm{Re}\,(\widetilde{u}_j(t, \varepsilon, \mu) - \widetilde{u}_k(t, \varepsilon, \mu))| > 0 \quad (j \neq k). \quad (27)$$

Therefore for these values μ exists the matrix $L_1(t, \varepsilon, \mu)$ with elements from $S(m, \varepsilon_1)$ such that $\inf\limits_{G(\varepsilon_1)} |\mathrm{det} L_1(t, \varepsilon, \mu)| > 0$ and

$$L_1^{-1}(U_0 + \mu\widetilde{U})L_1 = \Lambda_2(t, \varepsilon, \mu) = \mathrm{diag}(\widetilde{u}_1(t, \varepsilon, \mu), \ldots, \widetilde{u}_N(t, \varepsilon, \mu)).$$

We make in system (23) the substitution:

$$y = L_1(t, \varepsilon, \mu)y^1, \quad (28)$$

where y^1 – the new unknown N-dimesional vector. We obtain:

$$\frac{dy^1}{dt} = (\mu\Lambda_2(t,\varepsilon,\mu) + \varepsilon K(t,\varepsilon,\mu) + \varepsilon^2 \widetilde{H}_1(t,\varepsilon,\theta,\mu) +$$

$$+\mu\varepsilon \widetilde{V}(t,\varepsilon,\theta,\mu))y^1, \tag{29}$$

where $K = L_1^{-1}\Lambda_1 L_1 - \frac{1}{\varepsilon} L_1^{-1}\frac{dL_1}{dt} \in S(m-1,\varepsilon_1)$, $\widetilde{H}_1 = L_1^{-1}H_1L_1 \in F(m-2,l,\varepsilon_1,\theta)$, $\widetilde{V} = L_1^{-1}VL_1 \in F(m-1,l,\varepsilon_1,\theta)$.

We make in system (29) the next substitution:

$$y^1 = \left(E - \frac{\varepsilon}{\mu}\, Q_1(t,\varepsilon,\mu)\right) y^2, \tag{30}$$

where

$$(Q_1)_{jj} \equiv 0, \quad (Q_1)_{jk} = \frac{(K(t,\varepsilon,\mu))_{jk}}{\widetilde{u}_j(t,\varepsilon,\mu) - \widetilde{u}_k(t,\varepsilon,\mu)} \quad (j \neq k).$$

$Q_1 \in S(m-1,\varepsilon_1)$. The transformation will be no-degenerate if the constant C_1 in condition of Lemma will be sufficiently small. Then we obtain the system of kind (26), in which

$$\Lambda_3 = \mathrm{diag}((K)_{11},\ldots,(K)_{NN}) \in S(m-1,\varepsilon_2(\mu)),$$

$$K_1 = \left(E - \frac{\varepsilon}{\mu}\, Q_1\right)^{-1} \left(Q_1\Lambda_3 - KQ_1 + \frac{1}{\varepsilon}\frac{dQ_1}{dt}\right) \in S(m-2,\varepsilon_2(\mu)),$$

$$H_2 = \left(E - \frac{\varepsilon}{\mu}\, Q_1\right)^{-1} \widetilde{H}_1 \left(E - \frac{\varepsilon}{\mu}\, Q_1\right) \in F(m-2,l,\varepsilon_2(\mu),\theta),$$

$$V_1 = \left(E - \frac{\varepsilon}{\mu}\, Q_1\right)^{-1} \widetilde{V} \left(E - \frac{\varepsilon}{\mu}\, Q_1\right) \in F(m-1,l,\varepsilon_2(\mu),\theta).$$

Lemma 2 are proved.

Lemma 3. *Suppose we are given a linear first-order differential equation:*

$$\frac{dx}{dt} = (\mu u(t,\varepsilon) + \varepsilon\alpha(t,\varepsilon))x + \varepsilon v(t,\varepsilon,\theta), \tag{31}$$

where $u \in S(m,\varepsilon_0)$, $\alpha \in S(m-1,\varepsilon_0)$, $v \in F(m-1,l,\varepsilon_0,\theta)$, $\mu \in (0,\mu_0)$. *And suppose that the following conditions:*

1) $\Gamma_0[v] \equiv 0$;

2) $\inf\limits_{G(\varepsilon_0)} |\mathrm{Re}\; u(t,\varepsilon)| = \gamma^* > 0$.

Then $\exists\ \mu^* \in (0,\mu_0)$, $\exists\ C^* \in (0,+\infty)$ *such that* $\forall\ \mu \in (0,\mu^*)$, $\forall\ \varepsilon \in (0,\varepsilon^*(\mu))$, *where* $\varepsilon^*(\mu) = C^*\mu$, *the equation (31) has a particular solution* $x(t,\varepsilon,\theta,\mu) \in F(m-1,l,\varepsilon^*,\theta)$, *and* $\exists\ M \in (0,+\infty)$ *such that*

$$\|x\|_{F(m-1,l,\varepsilon^*(\mu),\theta)} \leqslant M\|v\|_{F(m-1,l,\varepsilon^*(\mu),\theta)}.$$

Proof. By the condidtion 1) of Lemma we expand the function $v(t,\varepsilon,\theta)$ in Fourier-serie:

$$x = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} x_\nu(t,\varepsilon)\exp(i\nu\theta(t,\varepsilon)). \tag{32}$$

We substitute (32) in (31) and equate the coefficients at $\exp(i\nu\theta(t,\varepsilon))$. We obtain:

$$\frac{dx_\nu}{dt} = \sigma_\nu(t,\varepsilon,\mu)x_\nu + \varepsilon v_\nu(t,\varepsilon), \quad \nu \neq 0, \tag{33}$$

where $\sigma_\nu(t,\varepsilon,\mu) = \mu u(t,\varepsilon) + \varepsilon\alpha(t,\varepsilon) - i\nu\varphi(t,\varepsilon)$.

We denote: $u^* = \sup\limits_{G(\varepsilon_0)} |u(t,\varepsilon)|$, $\alpha^* = \sup\limits_{G(\varepsilon_0)} |\alpha(t,\varepsilon)|$. We choose $\mu^* \in (0,\mu_0)$, $C^* \in (0,+\infty)$ so that $\forall\, \mu \in (0,\mu^*)$, $\forall\, \varepsilon \in (0, C^*\mu))$ holds $\inf\limits_{G(\varepsilon_0)} \varphi(t,\varepsilon) - \mu u^* - \varepsilon\alpha^* \geqslant \gamma_1 > 0$. Then obviously, that $\forall\, \nu \neq 0$ holds

$$|\sigma_\nu(t,\varepsilon,\mu)| \geqslant |\nu| \inf_{G(\varepsilon_0)} \varphi(t,\varepsilon) - \mu u^* - \varepsilon\alpha^* \geqslant |\nu|\gamma_1. \tag{34}$$

Let $f(t,\varepsilon) \in S(m-1,\varepsilon_0)$. We consider integral:

$$I_\nu(t,\varepsilon,\mu) = \varepsilon \int\limits_{\pm\frac{L}{\varepsilon}}^{t} f(\tau,\varepsilon) \exp\left(\int\limits_{\tau}^{t} \sigma_\nu(s,\varepsilon,\mu)ds \right) d\tau, \ \nu \neq 0,$$

where the sign in the lower limit of integration coincides with the sign of $\mathrm{Re}u(t,\varepsilon)$. Let for certainty $\mathrm{Re}u(t,\varepsilon) \leqslant -\gamma^* < 0$. We estimate:

$$|I_\nu(t,\varepsilon,\mu)| = \left| \varepsilon \int\limits_{-\frac{L}{\varepsilon}}^{t} f(\tau,\varepsilon) \exp\left(\int\limits_{\tau}^{t} \sigma_\nu(s,\varepsilon,\mu)ds \right) d\tau \right| \leqslant$$

$$\leqslant \varepsilon \sup_{G(\varepsilon_0)} |f(t,\varepsilon)| e^{2L\alpha^*} \int\limits_{-\frac{L}{\varepsilon}}^{t} \exp\left(\mu \int\limits_{\tau}^{t} \mathrm{Re}u(s,\varepsilon)ds \right) d\tau \leqslant$$

$$\leqslant \varepsilon e^{2L\alpha^*} \sup_{G(\varepsilon_0)} |f(t,\varepsilon)| \int\limits_{-\frac{L}{\varepsilon}}^{t} \exp\left(-\mu\gamma^*(t-\tau)\right) d\tau =$$

$$= e^{2L\alpha^*} \sup_{G(\varepsilon_0)} |f(t,\varepsilon)| \frac{\varepsilon}{\mu\gamma^*} \left(1 - \exp\left(-\mu\gamma^* \left(t + \frac{L}{\varepsilon} \right) \right) \right) \leqslant$$

$$= e^{2L\alpha^*} \sup_{G(\varepsilon_0)} |f(t,\varepsilon)| \frac{\varepsilon}{\mu\gamma^*} \left(1 - \exp\left(-\frac{2L\mu\gamma^*}{\varepsilon} \right) \right) \leqslant 2Le^{2L\alpha^*} \sup_{G(\varepsilon_0)} |f(t,\varepsilon)|. \tag{35}$$

The analogous estimation is true in case $\mathrm{Re}u(t,\varepsilon) \geqslant \gamma^* > 0$.

Taking into account the estimation (34), we introduce the operators:

$$D(v_\nu) = -\frac{d}{dt}\left(\frac{v_\nu}{\sigma_\nu}\right), \quad D^q(v_\nu) = D(D^{q-1}(v_\nu)), \quad D^0(v_\nu) = v_\nu.$$

Consider the following solution of equation (33):

$$x_\nu = \exp\left(\int\limits_{0}^{t} \sigma_\nu(s,\varepsilon,\mu)ds \right) \left(c_\nu + \int\limits_{\pm\frac{L}{\varepsilon}}^{t} v_\nu(\tau,\varepsilon) \exp\left(-\int\limits_{0}^{\tau} \sigma_\nu(s,\varepsilon,\mu)ds \right) d\tau \right), \tag{36}$$

where the sign in the lower limit of integration coincides with the sign of $\mathrm{Re}u(t,\varepsilon)$, and the constant c_ν respectively are defined by formula:

$$c_\nu = \exp\left(-\int\limits_0^{\pm\frac{L}{\varepsilon}} \sigma_\nu(s,\varepsilon,\mu)ds\right)\sum_{q=0}^{m-2}(-1)^q \frac{D^q(v_\nu(t,\varepsilon))}{-\sigma_\nu(t,\varepsilon,\mu)}\bigg|_{t=\pm\frac{L}{\varepsilon}},$$

We apply to (36) the m-fold integration by parts. We obtain:

$$D^{m-2}\left(\frac{dx_\nu}{dt}\right) = -\varepsilon\sigma(t,\varepsilon,\mu)\int\limits_{\pm\frac{L}{\varepsilon}}^{t} D^{m-1}(v_\nu(\tau,\varepsilon))\exp\left(\int\limits_{\tau}^{t}\sigma_\nu(s,\varepsilon,\mu)ds\right)d\tau.$$

Using the estimate (35), we obtain the required.

Lemma 3 are proved.

3. Principal Results.

Theorem. *Let the eigenvalues* $u_1(t,\varepsilon),\ldots,u_N(t,\varepsilon)$ *of matrix* $U_0(t,\varepsilon)$ *satisfy the condition (24). Then* $\exists\ \mu^{**}\in(0,\mu_0)$, $C^{**}\in(0,+\infty)$ *such that* $\forall\ \mu\in(0,\mu^{**})$, $\forall\ \varepsilon\in(0,\varepsilon^{**}(\mu))$, *where* $\varepsilon^{**}(\mu)) = C^{**}\mu$, *exists the transformation of kind (2), where* $\Phi\in F(m-2,l,\varepsilon^{**}(\mu),\theta)$, *which reducing the system (4) to kind (3), where* $\widetilde{A}(t,\varepsilon,\mu)\in S(m-2,\varepsilon^{**}(\mu))$.

Proof. Based on Lemma 1 we reduce the system (4) to kind (23). Now we construct the transformation

$$y^2 = (E+\mu Z(t,\varepsilon,\theta,\mu))z, \tag{37}$$

which reducing the system (26) to kind:

$$\frac{dz}{dt} = (\mu\Lambda_2(t,\varepsilon,\mu)+\varepsilon\Lambda_3(t,\varepsilon,\mu)+\frac{\varepsilon^2}{\mu}\ K_1(t,\varepsilon,\mu)+\mu\varepsilon B(t,\varepsilon,\mu))z, \tag{38}$$

where the matrix $B(t,\varepsilon,\mu)$ must be defined. Then for the matrix Z we obtain the following equation:

$$\frac{dZ}{dt} = \widetilde{\Lambda}(t,\varepsilon,\mu)Z - Z\widetilde{\Lambda}(t,\varepsilon,\mu)+\frac{\varepsilon^2}{\mu}\ H_2(t,\varepsilon,\theta,\mu)+\varepsilon(V_1(t,\varepsilon,\theta,\mu)-B(t,\varepsilon,\mu))+$$
$$+\frac{\varepsilon^2}{\mu}(K_1(t,\varepsilon,\mu)Z - ZK_1(t,\varepsilon,\mu))+\mu\varepsilon(V_1(t,\varepsilon,\theta,\mu)Z - ZB(t,\varepsilon,\mu))+$$
$$+\varepsilon^2 H_2(t,\varepsilon,\theta,\mu)Z, \tag{39}$$

where $\widetilde{\Lambda} = \mu\Lambda_2+\varepsilon\Lambda_3 = \mathrm{diag}(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_N)$.

Together with the equation (39) we consider the truncated equation:

$$\frac{dZ_0}{dt} = \widetilde{\Lambda}(t,\varepsilon,\mu)Z_0 - Z_0\widetilde{\Lambda}(t,\varepsilon,\mu)+\varepsilon(V_1(t,\varepsilon,\theta,\mu)-B_0(t,\varepsilon,\mu)), \tag{40}$$

where the matrix $B_0(t,\varepsilon,\mu)$ must be defined. We write the equation (40) in the component form:

$$\frac{d((Z_0)_{jk})}{dt} = \left(\widetilde{\lambda}_j(t,\varepsilon,\mu)-\widetilde{\lambda}_k(t,\varepsilon,\mu)\right)(Z_0)_{jk}+$$

$$+\varepsilon((V_1(t,\varepsilon,\theta,\mu))_{jk}-(B_0(t,\varepsilon,\mu))_{jk}),\ j,k=\overline{1,N}. \quad (41)$$

At first we consider the case $j=k$. We have:

$$\frac{d((Z_0)_{jj})}{dt}=\varepsilon((V_1(t,\varepsilon,\theta,\mu))_{jj}-(B_0(t,\varepsilon,\mu))_{jj}),\ j=\overline{1,N}. \quad (42)$$

We set $(B_0)_{jj}=\Gamma_0[(V_1)_{jj}]$ $(j=\overline{1,N})$. Then based on the results [1] we can state that the equation (42) has a particular solution $(Z_0)_{jj}$ from class $F(m-1,l,\varepsilon_2(\mu),\theta$, and $\exists\ K_4\in(0,+\infty)$ such that

$$\|(Z_0)_{jj}\|_{F(m-1,l,\varepsilon_2(\mu),\theta)}\leqslant K_4\|(V_1)_{jj}\|_{F(m-1,l,\varepsilon_2(\mu),\theta)}. \quad (43)$$

Now let $j\neq k$. We write the equation (41) in kind:

$$\frac{d((Z_0)_{jk})}{dt}=(\mu(\widetilde{u}_j(t,\varepsilon,\mu)-\widetilde{u}_k(t,\varepsilon,\mu))+$$
$$+\varepsilon((K(t,\varepsilon,\mu))_{jj}-(K(t,\varepsilon,\mu))_{kk}))(Z_0)_{jk}+$$
$$+\varepsilon((V_1(t,\varepsilon,\theta,\mu))_{jk}-(B_0(t,\varepsilon,\mu))_{jk}),\ j,k=\overline{1,N},\ (j\neq k). \quad (44)$$

We set $(B_0)_{jk}=\Gamma_0[(V_1)_{jk}]$ $(j,k=\overline{1,N};\ j\neq k)$. Then based on (27) the equation (44) is satisfy to all condidtions of Lemma 3, and according its $\exists\ \mu_4\in(0,\mu_0)$, $C_4\in(0,+\infty)$ such that $\forall\ \mu\in(0,\mu_4)$, $\forall\ \varepsilon\in(0,\varepsilon_4(\mu))$, where $\varepsilon_4(\mu)=C_4\mu$ the equation (44) has a particular solution $(Z_0)_{jk}\in F(m-1,l,\varepsilon_4(\mu),\theta)$, and $\exists\ K_5\in(0,+\infty)$ such that

$$\|(Z_0)_{jk}\|_{F(m-1,l,\varepsilon_4(\mu),\theta)}\leqslant K_5\|(V_1)_{jk}\|_{F(m-1,l,\varepsilon_4(\mu),\theta)}.$$

As performed $F(m-1,l,\varepsilon_4(\mu),\theta)\subset F(m-2,l,\varepsilon_4(\mu)$, then $\exists\ \mu_5\in(0,\mu_0)$, $C_5\in(0,+\infty)$ such that $\forall\ \mu\in(0,\mu_5)$, $\forall\ \varepsilon\in(0,\varepsilon_5(\mu))$, where $\varepsilon_5(\mu)=C_5\mu$, the equation (40) has a particular solution $Z_0(t,\varepsilon,\theta,\mu)\in F(m-2,l,\varepsilon_5(\mu),\theta)$, and $\exists K_6\in(0,+\infty)$ such that

$$\|Z_0\|^*_{F(m-2,l,\varepsilon_5(\mu),\theta)}\leqslant K_5\|V_1\|^*_{F(m-2,l,\varepsilon_5(\mu),\theta)}.$$

In this $B_0(t,\varepsilon,\mu)=(\Gamma_0[(V_1)_{jk}])_{j,k=\overline{1,N}}$.

Now we return to the equation (39). Draw the arguments similar to those in the [1] with using of the method of successive approximations. Then we can state that $\exists\ \mu_6\in(0,\mu_0)$, $C_6\in(0,+\infty)$ such that $\forall\ \mu\in(0,\mu_6)$, $\forall\ \varepsilon_6(\mu))$, where $\varepsilon_6(\mu)=C_6\mu$, the equation (39) has a particular solution $Z(t,\varepsilon,\theta,\mu)\in F(m-2,l,\varepsilon_6(\mu),\theta)$. In this $B(t,\varepsilon,\mu)\in S(m-2,\varepsilon_6(\mu))$. Thus the theorem are proved.

Conclusion. Thus, for the system (4) with the oscillating coefficients the sufficient conditions of the existence of the transformation, which reducing this system to a system with slowly varying coefficients are obtained in a resonance case.

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