ISSN 2304-1579. Visnyk Odesk. Nats. Univers. Mat. i Mekh.–2014 .–V.19,Is.4(24).–P.45–57

Mathematical Subject Classification: 65L60
UDC 519.632.4

R. Drebotiy, H. Shynkarenko
Ivan Franko National University of Lviv
Opole Univesity of Technology, Poland

COMPARISON OF ERROR INDICATORS AND REFINEMENT CRITERIA FOR *HP*-ADAPTATION ALGORITHM FOR FINITE ELEMENT METHOD

Дреботій Р. Г., Шинкаренко Г. А. Порівняння апостеріорних оцінювачів похибки та критеріїв згущення сітки для hp-адаптивного алгоритму методу скінченних елементів. В цій статті ми коротко розглядаємо *hp*-адаптивний метод скінченних елементів для одновимірної задачі конвекції-дифузії-реакції і проводимо порівняльний аналіз чисельних результатів, одержаних за допомогою комбінування розглядуваного алгоритму з різними оцінювачами похибки та критеріями покращення елементів.
Ключові слова: метод скінченних елементів (МСЕ), метод Гальоркіна, критерій Пекле, критерій Струхаля, метод конденсації внутрішніх параметрів, hp-адаптивність, апостеріорний оцінювач похибки (АОП), коректна задача, крайова задача дифузії-адвекції-реакції, контрольний розв'язок.

Дреботий Р. Г., Шинкаренко Г. А. Сравнение апостериорных оценок погрешности и критериев сгущения сетки для hp-адаптивного алгоритма метода конечных элементов. В этой статье мы коротко рассматриваем *hp*-адаптивный метод конечных элементов для одномерной задачи конвекции-диффузии-реакции и сравниваем численные результаты, полученные с помощью комбинирования рассматриваемого алгоритма с разными оценивателями погрешности и критериями улучшения конечных элементов.
Ключевые слова: метод конечных элементов (МКЭ), метод Галеркина, критерий Пекле, критерий Струхаля, метод конденсации внутренних параметров, hp-адаптивность, апостериорный оцениватель погрешности (АОП), корректная задача, краевая задача диффузии-адвекции-реакции, контрольное решение.

Drebotiy R., Shynkarenko H. Comparison of error indicators and refinement criteria for hp-adaptation algorithm for finite element method. In this paper we consider *hp*-adaptive finite element method for 1D convection-diffusion-advection boundary value problem and present comparative analysis of numerical results obtained using combination of introduced algorithm with different kinds of a posteriori error estimators, and element refinement criteria.
Key words: finite element method, Galerkin method, Peclet criteria, Strouhal criteria, static condensation, hp-adaptivity, a posteriori error estimator, well posed problem, diffusion-advection-reaction boundary value problem.

Introduction. Finite element method is an universal tool for solving boundary value problems for partial differential equations (see [1]). It is applicable for problems on very complex domains in 2- and 3-dimensional spaces. During last years the main focus is on the adaptive algorithms for FEM. The main idea is to adapt mesh

(h-adaptivity), element polynomial order (p-adaptivity) or both mesh and order (hp-adaptivity) to minimize computational cost needed for solving the given problem. Such algorithms are implemented using local a posteriori error estimators. It's naturally to interpret hp-schemes as most advanced as they give us most wide approximation capabilities. Theoretically it is proven that they can produce exponentially convergent sequences of approximations to original solution of boundary value problem [2].

In this work we recall hp-adaptive algorithm for solving the diffusion-advection-reaction boundary value problems with self-adjoint operators introduced in [11]. We extend the step of selection of elements for refinement with other error indicator and other selection criteria than were used in initial version of algorithm, providing numerical results for each of them.

The paper is structured according to the following order: in section 2 we define model problem; in section 3 we provide some specific problem transformations; in section 4 we construct variational formulation; in section 5 we make review of general finite element method schemes; section 6 we provide error estimators which will be used in adaptation algorithm; in section 7 we introduce various element selection criteria; in section 8 hp-adaptation algorithm is described. In section 9 we demonstrate some numerical results. Final conclusions are given in the last section.

Main Results

1. Model problem. We consider the following convection-diffusion-reaction boundary value
problem:

$$\begin{cases} \textit{given } \bar\mu = \bar\mu(x),\ \bar\beta = \bar\beta(x),\ \bar\sigma = \bar\sigma(x),\ \bar f = \bar f(x); \\ \textit{and } \bar\alpha,\ \bar\gamma,\ \bar g_0,\ \bar g_L \in \mathbb{R}, \\ \textit{find function } u = u(x) \textit{ such that} \\ -\dfrac{d}{dx}\left(\bar\mu \dfrac{du}{dx}\right) + \bar\beta \dfrac{du}{dx} + \bar\sigma u = \bar f \text{ on } G = (0, L); \\ \bar\mu \dfrac{du}{dx}\Big|_{x=0} = \bar\alpha[u(0) - \bar u_0],\ -\bar\mu \dfrac{du}{dx}\Big|_{x=L} = \bar\gamma[u(L) - \bar u_L]. \end{cases} \tag{1.1}$$

2. Scaling of variables. In order to show specific of the boundary value problem (1.1) we introduce a scaled variable $t \in [0, 1]$, in such way that $x := Lt$ transforming dependent variables

$$\begin{cases} \mu := \bar\mu \|\bar\mu\|^{-1}_{\infty,G},\ \beta := \bar\beta \|\bar\beta\|^{-1}_{\infty,G},\ \sigma := \bar\sigma \|\bar\sigma\|^{-1}_{\infty,G},\ f := \bar f \|\bar f\|^{-1}_{\infty,G}, \\ \alpha := \bar\alpha L \|\bar\mu\|^{-1}_{\infty,G},\ \gamma := \bar\gamma L \|\bar\mu\|^{-1}_{\infty,G},\ g_0 := L\|\bar\mu\|^{-1}_{\infty,G} \bar u_0,\ g_1 := L\|\bar\mu\|^{-1}_{\infty,G} \bar u_L, \end{cases}$$

and after small algebra we rewrite problem (1.1) in the following form

$$\begin{cases} \textit{find function } u = u(t) \textit{ such that} \\ -(\mu u')' + Pe[\beta u' + St\sigma u] = f \text{ in } \Omega = (0, 1); \\ (\mu u' - \alpha u)|_{t=0} = g_0,\ -(\mu u' - \gamma u)|_{t=1} = g_1, \end{cases} \tag{2.1}$$

where $v' := \frac{dv}{dt}$, and dimensionless numbers

$$Pe := \frac{L\|\bar\beta\|_{\infty,G}}{\|\bar\mu\|_{\infty,G}},\ St := \frac{L\|\bar\sigma\|_{\infty,G}}{\|\bar\beta\|_{\infty,G}} \tag{2.2}$$

are well-known Peclet criteria and Strouhal criteria respectively.

3. Variational formulation. The boundary value problem (2.1) admits the following variational formulation

$$\begin{cases} \textit{find } u \in V := H^1(\Omega) \textit{ such that} \\ c_\Omega(u, v) = \langle l_\Omega, v\rangle \quad \forall v \in V, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} c_\Omega(u, v) :=& (\mu u', v')_{L^2(\Omega)} + Pe[(\beta u', v)_{L^2(\Omega)} + St(\sigma u, v)_{L^2(\Omega)}] + \\ &+ \alpha uv|_{t=0} + \gamma uv|_{t=1} \;\; \forall u, v \in V, \end{aligned} \tag{3.2}$$

$$\langle l_\Omega, v\rangle := (f, v)_{L^2(\Omega)} + \alpha g_0 v(0) + \gamma g_1 v(1) \;\forall v \in V. \tag{3.3}$$

4. Petrov-Galerkin finite element scheme. Let $\{V_h\}_{h>0}$ be a family of finite element approximation subspaces $V_h \subset V$, $\dim V_h = N_h < +\infty$ is dense in V, moreover, for each $v \in V$ there is $v_h \in V_h$ such that

$$\|v - v_h\|_{H^1(\Omega)} \le Ch^k \|v\|_{H^{k+1}(\Omega)}, C = const > 0. \tag{4.1}$$

Let $\{W_h\}_{h>0}$ be the family of spaces with the same properties as $\{V_h\}_{h>0}$. In general case Petrov-Galerkin finite element scheme deal with following discrete variational problem

$$\begin{cases} \textit{given subspaces } V_h \subset V \textit{and } W_h \subset V; \\ \dim V_h = \dim W_h = N_h < +\infty, \\ \textit{find } u_h \in V_h \textit{ such that} \\ c_\Omega(u_h, v) = \langle l_\Omega, v\rangle \;\forall v \in W_h. \end{cases} \tag{4.2}$$

Let $\{\phi_j\}_{j=1}^N$ and $\{w_m\}_{m=1}^N$ be basis of and spaces respectively. Then problem (4.2) results to system of linear algebraic equations

$$\begin{cases} \textit{given subspaces } V_h \subset V \textit{and } W_h \subset V; \\ \dim V_h = \dim W_h = N_h < +\infty, \\ \textit{find } u_h = \sum\limits_{k=1}^{N} q_k \phi_k \in V_h \textit{ such that} \\ \sum\limits_{k=1}^{N} q_k c_\Omega(\phi_k, w_i) = \langle l_\Omega, w_i\rangle \; i = 1, ..., N. \end{cases} \tag{4.3}$$

In the current paper we use classic Galerkin method i.e. $V_h = W_h$. In general we can use Petrov-Galerkin method to obtain system of linear equations with symmetric matrix by choosing $w_i := z\phi_i$, where

$$z(x) := exp[-Pe \int\limits_0^x \mu^{-1}\beta \, dt], \forall x \in [0, 1]. \tag{4.4}$$

5. A posteriori error estimators of finite element approximations. Let us describe different ways for obtaining error level on each finite element. We can use explicit estimator - an explicit formula which gives us upper bound of error estimate as one number or implicit estimator which is obtained as function which approximates actual error of finite element discretization. To obtain it we solve auxiliary variational problem.

5.1. Explicit error estimator. To estimate error level we can use explicit error estimator i.e. explicit formula which gives us upper bound to approximation error on each finite element. Let us define the *approximation error* $e = u - u_h \in V_h \subset V$, the residual

$$R[u_h] := f + (\mu u_h')' - \beta Peu_h' - \sigma PeStu_h, \tag{5.1}$$

and bubble function

$$\omega_K(x) := (x_k - x)(x - x_{k-1}), supp\omega_K = K, \forall K \in \Im_h. \tag{5.2}$$

Then for each element we can compute the following error indicator [11]

$$\eta_K = \frac{C}{\sqrt{p_K(p_K+1)}} \left\| \sqrt{\omega_K} R\left[u_h^K\right] \right\|_{L^2(K)}, \tag{5.3}$$

where $p_K = \deg(u_h \| K)$ and

$$C = 4[\min(\mu_0, c_0)]^{-1},\ \mu(t) \geq \mu_0 = const > 0,$$
$$\sigma(t) - Pe\beta'(t) \geq c_0 = const > 0 \ a.e.\ in\ \Omega.$$

5.2. Implicit error estimator. Described estimator gives us only one number per finite element which we interpret as error level. To combine h- and p- refinements of elements we need to construct some type of estimator which:

i. will give us distinct error estimate for each of available refinement patterns of element;

ii. will be simply computable on each element.

For this purposes we will solve auxiliary variational problem for error on each element, using finite element method for different finite element spaces. Let us define $X^p(a,b)$ as a space of all polynomials of order p on closed interval $[a,b]$. For all refinement patterns we may define corresponding approximation spaces. We will use only two refinements: division of the element into two elements with the same polynomial orders and increasing element order by one. Corresponding spaces are the following:

$$\begin{aligned} V_{hp}^1(K) &:= \left\{ v \in C_0(K) |\ v \in X^{p_K}(x_{k-1}, x_{k-1/2}), \right. \\ &\qquad\qquad \left. v \in X^{p_K}(x_{k-1/2}, x_k) \right\}, \\ V_{hp}^2(K) &:= \left\{ v \in X^{p_K+1}(K) |\ v|_{\partial K} = 0 \right\}. \end{aligned} \tag{5.4}$$

To obtain error estimate as a single number for finite element error approximation on each space for $m = 1, 2$ we solve the next problems for local error:

$$\begin{cases} \text{find function } e_h^m \in V_{hp}^m(K) \text{ such that} \\ c_\Omega(e_h^m, v_h) = \int\limits_K R\left[u_h^K\right] v_h \, dx, \quad \forall v_h \in V_{hp}(K). \end{cases} \tag{5.5}$$

then error estimates for given two refinements are defined as $r_m = \|e_h^m\|_E, m = 1, 2$.

5.3. Error estimator based on fundamental solution. For error indicator η_K, introduced in 6.1, instead of using explicit formula, defined by error estimate (5.3) we can use implicit indicator in the form of problem (5.5) but with special assumptions [12]:

i. for local error approximation we use 1D space of finite functions;

ii. those functions are constructed for element $K = [t_{k-1}, t_k]$ in the form:

$$\varphi_K(x) = \begin{cases} \varphi_1(t) = c_{11}\varphi_{11}(t) + c_{12}\varphi_{12}(t) \ \ on\ t \in [t_{k-1}, t_{k-1/2}], \\ \qquad \varphi_1(t_{k-1}) = 0, \varphi_1(t_{k-1/2}) = 1 \\ \varphi_2(t) = c_{21}\varphi_{21}(t) + c_{22}\varphi_{22}(t) \ \ on\ t \in [t_{k-1/2}, t_k], \\ \qquad \varphi_2(t_{k-1/2}) = 1, \varphi_2(t_k) = 0, \end{cases} \tag{5.6}$$

where $\{\varphi_{1i}(t)\}, \{\varphi_{2i}(t)\}$ are the sets of fundamental solutions for equation

$$-(\tilde{\mu}u')' + Pe[\tilde{\beta}u' + St\tilde{\sigma}u] = 0 \tag{5.7}$$

with constant coefficients (selected as mean values of corresponding functions) on corresponding intervals $[t_{k-1}, t_{k-1/2}]$ and $[t_{k-1/2}, t_k]$. Then we solve (5.5) and use the energy norm of obtained approximation as an error indicator η_K. To find corresponding fundamental solutions we solve quadratic equation

$$-\tilde{\mu}\lambda^2 + Pe\tilde{\beta}\lambda + PeSt\tilde{\sigma} = 0. \tag{5.8}$$

For the roots λ_1, λ_2 we have three cases possible:

i. if $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2$ then $\varphi_{i1}(t) = \exp(\lambda_1 t), \varphi_{i2}(t) = \exp(\lambda_2 t)$;

ii. if $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 = \lambda_2$ then $\varphi_{i1}(t) = \exp(\lambda_1 t), \varphi_{i2}(t) = t\exp(\lambda_1 t)$

iii. if $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R}, \lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$ then
$\varphi_{i1}(t) = \exp(\alpha t)\sin(\beta t), \varphi_{i2}(t) = \exp(\alpha t)\cos(\beta t)$.

6. Element selection criteria.

To drive adaptation algorithm we need some rule using which we will be able to choose elements for the further refinement procedure. The key and most wide-used idea is to choose elements with greater error, in some sense. Suppose we have calculated error indicators $\{\eta_K\}$ for each elements on the mesh. In this paper we use two selection criteria:

i. (*"maximum"* criteria) element K is refined if

$$\eta_K > (1-\theta)\eta_{max}, \tag{6.1}$$

where $\eta_{max} = \max_K \eta_K$ and $\theta \in (0, 1)$ is fixed value;

ii. (*"average"* criteria) element K is refined if

$$\frac{\sqrt{N}\eta_K}{\sqrt{\|u_h\|_E^2 + \sum_{K'} \eta_{K'}^2}} 100\% > \varepsilon, \tag{6.2}$$

where ε is is acceptable tolerance in % for average error level over finite element, N is element count.

7. Adaptation algorithm. Let us define: TOL - acceptable relative error level in percent, p_{max} - maximum element order (polynomial degree).

Step 1: *Find finite element solution* u_h *on the current mesh* $\Im_h$;

Step 2: *Stop condition check.* For all elements K compute η_K as in section 6.1 (or 6.3). Define $\eta := \left(\sum_{K \in \Im_h} \eta_K^2 \right)^{1/2}$. If $\eta \|u_h\|_E^{-1} \times 100\% < TOL$ we stop the algorithm (TOL is acceptable relative error level in percent), else:

Step 3: *Choose elements for refinement* using (6.1) (or (6.2)). The set of all selected elements we name as A_θ.

Step 4: *Mesh modification.* For all selected elements $K = [x_{k-1}, x_k]$ ($p_K := \deg(u_h|_K)$) choose between bisection and increasing of polynomial degree on it by one. Compute values $r_m = \|e_h^m\|_E, m = 1, 2$ by solving problems (5.5). Consider the difference $\Delta = r_2 - r_1$. If $\Delta > \delta$, where $\delta > 0$ is predefined value, then we increase element order by 1, otherwise we bisect it into two elements with approximation polynomial orders (p_K, p_K).

Step 5: *Go to* Step 1.

Remark 1. *From the theoretical point of view we should set* $\delta = 0$ *but in practice, according to errors in numerical quadratures and round-off errors, furthermore the maximum order of approximating polynomial is bounded - so it's logically to use bisection in case when* Δ *is very small. For this purposes we choose small number* $\delta > 0$.

8. Numerical results. In this chapter we present results of our algorithm for some singular perturbed problems. Parameters δ and θ are equal for all iterations and are selected using search from several values to provide "optimal" values which minimize final number of iterations and final count of degrees of freedom.

We will analyze two sample problems and for each of them we obtain numerical results for four combinations derived from two estimators described in s.6.1 (explicit estimator) and s.6.3 (estimator based on fundamental solution) and two selection criteria (6.1) (or "maximum" criteria) and (6.2) which we will call "average" criteria. Initial mesh size is 7 elements. Average convergence rate was calculated using least squares method.

Problem 1: We consider boundary value problem (1.1) with the following data

$$\mu = 1, \beta = 0, \sigma = 10^5 e^x, f = 10^5, \alpha = \gamma = 10^8, \bar{u}_0 = \bar{u}_L = 0, L = 1.$$

Algorithm parameters are: $TOL = 1\%, p_{max} = 9, \delta = 2, \theta = 0.6$. Figure 1 and table 1 demonstrates algorithm behavior for Problem 1 when using indicators from sections 6.3 and 6.1 in combination with "maximum" criteria (6.1).

We can observe from table 1 than indicator based on fundamental solution can be more efficient but its error is not monotonically decreasing as in example with explicit error indicator.

Figure 2 and table 2 demonstrates algorithm behavior for Problem 1 when using indicators from sections 6.3 and 6.1 in combination with "average" criteria (6.2).

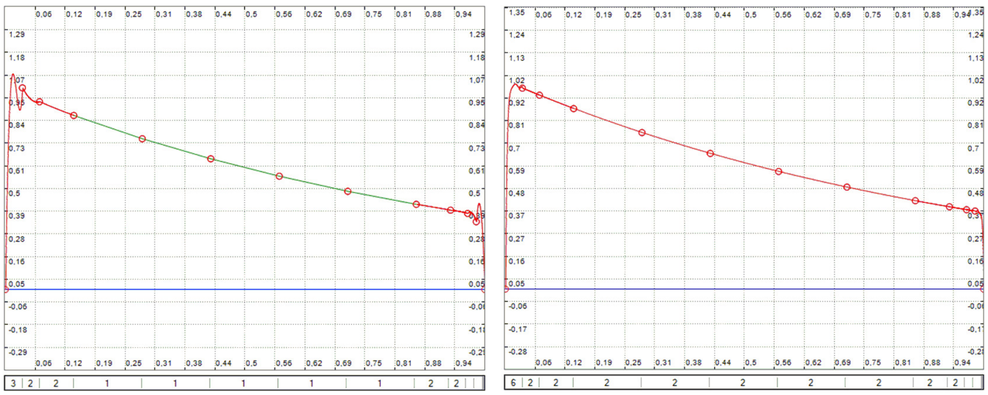

Figure 1: Approximation to solution of Problem 1 using implicit error indicator based on fundamental solution basis which was introduced in section 6.3 (left part) and explicit indicator from 6.1 (right part). Both combined with the "maximum" criteria (6.1). The row in the bottom of each plot shows polynomial degrees on each finite element.

Fundamental solution indicator						Explicit indicator					
n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n	n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n
0	7	8	20.58	8.58		0	7	8	2124.36	886.20	
1	7	10	7.51	3.04	4.51	1	7	10	669.04	271.33	5.17
2	9	14	3.00	1.39	2.29	2	9	14	221.91	89.12	3.27
3	10	16	2.92	1.17	1.29	3	11	18	77.71	31.07	4.17
4	12	20	3.42	1.37	-0.71	4	12	21	29.82	11.90	6.21
5	12	21	1.92	0.76	11.84	5	12	23	12.94	5.16	9.17
						6	12	25	6.10	2.68	7.84
						7	12	32	1.77	0.70	5.39
average rate of convergence 2.37, execution time 208ms						average rate of convergence 5, execution time 330ms					

Table 1: Convergence history for Problem 1 with the "maximum" criteria (6.1): n is an iteration number, N element count, $N_{dof}^{(n)}$ count of degrees of freedom, $\epsilon_n^\Omega = \eta$ absolute error indicator, $r_n^\Omega = \eta \|u_h\|_E^{-1} \times 100\%$ relative error, $p_n = -\left(\ln \epsilon_n^\Omega - \ln \epsilon_{n-1}^\Omega\right) \times \left(\ln N_{dof}^{(n)} - \ln N_{dof}^{(n-1)}\right)^{-1}$ rate of convergence.

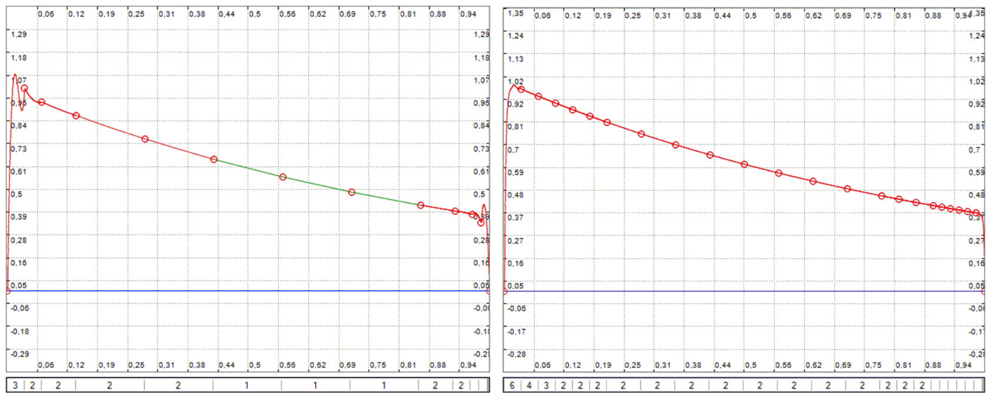

Figure 2: Approximation to solution of Problem 1 using implicit error indicator based on fundamental solution basis which was introduced in section 6.3 (left) and explicit indicator from 6.1 (right). Both combined with the "average" criteria (6.2). The row in the bottom of each plot shows polynomial degrees on each finite element.

Fundamental solution indicator						Explicit indicator					
n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n	n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n
0	7	8	21.43	8.94		0	7	8	2124.30	886.20	
1	7	12	4.08	1.65	4.08	1	7	15	627.39	254.32	1.94
2	9	16	5.44	2.18	-0.99	2	14	29	220.92	88.72	1.58
3	11	20	3.96	1.58	1.42	3	20	41	77.51	30.99	3.02
4	12	23	0.84	0.33	11.04	4	23	50	29.30	11.69	4.90
						5	23	54	12.02	4.79	11.57
						6	23	56	4.82	1.92	25.11
						7	23	58	1.77	0.70	28.48
average rate of convergence 1.88, execution time 197ms						average rate of convergence 3.08, execution time 504ms					

Table 2: Convergence history for Problem 1 with the "average" criteria (6.2).

The results for fundamental solution indicator in table 2 are almost the same as in table 1. Furthermore the behavior of the error has the same specialty with small peak inside. In general for Problem 1:

i. Comparing table 2 with example from table 1 we can see explicit indicator was twice more deficient in according to iterations, element and d.o.f. counts.

ii. Error is monotonic in relation to count of d.o.f.

iii. From examples above we can clearly see that convergence rate is monotonically increasing so this is the fact that approves the possibility of presence of exponential convergence.

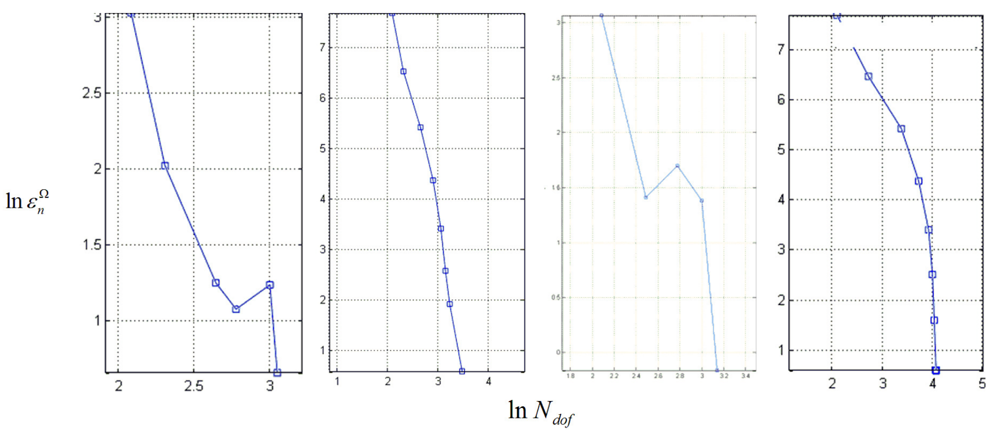

Figure 3: Dependency between absolute error indicator ϵ_n^{Ω} and number of degrees of freedom $N_{dof}^{(n)}$ in logarithmic scale for Problem 1. a) for algorithm with indicator based on fundamental solution described in section 6.3 and "maximum" criteria (6.1); b) for algorithm with explicit error indicator from section 6.1 and "maximum" criteria (6.1); c) for algorithm with indicator based on fundamental solution described in section 6.3 and "average" criteria (6.2); d) for algorithm with explicit error indicator from section 6.1 and "average" criteria (6.1).

Also we can conclude from numerical examples for Problem 1: fundamental solution-based indicator is a bit more efficient but it is not providing monotonically convergent error. For the last example we possibly observed exponential convergence. Comparing two criteria for element selection leads to conclusion that "maximum" criteria from (6.1) is a bit more efficient than "average" criteria (6.2) but in general they are almost identical in according to numerical results.

Problem 2: We consider boundary value problem (1.1) with the following data

$$\mu = 1, \beta = e^{2x} - 2 \cdot 10^2, \sigma = 10^2(\cos x + 2), f = 10^3 e^{-10^2(x-0.5)^2},$$
$$\alpha = \gamma = 10^8, \bar{u}_0 = \bar{u}_L = 0, L = 1.$$

Algorithm parameters are: $TOL = 1\%, p_{max} = 9, \delta = 0, \theta = 0.6$. Figure 4 and table 3 demonstrates algorithm behavior for Problem 2 when using indicators from sections 6.3 and 6.1 in combination with "maximum" criteria (6.1).

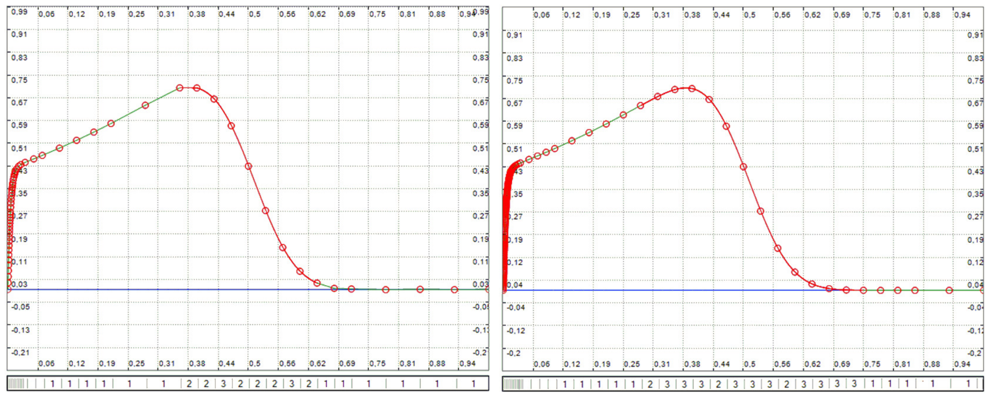

Figure 4: Approximation to solution of Problem 2 using implicit error indicator based on fundamental solution basis which was introduced in section 6.3 (left) and explicit indicator from 6.1 (right). Both combined with the "maximum" criteria (6.1). The row in the bottom of each plot shows polynomial degrees on each finite element.

Fundamental solution indicator						Explicit indicator					
n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n	n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n
1	7	8	19.97	118.83		0	7	8	53.86	320.53	
1	12	13	16.62	96.31	0.37	1	11	12	33.56	194.66	1.16
2	15	16	3.8	27.81	7.09	2	14	15	18.85	135.13	2.58
3	17	18	5.07	48.95	-2.42	3	16	17	11.07	104.86	4.25
4	18	19	1.66	18.85	20.65	4	17	19	5.09	57.91	6.96
5	19	20	1.15	12.1	7.1	5	18	20	3.44	36.14	7.64
6	20	23	0.65	6.86	4.04	6	20	26	1.54	16.25	3.04
7	26	31	0.32	3.33	2.39	7	24	30	0.92	9.63	3.61
8	33	40	0.32	3.3381	0	8	35	44	0.48	5.02	1.7
9	37	44	0.27	2.88	1.59	9	56	69	0.22	2.34	1.69
10	37	46	0.21	2.26	5.51	10	85	101	0.11	1.24	1.6
11	42	53	0.15	1.57	2.53	11	138	160	0.06	0.66	1.35
12	58	69	0.08	0.93	1.98						
average rate of convergence 2.72, execution time 825ms						average rate of convergence 2.29, execution time 1248ms					

Table 3: Convergence history for Problem 2 with the "maximum" criteria (6.1).

As in the Problem 1 results error is not monotonic for indicator based on fundamental solution and it is monotonic for explicit indicator i.e. exactly the same behavior as in the Problem 1.

Figure 5 and table 4 demonstrates algorithm behavior for Problem 1 when using indicators from sections 6.3 and 6.1 in combination with "average" criteria (6.2).

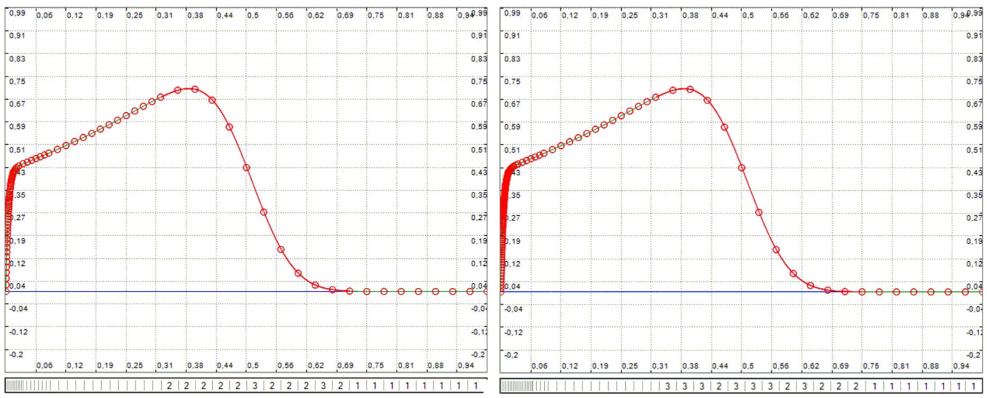

Figure 5: Approximation to solution of Problem 2 using implicit error indicator based on fundamental solution basis which was introduced in section 6.3 (left) and explicit indicator from 6.1 (right). Both combined with the "average" criteria (6.2). The row in the bottom of each plot shows polynomial degrees on each finite element.

Fundamental solution indicator						Explicit indicator					
n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n	n	N	$N_{dof}^{(n)}$	ϵ_n^Ω	r_n^Ω	p_n
0	7	8	9.17651	54.6	-	0	7	8	53.86	320.53	
1	14	15	11.58043	67.35	0.37	1	14	15	33.33	193.88	0.76
2	25	29	10.38023	75.74	0.16	2	25	29	17.3	126.26	0.99
3	36	45	3.89025	37.63	2.23	3	37	48	8.75	84.66	1.35
4	41	51	1.96705	22.38	5.44	4	42	56	4.38	49.93	4.47
5	44	58	1.03781	10.91	4.97	5	44	60	2.19	23.11	10.02
6	49	63	0.4601	4.83	9.83	6	49	65	1.09	11.54	8.64
7	58	72	0.23279	2.43	5.1	7	60	77	0.55	5.76	4.08
8	69	83	0.11914	1.24	4.71	8	77	95	0.27	2.89	3.27
9	84	98	0.06066	0.63	4.06	9	100	119	0.14	1.48	2.96
						10	129	149	0.07	0.83	2.56
average rate of convergence 2.58, execution time 1222ms						average rate of convergence 2.44, execution time 1719ms					

Table 4: Convergence history for Problem 2 with the "average" criteria (6.2).

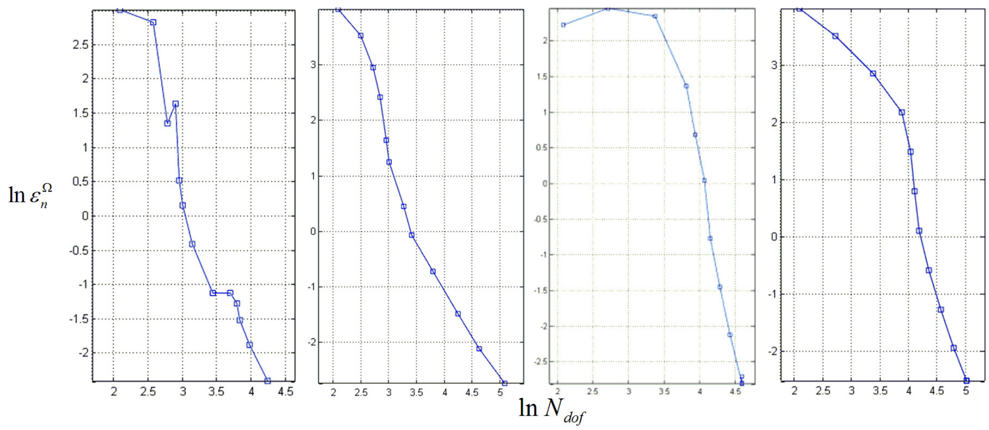

Figure 6: Dependency between absolute error indicator ϵ_n^Ω and number of degrees of freedom $N_{dof}^{(n)}$ in logarithmic scale for Problem 2. a) for algorithm with indicator based on fundamental solution described in section 6.3 and "maximum" criteria (6.1); b) for algorithm with explicit error indicator from section 6.1 and "maximum" criteria (6.1); c) for algorithm with indicator based on fundamental solution described in section 6.3 and "average" criteria (6.2); d) for algorithm with explicit error indicator from section 6.1 and "average" criteria (6.1).

In general we can see from provided numerical examples that: 1) the better choice in according to count of elements, iterations and d.o.f. reached is a combination of the indicator from chapter 6.3 based on fundamental solution and "maximum" criteria (6.1); 2) there is no large difference between "maximum" and "average" selection criteria; 3) if we need to have monotonic error decreasing we need to choose explicit indicator from 6.1.

Conclusion. In this work we constructed *hp*-adaptive algorithm for solving the diffusion-advection-reaction boundary value problems with self-adjoint operators. We proved the optimality in some sense of refinement selection step used in algorithm. Also we introduced symmetrization procedure which can be used to transform given nonsymmetrical variational problem to equivalent symmetric problem, therefore making possible application of constructed algorithm to nonsymmetrical problems too. Also we studied precisely conditions which problem data needs to satisfy to make boundary problem well-posed.

To drive adaptation process we introduce two a posteriori error estimators. For element selection for refinement procedure we use explicit estimator, i.e. explicit formula which gives upper bound of actual error on finite element. After elements for refinement were selected we need to choose on each element refinement pattern: bisection with original element order preservation or increment of polynomial degree on element by one. For this purpose we use classic implicit error estimator (i.e. in the form auxiliary variational problem for error function). Using explicit estimator gives us way of homogeneous computation of per-element error, needed for proper selection elements for refinement. Respectively, using auxiliary error problem gives us elegant way to choose between different types of elements refinement.

In the end we present comparative analysis of numerical results obtained using combination of introduced algorithm with different kinds of a posteriori error estimators, and element refinement criteria.

1. **Brenner S.** The Mathematical Theory of Finite Element Methods / S. Brenner, L. Scott. – Springer, 2008, 3ed. – 404p.
2. **Schwab Ch.** p and hp-Finite Element Methods // Clarendon Press, 1998.
3. **Demkowicz L.** Computing with hp-adaptive finite elements. I. One- and Two Dimensional Elliptic and Maxwell Problems / L. Demkowicz. – Austin, 2005. – 418 p.
4. **Doolan E.P.** Uniform numerical methods for problems with initial and boundary layers / E.P. Doolan, J.J.H. Miller, W.H.A. Schilders. – BOOLE PRESS, 1980. – 198 p.
5. **Dorfler W.** Dorfler W. Convergence of an adaptive hp finite element strategy in one space dimension. / W. Dorfler, V. Heuveline // Applied Numerical Mathematics. – 2007. – No.57, 1108–1124.
6. **Logan J. D.** Transport modeling in hydrogeochemical systems. New York: Springer, 2001. – 226 p.
7. **Sinchuk Y. O.** Finite element method approximation with exponential weighting functions / Y. O. Sinchuk, G. A. Shynkarenko // App. Problems Mech. Math. – 2007. – 5. – P. 61–70 (in Ukrainian).
8. **Solin P.** Higher-Order Finite Element Methods / P. Solin, K. Segeth, I. Dolezel. – Chapman & Hall, 2003. – 388 p.
9. **Solin P.** Static Condensation, Partial Orthogonalization of Basis Functions, and ILU Preconditioning in the hp-FEM / P. Solin, T. Vejchodsky // The University of Texas at El Paso, Department of Mathematical Sciences Research Reports Series, Research Report. – 2007. – No.4.
10. **Rektorys K.** Variational Methods in Mathematics, Science and Engineering. Second edition. – Dr. Reidel Publishing Company, 1980. – 589 p.
11. **Drebotiy R.** HP-adaptive finite element method for 1d diffusion-convection-reaction boundary value problems / R.Drebotiy, H.Shynkarenko. – Opole 2014, in preparation.
12. **Сінчук Ю. О.** Апостеріорний оцінювач похибки апроксимацій МСЕ для задач конвекції-дифузії / Ю. О. Сінчук, Г. А. Шинкаренко // Вісник Тернопільського держ. техн. ун-ту. – 2008. – Т. 13, №2. – С.169–174.

Received 08.08.2014