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Background of the multimodal method in the sloshing problem *

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Мультімодальний метод зводить задачу з вільною поверхнею про коливання рідини в баках до скінченовимірних (модальних) систем звичайних нелінійних диференціальних рівнянь. Метод передбачає ідеальну нестисліву рідину з безвихоровими течіями, але здатен враховувати дисипацію. Його було запропоновано для неімпульсних гідродинамічних навантажень на бак, але останні дослідження показали, що його можна вдало комбінувати з іншими аналітичними методами для моделювання імпульсних навантажень у цілому і сламінгу, зокрема. В 50–60 роках XX сторіччя метод розвивався як засіб обчислювальної гідродинаміки, але з 1990 років він програє іншим спеціалізованим чисельним алгоритмам. На сьогодні він має подвійне значення. З одного боку, метод є унікальним аналітичним інструментом для параметричних досліджень нелінійних режимів коливання рідини, їх стійкості, оцінки виникнення хаосу. Його використовуть для прямих розрахунків, коли традиційні чисельні схеми не працюють, приміром, для баків з перфорованими екранами. З іншого боку, існуючі слабонелінійні модальні системи є аналогом рівнянь Кордевега–де–Вріза, Бусінеска, чи аналогічних рівнянь теорії поверхневих хвиль, тому є цікавими як математичний об'єкт.

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Мультимодальный метод сводит задачу со свободной поверхностью про колебания жидкости в баках к конечномерным (*модальным*) системам обыкновенных нелинейных дифференциальных уравнений. Метод предполагает идеальную несжимаемую жидкость (безвихревые течения), однако способен учитывать диссипацию. Его предложили для неимпульсных гидродинамических нагрузок на стенки бака, но последние исследования показали, что он успешно комбинируем с другими аналитическими методами для моделирования импульсных нагрузок, в целом, и сламминга, в частности. В 50-60 годы метод развивался как средство вычислительной гидродинамики, но с 90-00 годов он проигрывает специализированным численным алгоритмам. В настоящее время метод играет двойную роль. *С одной стороны*, он является уникальным аналитическим инструментом для параметрических исследований нелинейных режимов, их устойчивости, оценки возникновения хаоса. Метод используется и для прямых расчетов, когда традиционные численные схемы не работают, например, для баков с перфорированными экранами. *С другой стороны*, существующие слабо-нелинейные модальные системы являются аналогом уравнений Кордевега-де-Вриза, Буссинеска, или подобных уравнений теории поверхностных волн, поэтому интересны как математический объект.

1. Genesis

The *coupled* "rigid tank–contained liquid" dynamics is associated with the so-called *hybrid mechanical systems* consisting of two coupled subsystems of diverse mechanical and mathematical nature. The first subsystem, the rigid tank, can move with six degrees of freedom and, therefore, its dynamic governing equations are a six-dimensional system of ordinary differential equations (ODEs). The hydromechanical subsystem, the contained liquid, is governed by a free-surface problem and, as consequence, implies an infinite number of degrees of freedom. Based on the free-surface problem or its Lagrange variational formulation, the multimodal method makes it possible to select, constructively, the hydrodynamic-type generalised coordinates and derives a system of ODEs which plays the role of approximate (Euler–Largange) hydrodynamic equations.

Etymology of the word *"multimodal"* in the liquid sloshing problems comes, most probably, from the sentence "nonlinear *multimodal* analysis" appearing in the title of the paper [16]. However, the multimodal method was not originated in 2000 but forty-fifty years behind, in the 50-60's, when aircraft, spacecraft and marine applications used to be a great challenge for applied mathematicians and engineers involved in

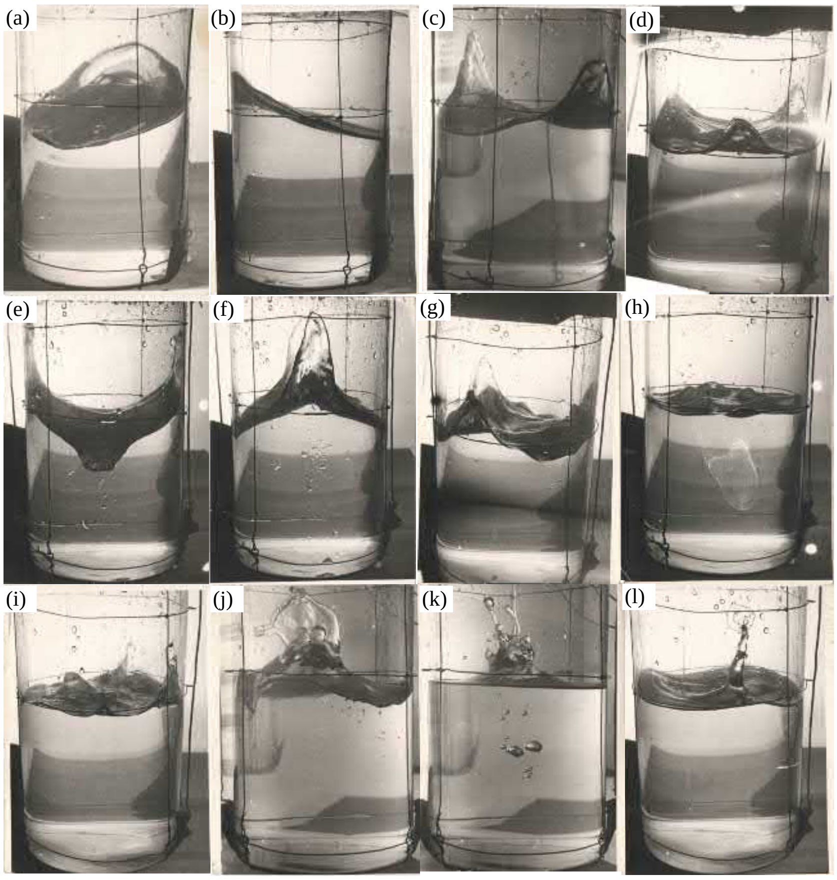

Fig 1. The photos made in the 60's by Kiev's research group demonstrate the instant free-surface patterns for diverse resonant tank excitations that illustrate the free-surface nonlinearity. The cases (a–d) depict a weakly-nonlinear sloshing when the lowest [degenerating] modes have the dominate character. The second and third antisymmetric modes are nonlinearly excited in the cases (c) and (d) but a stronger amplification of these modes is shown in (e), (f), and (g). Even though the contained liquid is almost at the rest (h), the nonlinearity may cause very steep wave profiles. The pictures (i-l) illustrate breaking waves, overturning, bubbling (k), and other strongly nonlinear phenomena leading to a free-surface fragmentation.

studying the dynamics of vehicles carrying a partially-filled container. All experimental, computational and theoretical aspects were of interest. An enthusiastic atmosphere of these years is well expressed in the memoir article by H. Abramson [2] who was a leader of the corresponding NASA research program. A scientific heritage of these years reported, partly, in the fundamental books and monographs issued, basically, in the USA [3, 25, 32, 86, 92, 93] and the Soviet Union [1, 24, 34, 50, 51, 61–63, 77].

The model tests made it possible to establish input parameters for which the free-surface nonlinearity matters. The USA collection of experiments is well reviewed in the NASA Report [3]. Systematic experiments of the Soviet scientists were conducted in Moscow, Kiev, Dnipropetrovsk, and Tomsk. These are badly known from the open literature. Fig. 1 shows the photos made in the 60's by Kiev' research group. They demonstrate an increasing importance of the free-surface nonlinearity becoming visible due to steep wave patterns, breaking, overturning, bubbling, and the free-surface fragmentations.

Linear multimodal method. Summarizing the main results of the 50-60's, one should emphasise, among the aforementioned books and reports, the papers [68, 73, 76] and the books [24, 50, 63] where the concept of the *linear multimodal method* was originated and developed for an ideal incompressible contained liquid with an irrotational flow. According to this concept, the coupled "rigid body–contained liquid" dynamics is treated as a conservative mechanical system with an infinite number of degrees of freedom whose generalised coordinates are explicitly chosen as the time-depending amplitude parameters at the natural sloshing modes. The linear multimodal method derives an infinite-dimensional system of linear ODEs with respect to the six generalised coordinates governing the small-magnitude rigid body motion and an infinite set of the *hydrodynamic* generalised coordinates responsible for linearly-perturbed (relative to the hydrostatic liquid shape) natural sloshing modes. The latter hydrodynamic-type subsystem was called the *linear modal system*. The derivation of the linear modal system requires to know the natural *sloshing modes and frequencies* and the so-called *Stokes–Joukowski potentials*.

The Stokes–Joukowski potentials were introduced by the "father of the Russian aviation", Nikolay Joukowski (1885) [33], who studied a spatially-moving rigid body with a cavity completely filled by an ideal incompressible liquid. He showed that the liquid moves as a rigid body due to the translatory body motions but the angular body motions lead to specific liquid flows which are described by the Stokes–Joukowski velocity potentials. The potentials are solutions of the corresponding Neumann boundary value problems. Finding the Stokes–Joukowski potentials for the linear sloshing problem implies solving these Neumann boundary problems in the mean (hydrostatic) liquid domain.

The spectral boundary problem on the natural sloshing modes and frequencies (the natural spectral sloshing problem, NSSP) is also formulated in the mean liquid domain so that the spectral parameter κ appears in the mean free-surface boundary condition and defines the natural sloshing frequencies by $\sigma = \sqrt{\kappa g}$ (g is the gravity acceleration). The natural sloshing modes determine the standing surface wave patterns; they are the eigenfunctions of the NSSP.

The standing waves (the natural sloshing modes) on the free surface in an upright circular basin were first described by Mikhail Ostrogradskii (his manuscript [74] was submitted to the Paris Academy of Sciences in 1826). These standing waves for some other basins were further analysed by Poisson and Rayleigh. However, a rigorous mathematical theory of the NSSP was created only in the 60's. The theory establishes the purely positive pointer spectrum consisting of an infinite set of eigenvalues. The eigenvalues have the only limited point at the infinity. This is in the contrast to the water wave theory which is characterised by a continuous spectrum. The latter fact is extremely important for understanding why the water wave theory dealing with the unlimited liquid volume yields approximate models in terms of partial differential equations but the same nonlinear boundary problem in a bounded liquid domain (sloshing problem) leads, after implementation of the multimodal method, to the ODEs appearing as the corresponding approximate mathematical models. Representative mathematical publications on the NSSP are [12] and Ch. VI in [24]. S. Krein [36] has generalised these spectral theorems for a viscous incompressible liquid, but N. Kopachevskii proved the corresponding spectral theorems for a capillary liquid (see, Part II in [66]).

Coupling the linear modal systems and the dynamic equations for the rigid container becomes a mathematically simple procedure provided by the so-called linearised *Lukovsky formulas* which express the hydrodynamic forces and moments in terms of the introduced hydrodynamic generalised coordinates. The *hydrodynamic coefficients* in both the linear modal system and the Lukovsky formulas are integrals over the Stokes–Joukowski potentials and the natural sloshing modes as well as their derivatives. After adding the corresponding initial conditions responsible for initial shapes and velocities of the contained liquid and the rigid body, the Runge–Kutta type simulations of the coupled linear ordinary differential equations enable to describe the linear transient "rigid body–contained liquid" dynamics. For the harmonic loads to the rigid tank, one can also find, analytically, the periodic (steady-state wave) solution.

A canonical description of the linear multimodal method is reported in [24, 44]. Faltinsen & Timokha [17], Ch. 5, presents a contemporary treatment of the linear multimodal method.

Because getting the hydrodynamic coefficients in the linear modal equations exclusively depends on whether the natural sloshing modes and the Stokes–Joukowski potentials are known, the majority of the 50-80's publications on the linear multimodal method were devoted to constructing approximate solutions of the NSSP and the Neumann boundary value problem on the Stokes–Joukowski potentials. A lot of brilliant ideas on how to construct the analytically approximate solutions are collected in [44]. One of these ideas consists of constructing these solutions as a decomposition of the harmonic polynomials whose completeness in the so-called star-shaped domains is proved in [94, 95]. This and other Trefftz type solution methods are well outlined in [23, 44]. The harmonic polynomials are a basis for the Harmonic Polynomial Cell (HPC) method proposed a few years ago [81].

Thanks to new computer facilities and methods developed in the 90's, numerical solutions of these spectral and Neumann boundary problems can effectively be found by using various open and/or commercial solvers based on, e.g., the finite element method. This means that the multimodal method gives, based these numerical solutions, an analytical solution of the linear sloshing problem and, employing that solution, of the *linear* coupled "rigid tank–contained liquid" problem. When can the aforementioned solvers be invalid? An example is when the natural sloshing modes are characterised by the singular behaviour at, e.g., sharp edges (screen's openings, baffles, ribs, etc.), causing a weak convergence of the standard numerical packages. Accounting for this behaviour may significantly improve the convergence [18–21, 23].

Nonlinear sloshing analysis of the 50-60's. The 50-60's studies founded most important directions in the *nonlinear sloshing* analysis. Under certain circumstances, these directions can be associated with original works by N.N. Moiseev [60], G.S. Narimanov [69], and L. Perko [64, 75].

N.N. Moiseev [60] constructed an asymptotic steady-state wave (periodic) solution of the nonlinear free-surface problem for an upright tank performing a prescribed horizontal and/or angular harmonic motion with the forcing frequency σ close to the lowest natural sloshing frequency σ_1. He suggested a finite liquid depth of an ideal incompressible liquid with irrotational flows. Moiseev *proved* that, if the nondimensional forcing amplitude is a small parameter of the order $\epsilon \ll 1$, the primary excited mode(s) amplification is characterised by the order $\epsilon^{1/3}$ and the matching asymptotics (the so-called Moiseev detuning) is $|\sigma^2 - \sigma_1^2|/\sigma_1^2 = O(\epsilon^{2/3})$. Moiseev implicitly assumed that there are no the so-called *secondary resonances*. The Moiseev type asymptotic steady-state solutions were analytically constructed for two-dimensional rectangular [15, 71] and some other [6, 32, 49, 85] tank shapes. Analytically, these were derived in terms of the natural sloshing modes which were known in analytical (or analytically approximate) form. The natural sloshing modes are analytically expandable over the mean free surface (together with higher-order derivatives) for these tank shapes. Getting the Moiseev asymptotic solutions leads to tedious analytical derivations.

Whereas Moiseev has exclusively focused on the resonant steady-state sloshing occurring due to a prescribed harmonic tank excitation, **G.S. Narimanov** [69], bearing in mind a simulation of both steady-state and transient sloshing, proposed a perturbation technique deriving a *weakly-nonlinear* analogy of the linear modal equations. The weakly-nonlinear modal equations also facilitate studying the coupled "tank-liquid" dynamics. To derive the weakly-nonlinear modal equations, Narimanov [69] *postulated* a set of asymptotic relations between the hydrodynamic type generalised coordinates. The latter coordinates are introduced in the same way as in the linear case. Even though Narimanov did not know Mosieev's results which were published one year later, the used asymptotic relations between the hydrodynamic generalised coordinates are, in fact, the same as those following from Moiseev's analysis if the Moiseev periodic solution is re-expressed in terms of the Fourier solution by the natural sloshing modes. The Narimanov type multimodal method was originally developed for an upright circular cylindrical tank, but V. Stolbetsov [87–90] and I.A. Lukovsky [45,70] generalised it to other tank shapes. Up to date, the Narimanov type modal systems are derived for upright tanks of circular, annular and rectangular cross-sections, conical and spherical tanks as well as for an upright circular cylindrical tank with a rigid-ring baffle [28, 45, 47, 70].

One should note that the original Narimanov expressions contain *algebraic errors*. The right (corrected) expressions are reported in Lukovsky' works published after 1975 (see, e.g., [45,47,70]). Unfortunately, as it has been mentioned in the Moiseev case, constructing the Narimanov-type weakly-nonlinear modal equations leads to difficult and tedious derivations that dramatically increase when introducing a long set of the hydrodynamic generalised coordinates. As a result, all the existing Narimanov-type modal systems are of a low dimension – they couple from two to five generalised coordinates only.

Due to a lack of computer facilities and suitable numerical methods, engineering computations based on the space-and-time discretisation of the free-surface sloshing problem are little presented in the literature of the 60-70's. They would appear *en masse* only in the 90's. A unique exception is the so-called **Perko** method [64,75] which can be interpreted as both a *numerical version* of the multimodal method and a CFD solver. The Perko method uses, on the one hand, the Galerkin projective scheme and, on the other hand, the natural sloshing modes in a Fourier-type presentation of the velocity potential. The Perko method is an origin of some other numerical and semi-analytical techniques developed in the forthcoming years. In the 00's [37, 80], the method was combined with the Bateman–Luke variational statement.

2. Origins of the nonlinear multimodal method

I.A. Lukovsky and J.W. Miles [46, 47, 52] employed the Bateman-Luke variational formulation [5, 43] to derive the fully-nonlinear modal equations and, applying the Narimanov–Moiseev asymptotic relations, their approximate weakly-nonlinear form. J.W. Miles [53, 54] generalised the Moiseev results to study an amplitude modulation of the weakly-nonlinear nearly steady-state sloshing (almost periodic solutions) for the case of a harmonically-excited tank, again, by using the Bateman-Luke variational formalism. Finally, O.S. Limarchenko [38–41] proposed a weakly-nonlinear version of the Perko method combining it with the classical Lagrange variational principle and a Galerkin projective scheme which solves the kinematic part of the sloshing problem. These works are the origins for different versions of the nonlinear multimodal method.

Lukovsky–Miles' modal equations. In 1976, Lukovsky and Miles have independently utilised the Bateman–Luke variational formulation for derivations of a fully-nonlinear modal system. In their original papers [47, 52], the authors considered a prescribed harmonic translatory tank motions but, later on, Lukovsky [16, 46] generalised the derivations for an arbitrary prescribed rigid tank motion and proposed the so-called nonconformal mapping technique [22, 45, 48] to get the fully-nonlinear modal equations for tanks with non-vertical walls. He also derived the so-called [fully-nonlinear] Lukovsky formulas for the hydrodynamic force and moment [47] (Ch. 7 of [17] gives an alternative derivation) and, furthermore, showed how to use the Bateman–Luke formalism in derivations of the dynamic equations of the coupled "rigid tank–contained liquid" mechanical system [47]. Because both kinematic and dynamic relations of the free-surface problem naturally follow from the Bateman–Luke variational formulation, these modal equations fall into kinematic and dynamic subsystems which appear as the first-order infinite-dimensional systems of ODEs coupling the hydrodynamic-type generalised coordinates and velocities. Pursuing an approximate finite-dimensional system of the second-order differential (modal) equations, as it has been in Narimanov's case, Lukovsky and some other authors adopted the Narimanov–Moiseev asymptotics for the generalised coordinates and "velocities". A series of those weakly-nonlinear modal systems was derived and used for analytical studies of steady-state resonant regimes and transient wave motions.

Derivations of the Lukovsky–Miles modal equations assume that we have got analytically approximate natural sloshing modes and the Stokes–Joukowski potentials analytically defined over the mean free surface. This is a serious limitation of the method preventing its generalisation to arbitrary tank shapes that explains why an extensive use of the fully-nonlinear modal equations and their finite-dimensional weakly-nonlinear versions did start only from 2000, exploiting a treasure of analytical methods which construct approximate natural sloshing modes. The methods have been worked out in the 70-90's.

Miles' equations. Bearing in mind a generalisation of Moiseev's results on the resonant steady-state wave regimes, Miles [53, 54] derived the so-called *Miles equations* which govern a slow-time variation of dominant amplitudes of an almost periodic sloshing occurring due to a small-amplitude horizontal harmonic excitations of an upright circular cylindrical tank; the forcing frequency is close to the lowest natural sloshing frequency. He adopted the Moiseev asymptotic ordering, the Moiseev detuning, and the multiple time scales technique. Separation of the fast and slow time scales was done directly in the Bateman–Luke action. According to the Narimanov–Moiseev asymptotics, there are four (or less) independent slowly-varying amplitudes for upright cylindrical tanks. These are of the $O(\epsilon^{1/3})$-order (ϵ is the nondimensional forcing amplitude). The Miles equations were later derived for upright tanks of the rectangular cross-section.

Using Miles equations is a rather popular approach in applied mathematical studies on classifying a nearly steady-state sloshing, detecting periodic orbits and clarifying the chaos in the hydrodynamic systems. Both horizontal and vertical (Faraday waves) harmonic excitations have been in focus [26, 27, 30, 31, 54–59].

Krasnopolskaya and Shvets [35, 82] extended the Miles technique to the case of the "rigid tank–contained liquid" mechanical system with a limited power supply forcing.

Employing the Perko-type method. In the 70's, the Perko method was rarely used. An exception is the works [13, 65] in which the Galerkin type numerical schemes were proposed that are rather similar to that by Perko. Instead, the Perko method was adopted as a component of computational versions of the multimodal method [9, 11, 38, 39]. The focus was on simulating weakly-nonlinear transient sloshing and the weakly-nonlinear tank–contained liquid dynamics. Those computational versions are well exemplified by that of O.S. Limarchenko [38–42] who combined the Perko method and the classical Lagrange variational formulation. This variational formulation exclusively leads to the dynamic boundary condition (the pressure balance on the free surface) but the kinematic relations should be considered as a constraint. Adopting a modal-type solution, Limarchenko solves the kinematic constraint by a Galerkin projective method (similar to that by Perko), but the dynamic modal equations are obtained with using the variational formulation as in the works by Lukovsky and Miles. Furthermore, the kinematic and dynamic equations are recombined to derive a finite-dimensional system of the second-order ODEs. These (i) are weakly-nonlinear, contain only second- and third-order polynomial nonlinearities, (ii) are based on *a priori* postulated dominant and higher-order generalised coordinates, (iii) include the zero-hydrodynamic coefficients which are difficult to select in an analytical way, and, therefore, (iv) are not applicable for analytical studies but rather for *ad hoc* simulations of transients by employing the Runge-Kutta solvers. In other words, the Perko-type weakly-nonlinear modal equations appear as a specific computer package (not an applied mathematical tool) which is not explicitly available for interested readers of the corresponding papers.

3. Computational fluid dynamics

Extensive CFD simulations of the nonlinear sloshing had actually started only in the 70-80's. The primary focus was on the finite difference (maker-and-cell, etc.) and finite element methods [4, 14, 67, 72, 78, 83, 91]. Along with publishing open-source algorithms, the FLOW-3D package was founded out to become one of the leading commercial Navier-Stokes solvers. A collection of FLOW-3D simulations was presented by F. Solaas [84] who discussed its advantages and drawbacks.

In 2000, the papers [37, 80] tried to breath a new life into the Perko method by combining it with the Bateman-Luke variational formalism. In fact, they employed the truncated fully-nonlinear Lukovsky–Miles modal system in the time-step integrations with an appropriate initial conditions. A drawback of this approach is that this modal system is unrealistically *stiff* so that special artificial damping terms should be incorporated to damp the rising parasitic higher harmonics. This is the same as discussed in [13].

The 90-10's have opened a new, computational era in simulating the nonlinear sloshing of a viscous liquid. Volume of Fluid (VoF), Smoothed Partitions Hydromechanics (SPH), and their modifications made it possible, using parallel computations, rather accurate and efficient computations of transient nonlinear sloshing and the coupled liquid-tank dynamics. Interested readers are referred to [10] which outlines the state-of-the-art of the 90's. The *recent advances* in the *numerical sloshing* are reported in [29, 79, 96].

Advantages of the contemporary CFD are that it is normally based on viscous and fully-nonlinear statement and allows for modelling specific free-surface phenomena associated with (i) the free-surface fragmentation, (ii) wave breaking, (iii) overturning (typically at the walls), (iv) roof and wall impacts, flip-through. The phenomena are partly illustrated in Fig. 1. Against these abilities, the existing modal systems and the Perko-type computational schemes look rather poor. They (i) are often of weakly-nonlinear nature and based on ideal potential liquid motions, (ii) unable to describe the aforementioned specific free-surface phenomena, (iii) lead to stiff time-step simulations, (iii) need an extensive validation by experiments due to the so-called secondary resonances to be correctly predicted and numerous physical and mathematical assumptions done during derivation of the finite-dimensional modal systems. Perhaps, this partly explains why the multimodal method was almost forgotten in the 90's when the main emphasis was placed on constructing efficient CFD solvers.

In the 90-10's, *one can say that the multimodal method loosed the competition to the CFD* as a computational tool for simulating the liquid sloshing dynamics. To some extend, the situation is similar to that in the water wave theory and associated Kordeweg–de Vries, Boussinesq, etc. equations which are currently of little interest in engineering computations where various CFD packages with parallel computing schemes play the practical role. This clarifies why the Perko-type methods, even in their modified form [37, 80], are no more developing in the literature. The scopes of the multimodal method remain the analytical (mostly applied mathematical) studies as well as exceptional cases when traditional CFD methods are inefficient. The latter is well exemplified by sloshing in tanks within a perforated screen.

4. Reincarnation

A new era of the nonlinear multimodal method, now as, basically, an analytical tool, has started in 2000, due to the paper [16]. The paper re-derived the fully-nonlinear Lukovsky modal equations and proposed a three-degrees-of-freedom weakly nonlinear modal system handling the resonant liquid sloshing due to horizontal excitations of the lowest natural sloshing frequency. The finite liquid depth and the so-called Narimanov–Moiseev intermodal asymptotics were assumed. Experimental model tests were done, in particular, for establishing limitations of the derived weakly-nonlinear modal system. The modal system becomes invalid for with decreasing the mean liquid depth, roof impact, increasing the forcing amplitude and, in some cases, due to passage to three-dimensional wave patterns. Both qualitative analytical studies of steady-state resonant sloshing and direct numerical simulations of transient waves by the weakly-nonlinear modal system were presented as two perspective directions in using the multimodal method.

As an analytical tool. The weakly-nonlinear modal equations derived by employing the multimodal method play the same role as approximate weakly-nonlinear water wave theories, e.g., by Kordeweg–de Vries and Boussinesq. Using these equations is efficient for analytical studies of the dynamics and stability of liquid sloshing, classification of steady-state wave regimes, and identifying the chaos as well as in parametric studies. In addition, the multimodal method is of great importance for studying novel sloshing problems for which the CFD remains less applicable.

As a numerical tool in exceptional cases. Generally speaking, the weakly-nonlinear modal systems are not applicable for strongly nonlinear and viscous phenomena which are now in focus of the CFD. As long as the free-surface fragmentation is of less importance and strongly viscous flow is of local character, the multimodal method may be applied provided by accounting for the associated damping. This is as in the domain decomposition method, the use of the multimodal method has more perspective in simulations than using a fully-viscous and nonlinear solver. A limitation is that model tests are required to evaluate whether the sloshing is really weakly-nonlinear and the damping models are applicable.

The forthcoming extended survey will focus on the nonlinear multimodal method and associated modal systems of the 00-10's. The focus is on the so-called heavy liquid when the surface tension does not matter. Interested readers are referred to [7, 8] who gives the state-of-the-art on the surface tension-affected sloshing.

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