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BOUNDARY-VALUE PROBLEM FOR DIFFERENTIAL EQUATION WITH MIXED RIEMANN—LIOUVILLE DERIVATIVE OF FRACTIONAL ORDER

Вітюк О. Н., Міхайленко А. В. Крайова задача для диференціального рівняння із змішаною похідною Рімана—Ліувілля дробового порядку. В роботі отримано достатні умови розв'язності крайовоі задачі для диференціального рівняння, що містить мішану похідну Рімана-Ліувілля дробового порядку.
Ключові слова: лівостороння змішана похідна Рімана"– Ліувілля, дробовий порядок, теорема існування та єдиності.

Витюк А. Н., Михайленко А. В. Краевая задача для дифференциального уравнения со смешанной производной Римана—Лиувилля дробного порядка. В работе получены достаточные условия разрешимости краевой задачи для диференциального уравнения со смешанной производной Римана-Лиувилля дробного порядка.
Ключевые слова: левосторонняя смешанная производная Римана—Лиувилля, дробный порядок, теорема существования и единственности.

Vityuk A. N., Mykhailenko A. V. Boundary-value problem for differential equation with mixed Riemann—Liouville derivative of fractional order. In this paper, we find a sufficient condition for solvability of boundary-value problem for differential equation with mixed Riemann-Liouville derivative of fractional order.
Key words: left-sided mixed Riemann—Liouville derivative, fractional order, existence and uniqueness theorem.

Introduction. Let $P = [0,a] \times [0,b], 0 < a, b < \infty, \alpha > 0, \beta > 0, r = (\alpha;\beta), \Theta = (0;0)$. By C(P) we denote the space of continuous functions $f : P \to R$ with the norm

$$\|f(x,y)\|_C = \max_P |f(x,y)|.$$

By $AC(P)$ and $L(P)$ we denote the space of absolutely continuous and summable by Lebesgue functions $f : P \to R$ accordingly. The following expression

$$\phi_\alpha(x) = I_0^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \phi(t)dt,$$

where $\Gamma(\cdot)$ is the gamma-function of Euler, is called [1,2] left-sided Riemann—Liouville integral of order α. Let $\alpha = [\alpha] + \alpha, n = [\alpha] + 1$. Then the expression

$$D_0^\alpha \phi(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\alpha-1} \phi(t)dt$$

is called [1, §2,p.42],[2, §1.1, p.9] the left-sided fractional derivative of Riemann—Liouville of order α of function $\phi : [0, a] \to R$. Left-sided mixed integral and derivative of order r are defined as follows [1, §24, p.340]:

$$I_{0,x}^{\alpha} f(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t, y) dt,$$

$$I_{0,y}^{\beta} f(x, y) = \frac{1}{\Gamma(\beta)} \int_0^y (y - s)^{\beta - 1} f(x, s) ds,$$

$$f_r(x, y) = I_{\Theta}^{(\alpha,\beta)} f(x, y) = I_{\Theta}^{r} f(x, y) = I_{0,x}^{\alpha} I_{0,y}^{\beta} f(x, y) =$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x - t)^{\alpha - 1} (y - s)^{\beta - 1} f(t, s) ds dt,$$

$$D_{0,x}^{\alpha} f(x, y) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dx} \right)^n \int_0^x (x - t)^{n - \alpha - 1} f(t, y) dt,$$

$$D_{\Theta}^{r} f(x, y) = D_{0,x}^{\alpha} D_{0,y}^{\beta} f(x, y) =$$

$$= \frac{1}{\Gamma(n - \alpha) \cdot \Gamma(m - \beta)} \cdot \frac{\partial^{n+m}}{\partial x^n \partial y^m} \int_0^x \int_0^y (x - t)^{n - \alpha - 1} (y - s)^{m - \beta - 1} f(t, s) ds dt,$$

where $n = [\alpha] + 1, m = [\beta] + 1$. Elliptic boundary-value problems with area boundary conditions are comprehensively studied. First results dedicated to the research of hyperbolic boundary-value problems with area boundary conditions were obtained in papers [3-6]. The research of following boundary-value problem

$$\frac{\partial^4 z(x, y)}{\partial x^2 \partial y^2} = F(x, y, z(x, y)), \tag{1.1}$$

$$z(i, y) = z(x, j) = 0, 0 \leqslant x, y \leqslant 1; i, j = 0, 1 \tag{1.2}$$

began in [3]. In [8] were established that this problem is equivalent to the solution of some Fredholm integral equation. Based on this equivalence in [7] the numerical method of solution of this problem was constructed.

In [9] the conditions of existence and uniqueness of solution of differential inclusion

$$\frac{\partial^4 z(x, y)}{\partial x^2 \partial y^2} \in F(x, y, z(x, y))$$

that satisfies the conditions (1.2) were obtained and its properties were studied.

In [6] the following boundary-value problem was considered,

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} - A(x, y)u = f(x, y), (0 < x < a, 0 < y < b),$$

$$u(0, y) = u(x, 0) = 0, \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{x=a} = \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{y=b} = 0,$$

and for this problem Green's function $G(x, y; t, s)$ was constructed such that

$$u(x, y) = \int_0^a \int_0^b G(x, y; t, s) f(t, s) ds dt.$$

Let now $0 < \alpha, \beta \leqslant 1, p = (1 + \alpha; 1 + \beta), q = (1 - \alpha; 1 - \beta)$. In [10] the following boundary-value problem were considered

$$D_\Theta^p u(x, y) = F(x, y, u(x, y), D_\Theta^r u(x, y)),$$

$$u_q(x, 0) = u_q(x, b) = u_q(0, y) = u_q(a, y) = 0, 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b.$$

The sufficient conditions of existence and uniqueness of solution of this problem were received. The present paper deals with conditions of solvability of the boundary-value problem

$$D_\Theta^p u(x, y) = F[u(x, y)] \equiv F(x, y, u(x, y)), \tag{1.3}$$

$$u(x, 0) = u(x, b) = u(0, y) = u(a, y) = 0; 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b. \tag{1.4}$$

Main Results.

2. Preliminaries. In this section we introduce notations, definitions and preliminary facts which are used thoughout this paper.

Definition 2.1 ([11]) *Continuous function $z(x, y) : P \to R$ is called absolutely continuous if and only if it can be presented as follows*

$$z(x, y) = z_1(x) + z_2(y) - z(0, 0) + \int_0^x \int_0^y v(t, s) ds dt,$$

where $z_1(x) \in AC([0, a]), z_2(y) \in AC([0, b]), v(x, y) \in L(P)$.

Definition 2.2 ([1, §1, p.21]) *Function $u(x) : J \to R, J = [a, b]$ belongs to the set $AC^n(J)$, if $u^{(k)}(x) \in AC(J), k = \overline{0, n-1}$.*

Let $D_{xy}^k u(x, y) = \frac{\partial^2 k u(x,y)}{\partial x^k \partial y^k}, k = 0, 1, \dots (D_{xy}^0 u(x, y) = u(x, y))$.

By $AC^n(P)$ we denote class of functions $v(x, y) : P \to R$ such that $D_{xy}^k v(x, y) \in AC(P), k = (0, n - 1)$.

Lemma 2.1 ([12]) *Let σ_1, σ_2 are any positive numbers and $0 \leqslant \tau \leqslant 1$. Then*

$$|\sigma_1^\tau - \sigma_2^\tau| \leq |\sigma_1 - \sigma_2|^\tau.$$

Lemma 2.2 *Let $u(x) \in AC([0, a]), v(y) \in AC([0, b])$ and $u(0) = v(0)$. Then $z(x, y) = u(x) \cdot v(y) \in AC(P)$.*

We consider a boundary-value problem

$$D_0^{1+\alpha} y(x) = f(x), \tag{2.1}$$

$$y(0) = y(a) = 0, \tag{2.2}$$

where $f(x)$ is a measurable function.

For the solution of boundary-value problem we name function $y(x) : [0, a] \to R$, so that $y(x) \in C([0, a]), y_{1-\alpha}(x) \in AC^2([0, a])$, satisfies the boundary-value conditions (2.2) and differential equation (2.1) for almost all $x \in [0; a]$.

Lemma 2.3 ([14]) *Let $f(x) : [0,a] \to R$ be a measurable function and $|f(x)| \leqslant M$. Then the solution of boundary-value problem (2.1), (2.2) is represented as follows:*

$$y(x) = \int_0^a G(x,t) f(t) dt,$$

where

$$G(x,t) = \left[\begin{array}{c} -\frac{(x(a-t))^\alpha - (a(x-t))^\alpha}{a^\alpha \Gamma(1+\alpha)}, 0 \leqslant t \leqslant x \\ -\frac{(x(a-t))^\alpha}{a^\alpha \cdot \Gamma(1+\alpha)}, x \leqslant t \leqslant a. \end{array} \right.$$

3. Existence of solutions. Consider a boundary-value problem

$$D^p_\Theta u(x,y) = w(x,y), w(x,y) \in C(P), \tag{3.1}$$

$$u(x,0) = u(x,b) = 0, 0 \leqslant x \leqslant a; u(0,y) = u(a,y) = 0, 0 \leqslant y \leqslant b. \tag{3.2}$$

Definition 3.1 *Continuous function $u(x,y) : P \to R$, so that $u_q(x,y) \in AC^2(P)$, satisfies the boundary-value conditions (3.2) and differential equation (3.1) for $(x,y) \in P$ is called the solution of boundary-value problem (3.1), (3.2).*

Lemma 3.1 The solution of boundary-value problem (3.1), (3.2) is represented as follows

$$u(x,y) = \int_0^a \int_0^b \Phi(x,y;t,s) w(t,s) ds dt,$$

where $\Phi(x,y;t,s) = G(x,t) \cdot G(y,s)$,

$$G(y,s) = \left[\begin{array}{c} -\frac{(y(b-s))^\beta - (b(y-s))^\beta)}{b^\beta \cdot \Gamma(1+\beta)}, 0 \leqslant s \leqslant y, \\ -\frac{(y(b-s))^\beta}{b^\beta \cdot \Gamma(1+\beta)}, y \leqslant s \leqslant b. \end{array} \right.$$

Proof. Assume that $u(x,y)$ is the solution of boundary-value problem (3.1), (3.2) and

$$v(x,y) = D^{1+\beta}_{0,y} u(x,y) = \frac{1}{\Gamma(1-\beta)} \frac{\partial^2}{\partial y^2} \int_0^y (y-s)^{-\beta} u(x,s) ds. \tag{3.3}$$

As $u(0,y) = u(a,y) = 0$ for $y \in [0,b]$ then

$$v(0,y) = v(a,y) = 0, 0 \leqslant y \leqslant b. \tag{3.4}$$

Take into account that $u(x,y)$ is the solution of boundary-value problem (3.1), (3.2). Then

$$D^p_\theta u(x,y) = D^{1+\alpha}_{0,x} D^{1+\beta}_{0,y} u(x,y) = D^{1+\alpha}_{0,x} v(x,y) = w(x,y), (x,y) \in P.$$

Consequently at any fixed $y \in [0,b]$ function $v(x,y)$ is the solution of the equation

$$D^{1+\alpha}_{0,x} v(x,y) = w(x,y). \tag{3.5}$$

Under lemma 2.3 the solution of boundary-value problem (3.5), (3.4) is represented as follows

$$v(x,y) = \int_0^a G(x,t) w(t,y) \equiv \delta(x,y), y \in [0,b]. \tag{3.6}$$

From (3.3), (3.2) it follows that at any $x \in [0, a]$

$$D_{0,y}^{1+\beta} u(x, y) = \delta(x, y), \tag{3.7}$$

$$u(x, 0) = u(x, b) = 0. \tag{3.8}$$

Thereby $u(x, y)$ under lemma 2.3 can be presented as follows

$$u(x, y) = \int_0^b G(y, s)\delta(x, s)ds = \int_0^b G(y, s) \left(\int_0^a G(x, t)w(t, s)dt \right) ds =$$

$$= \int_0^a \int_0^b \Phi(x, y; t, s)w(t, s)dsdt.$$

Consider a boundary-value problem (1.3), (1.4). The solution of this problem is such a continuous function $u(x, y) : P \to R$ that $u_q(x, y) \in AC^2(P)$ and that satisfies the condition (1.4) and differential equation (1.3) for $(x, y) \in P$.

Theorem 3.1 *Suppose that function $F(x, y, u) : P \times R \to R$ is continuous and $|F(x, y, u)| \leqslant M$. Then boundary-value problem (1.3), (1.4) is equivalent to the integral equation*

$$u(x, y) = \int_0^a \int_0^b \Phi(x, y; t, s)F(t, s, u(t, s))dsdt. \tag{3.9}$$

Proof. Let $u(x, y) \in C(P)$ be solution of boundary-value problem (1.3), (1.4). Then $F[u(x, y)] \in C(P)$ and according to lemma 3.1 $u(x, y)$ is represented by (3.9).

Suppose now that $u(x, y) \in C(P)$ is the solution of integral equation (3.9) and let's prove that $u(x, y)$ is the solution of boundary-value problem (1.3), (1.4). For $(x, y) \in (0, a) \times (0, b)$

$$u(x, y) = \frac{1}{a^\alpha b^\beta \Gamma(1+\alpha)\Gamma(1+\beta)} \left(\int_0^x \int_0^y ((x(a-t)^\alpha - (a(x-t))^\alpha) \cdot \left((y(b-s)^\beta - \right.\right.$$

$$\left. -(b(y-s))^\beta\right) F[u(t, s)]dsdt + \int_x^a \int_0^y (x(a-t))^\alpha \left((y(b-s))^\beta - (b(y-s))^\beta\right) F[u(t, s)]dsdt +$$

$$+ \int_0^x \int_y^b ((x(a-t))^\alpha - (a(x-t))^\alpha) (y(b-s))^\beta F[u(t, s)]dsdt +$$

$$+ \int_x^a \int_y^b (x(a-t))^\alpha (y(b-s))^\beta F[u(t, s)]dsdt =$$

$$= \frac{1}{a^\alpha b^\beta \Gamma(1+\alpha)\Gamma(1+\beta)} \left[\int_0^x \int_0^y (x(a-t))^\alpha (y(b-s))^\beta F[u(t, s)]dsdt - \right.$$

$$- \int_0^x \int_0^y (x(a-t))^\alpha (b(y-s))^\beta F[u(t, s)]dsdt - \int_0^x \int_0^y (a(x-t))^\alpha (y(b-s))^\beta F[u(t, s)]dsdt +$$

$$+ \int_0^x \int_0^y (a(x-t))^\alpha (b(y-s))^\beta F[u(t, s)]dsdt + \int_x^a \int_0^y (x(a-t))^\alpha (y(b-s))^\beta F[u(t, s)]dsdt -$$

$$- \int_x^a \int_0^y (x(a-t))^\alpha (b(y-s))^\beta F[u(t, s)]dsdt? + \int_0^x \int_y^b (x(a-t))^\alpha (y(b-s))^\beta F[u(t, s)]dsdt -$$

$$-\int_0^x \int_y^b (a(x-t))^\alpha (y(b-s))^\beta F[u(t,s)]dsdt+$$
$$+\int_x^a \int_y^b (x(a-t))^\alpha (y(b-s))^\beta F[u(t,s)]dsdt \tag{3.10}$$

In (3.10) the summands are grouped like this: the first, the fifth, the seventh and the ninth; the second and the sixth; the third and the eighth; the fourth. We receive

$$u(x,y) = \frac{x^\alpha y^\beta}{a^\alpha b^\beta \Gamma(1+\alpha)\Gamma(1+\beta)} \int_0^a \int_0^b (a-t)^\alpha (b-s)^\beta F[u(t,s)]dsdt-$$
$$\frac{x^\alpha}{a^\alpha \Gamma(1+\alpha)\Gamma(1+\beta)} \int_0^a \int_0^y (a-t)^\alpha (y-s)^\beta F[u(t,s)]dsdt-$$
$$-\frac{y^\beta}{b^\beta \Gamma(1+\alpha)\Gamma(1+\beta)} \int_0^x \int_0^b (x-t)^\alpha (b-s)^\beta F[u(t,s)]dsdt+$$
$$+\frac{1}{\Gamma(1+\alpha)\Gamma(1+\beta)} \int_0^x \int_0^y (x-t)^\alpha (y-s)^\beta F[u(t,s)]dsdt. \tag{3.11}$$

Then we prove that $u_q(x,y) = I_\theta^{(1-\alpha;1-\beta)} u(x,y) \in AC^2(P)$. Direct computation results

$$u_q(x,y) = xy\gamma - \frac{x}{a^\alpha} \int_0^y (y-\tau) \left(\int_0^a (a-z)^\alpha F[u(z,\tau)]dz \right) d\tau-$$
$$-\frac{y}{b^\beta} \int_0^x (x-z) \left(\int_0^b (b-\tau)^\beta F[u(z,\tau)]d\tau \right) dz + I_\theta^{(2;2)} F[u(x,y)] \equiv A_1 - A_2 - A_3 + A_4, \tag{3.12}$$

where

$$\gamma = \int_0^a \int_0^b (a-t)^\alpha (b-s)^\beta F[u(t,s)]dsdt,$$
$$I_\theta^{(2;2)} F[u(x,y)] = \int_0^x \int_0^y (x-t)(y-s) F[u(t,s)]dsdt.$$

If $\phi(\tau) = \int_0^a (a-z)^\alpha F[u(z,\tau)]dz$, then $A_2 = \frac{x}{a^\alpha} \int_0^y (t-\tau)\phi(\tau)d\tau$.

Since $\phi(\tau) \in C([0,b])$, then $\int_0^y (y-\tau)\phi(\tau)d\tau \in AC([0,b])$ and by lemma 2.2 $A_2 \in AC(P)$. It is obvious that $A_1, A_3, A_4 \in AC(P)$. Consequently, $u_q(x,y) \in AC(P)$. Next, prove that $D_{xy} u_q(x,y) \in AC(P)$. Really, by (3.12) we receive

$$D_{xy} u_q(x,y) = \gamma - \frac{1}{a^\alpha} \int_0^y \phi(\tau)d\tau - \frac{1}{b^\beta} \int_0^x \psi(z)dz + \int_0^x \int_0^y F[u(t,s)]dsdt,$$

where

$$\psi(z) = \int_0^b (b-\tau) F[u(z,\tau)]d\tau.$$

It is obvious that $D_{xy} u_q(x,y) \in AC(P)$. Consequently, $u_q(x,y) \in AC^2(P)$.Applying (3.11) we will convince that $u(x,u)$ satisfy boundary-value condition (1.4). In addition, $D_\theta^p u(x,y) = D_{xy}^2 u(x,y) = F[u(x,y)]$ for $(x,y) \in P$. Theorem 3.1 is proved.

Theorem 3.2 *Assume that function $F(x,y,u) : P \times R \to R$ satisfies conditions of theorem 3.1 and Lipschitz condition $|F(x,y,u) - F(x,y,v)| \leqslant L|u-v|$, at that*

$$\rho = \frac{a^{1+\alpha} b^{1+\beta} \cdot L}{4^{\alpha+\beta} \cdot \Gamma(1+\alpha) \cdot \Gamma(1+\beta)} < 1.$$

Then in the area P the unique solution of boundary-value problem (1.3), (1.4) exists.

Proof: For $u(x,y) \in C(P)$ we define operator Tu

$$Tu(x,y) = \int_0^a \int_0^b \Phi(x,y;s,t) \cdot F(t,s,u(t,s)) ds dt.$$

Now prove that $T : C(P) \to C(P)$. Let $w(x,y) = Tu(x,y)$ and $0 \leqslant x_1 < x_2 \leqslant a, 0 \leqslant y \leqslant b$. Then

$$|w(x_2,y) - w(x_1,y)| \leqslant M \int_0^a \int_0^b |G(x_2,t) - G(x_1,t)| \cdot |G(y,s)| ds dt,$$

at that

$$\int_0^b |G(y,s)| ds = \int_0^y |G(y,s)| ds + \int_y^b |G(y,s)| ds \equiv B_1 + B_2.$$

Applying lemma 2.1 we receive

$$B_1 = \frac{1}{b^\beta \cdot \Gamma(1+\beta)} \int_0^y \left| (y(b-s))^\beta - (b(y-s))^\beta \right| ds \leqslant \frac{1}{b^\beta \cdot \Gamma(1+\beta)} \int_0^y ((b-y)s)^\beta ds \leq$$

$$\leqslant \frac{1}{b^\beta \cdot \Gamma(1+\beta)} \int_0^y b^\beta s^\beta ds = \frac{y^{\beta+1}}{(\beta+1) \cdot \Gamma(\beta+1)} \leqslant \frac{b^{\beta+1}}{\Gamma(\beta+2)};$$

$$B_2 = \frac{1}{b^\beta \cdot \Gamma(\beta+1)} \int_y^b (y(b-s))^\beta \leqslant \frac{1}{\Gamma(\beta+1)} \int_y^b (b-s)^\beta ds = \frac{(b-y)^{\beta+1}}{\Gamma(\beta+1)} \leqslant \frac{b^{\beta+1}}{\Gamma(\beta+2)}.$$

Consequently,

$$\int_0^b |G(y,s)| ds \leqslant B, B = \frac{2b^{\beta+1}}{\Gamma(\beta+2)}.$$

Now

$$|w(x_2,y) - w(x_1,y)| \leq \frac{2Mb^{\beta+1}}{\Gamma(\beta+2)} \left(\int_0^{x_1} |G(x_2,t) - G(x_1,t)| dt + \right.$$

$$\left. + \int_{x_1}^{x_2} |G(x_2,t) - G(x_1,t)| dt + \int_{x_2}^b |G(x_2,t) - G(x_1,t)| dt \right) \equiv K_1 + K_2 + K_3.$$

On application of lemma 2.1 we receive

$$K_1 \leqslant \frac{M}{a^\alpha \cdot \Gamma(1+\alpha)} \int_0^{x_1} (((x_2(a-t))^\alpha - (x_1(a-t))^\alpha) + ((a(x_2-t))^\alpha \ -$$

$$- (a(x_1-t))^\alpha)) dt \leq \frac{2MBa^{\alpha+1}(x_2-x_1)^\alpha}{a^\alpha \Gamma(\alpha+1)} = \frac{2MBa}{\Gamma(\alpha+1)}(x_2-x_1)^\alpha;$$

$$K_2 \leqslant \frac{MB}{a^\alpha \cdot \Gamma(\alpha+1)} \int_{x_1}^{x_2} |(x_2(a-t))^\alpha + (a_1(x_2-t))^\alpha - (x_1(a-t)^\alpha)|dt \leq$$

$$\leqslant \frac{MB}{a^\alpha \cdot \Gamma(\alpha+1)} \int_{x_1}^{x_2} ((a-t)^\alpha (x_2^\alpha - x_1^\alpha) + a^\alpha (x_2-t)^\alpha) dt \leq$$

$$\leqslant \frac{MB}{\Gamma(1+\alpha)} \int_{x_1}^{x_2} ((x_2-x_1)^\alpha + (x_2-t)^\alpha) dt \leq \frac{MB}{\Gamma(\alpha+1)} (a(x_2-x_1)^\alpha +$$

$$+\frac{(x_2-x_1)^{\alpha+1}}{\Gamma(\alpha+1)} \leq \frac{MB}{\Gamma(1+\alpha)} (a(x_2-x_1)^\alpha + a(x_2-x_1)^\alpha) \leq \frac{2aMB}{\Gamma(\alpha+1)};$$

$$K_3 \leqslant \frac{MB}{a^\alpha \Gamma(1+\alpha)} \int_{x_2}^{a} (x_2^\alpha - x_1^\alpha)(a-t)^\alpha dt \leqslant \frac{MBa}{\Gamma(1+\alpha)} (x_2^\alpha - x_1^\alpha) \leqslant \frac{MBa}{\Gamma(1+\alpha)} (x_2-x_1)^\alpha.$$

So,

$$|w(x_2,y) - w(x_1,y)| \leq \frac{5MBa}{\Gamma(1+\alpha)} (x_2-x_1)^\alpha.$$

By analogy for $0 \leqslant y_1 < y_2 \leqslant b, 0 \leqslant x \leqslant b$ we receive

$$|w(x,y_2) - w(x,y_1)| \leqslant \frac{5MAb}{\Gamma(1+\beta)} (y_2-y_1)^\beta, A = \frac{2a^{\alpha+1}}{\Gamma(2+\alpha)}.$$

For $\varepsilon > 0$, if $|x_1 - x_2| \leqslant \delta, |y_1 - y_2| \leqslant \delta$, where

$$\delta = \max\left(\frac{\varepsilon\Gamma(1+\alpha)}{10aMB}, \frac{\varepsilon\Gamma(1+\beta)}{10bMA}\right),$$

then $|w(x_1,y_1) - w(x_2,y_2)| \leq \varepsilon$. In theorem 3.1 it is proved that if $u(x,y)$ is the solution of integral equation (3.10), then $u_q(x,y) \in AC^2(P)$. By analogy it is possible to prove that $w_q(x,y) \in AC^2(P)$.

Consequently, the fixed point of operator T will be the solution of boundary-value problem (1.3), (1.4). Next, prove that operator $T : C(P) \to C(P)$ is compressive operator. Let $u_k(x,y) \in C(P), w_k(x,y) = Tu_k(x,y), k = 1, 2$.

Then for $(x,y) \in P$

$$|w_1(x,y) - w_2(x,y)| \leq \frac{La^{\alpha+1} \cdot b^{\beta+1}}{4^{\alpha+\beta}\Gamma(1+\alpha)\Gamma(1+\beta)} \|u_1(x,y) - u_2(x,y)\|_C, \qquad (3.13)$$

since

$$|G(x,t)| \leqslant \frac{a^\alpha}{4^\alpha \cdot \Gamma(1+\alpha)}, |G(y,s)| \leqslant \frac{b^\beta}{4^\beta \cdot \Gamma(1+\beta)}.$$

Hereby, from (3.13)

$$\|Tu_1(x,y) - Tu_2(x,y)\|_C \leqslant \rho \|u_1(x,y) - u_2(x,y)\|_C,$$

notably operator $T : C(P) \to C(P)$ is compressive operator. Consequently, operator T has the unique fixed point that will be the solution of boundary-value problem (1.3), (1.4). Theorem 3.2 is proved.

Conclusion. In this paper was found a sufficient condition for solvability of boundary-value problem for differential equation with mixed Riemann—Liouville derivative of fractional order.

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