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A NONLOCAL MULTIPOINT PROBLEM FOR A DIFFERENTIAL OPERATOR EQUATION OF SECOND ORDER

Встановлено розв'язність нелокальної багатоточкової задачі для еволюційного рівняння другого порядку відносно часової змінної з оператором, спектр якого дискретний. Нелокальна умова задовольняється в слабкому розумінні у просторі формальних рядів Фур'є, які ототожнюються з лінійними неперервними функціоналами (узагальненими елементами на просторі, пов'язаному з даним оператором).

We establish the solvability of a nonlocal multipoint problem for a second order evolution equation with respect to time variable with an operator having discrete spectrum. A nonlocal condition is considered to be satisfied in the weak sense in the space of formal Fourier series that are identified with continuous linear functionals (generalized elements) on some space connected with a given operator.

1. Introduction

The theory of nonlocal boundary value problems, as a part of general partial boundary value problems theory, has been extensively developed from the 70th of the previous century. The study of such problems is due to a number of applications in mechanics, physics, chemistry, biology, ecology and other natural sciences that appear in mathematical modeling of different processes [1–7]. Dezin [8] was the first who pointed out on the advisability to use nonlocal conditions in view of general boundary value problems theory. He has been investigated solvable extensions of differential operators generated by a differentiation operation with constant coefficients. He has shown that, in order to set a well posed boundary value problem, one has to use both local and nonlocal conditions. Afterwards, the investigations started by Dezin have been developed by Romanko [9], Junusov [10], Mamyan [11], Makarov [12] and others.

Nonlocal boundary value problems in different contexts have been investigated by many mathematicians, who have used different methods and approaches (Nakhushev [13], Samarsky [14], Ptashnyk [15], Chesalin [16], Skubachevsky [17] and others). They have obtained important results concerning the setting, well solvability and construction of solutions, investigated dependence of the solvability type on the behavior of operation symbols, stated the regularity and non-regularity conditions of boundary value problems for essential types of differential-operator equations. As a problem of the kind, one can consider a nonlocal multipoint with respect to time problem which generalizes a Cauchy problem, where the initial condition $u(t,\cdot)|_{t=0} = f$ is replaced with the following one

$$\sum_{k=0}^{m} \alpha_k u(t,\cdot)|_{t=t_k} = f, \qquad (1)$$

$$t_0 = 0, \{t_1, \ldots, t_m\} \subset (0,T],$$

where $\{\alpha_0, \alpha_1, \ldots, \alpha_m\} \subset \mathbb{R}$ and $m \in \mathbb{N}$ are fixed numbers (if $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$ then we obtain a Cauchy problem).

In this paper we study a nonlocal multipoint problem with condition (1) for a second order differential-operator equation with a nonnegative self-adjoint operator on a Hilbert space with discrete spectrum. The positive and negative spaces corresponding to such an operator are contained in the space of the formal Fourier series that are identified with certain continuous linear functionals (generalized elements). We define an abstract convolution operator on the space of formal Fourier series via which the above self-adjoint operator is considered as a convolution operator. On the base of such an approach we prove the solvability of a nonlocal multipoint problem, build a fundamental solution $G(t)$, $t \in (0, T]$ and study its properties and structure. A solution $u(t)$ is represented as the convolution $G(t) * g$, where g is a continuous linear functional on a suitable subspace X of main elements ($X \subset H \subset X'$, H a Hilbert space, X' is the topologically conjugate space to X). Remark that $u(t)$, $G(t) \in X$ for all $t \in (0, T]$, however (1) is satisfied for $u(t)$ in the weak sense, that is, $\sum_{k=0}^{m} \alpha_k \lim_{t \to t_k} u(t) = f$, $f \in X'$, where the limits are taken in the space X' which is, in certain sense, "maximal" space of elements (continuous linear functionals) for setting of a nonlocal multipoint problem, every solution of which has the same properties as the fundamental solution.

2. Spaces of main and generalized elements. Formal Fourier series

Let H be an infinite dimensional separable Hilbert space with an inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, A a nonnegative self-adjoint operator with a dense in H domain $\mathcal{D}(A)$ and discrete spectrum, $(\lambda_k)_{k=1}^{\infty}$ a nondecreasing sequence of eigenvalues of A, $\lambda_k \geq 0$, $\lim_{k \to +\infty} \lambda_k = +\infty$, in this case the condition $\sum_{k:\lambda_k \neq 0} \lambda_k^{-p} < \infty$ with some $p > 0$ is true; $(e_k)_{k=1}^{\infty}$ the orthonormal basis of H consisting of the corresponding eigenvectors of A.

We set

$$\Phi_m = \Big\{ \varphi \in H : \ \varphi = \sum_{k=1}^{m} c_{k,\varphi} e_k, \ c_{k,\varphi} \in \mathbb{C} \Big\},$$

$$\Phi = \lim_{m \to \infty} \operatorname{ind} \Phi_m$$

(the subspace Φ is dense in H and invariant with respect to A). Denote by Φ' the space of all anti-linear continuous functionals on Φ with the weak convergence

$$\Big(\Phi' \ni f_n \xrightarrow[n \to \infty]{\Phi'} f \in \Phi' \Big) \Leftrightarrow$$

$$\Big(\langle f_n, \varphi \rangle \xrightarrow[n \to \infty]{} \langle f, \varphi \rangle, \quad \forall \varphi \in \Phi \Big)$$

(the symbol over the arrow denotes the space in which the convergence is considered, $\langle f, \varphi \rangle$ means the action of f at an element φ).

By the correspondence

$$H \ni \varphi \longrightarrow f_{\varphi} \in \Phi' : \ \langle f_{\varphi}, \psi \rangle = (\varphi, \psi),$$

$$\forall \psi \in \Phi,$$

we define an embedding $H \subset \Phi'$. Thus, $\Phi \subset H \subset \Phi'$ with dense continuous embeddings. Elements of Φ' are called *generalized elements*.

Let s be the space of all numerical sequences $(c_k)_{k=1}^{\infty}$, $c_k \in \mathbb{C}$, with the coordinate-wise convergence. The isomorphism

$$F : \ \Phi' \ni f \to (c_k = \langle f, e_k \rangle)_{k=1}^{\infty} \in s$$

of Φ' onto s sends Φ onto the set of all finitely nonzero sequences from s, and H onto ℓ_2. By means of the above isomorphism, the operator A induces the operation $(c_k)_{k=1}^{\infty} \to (\lambda_k c_k)_{k=1}^{\infty}$, and we can extend A on Φ' to a continuous operator $\hat{A}$ by $\hat{A} f = F^{-1} (\lambda_k c_k)_{k=1}^{\infty}$, $\Phi' \ni f = \sum_{k=1}^{\infty} c_k e_k$ [18, p. 8–22].

Let $f \in \Phi'$. The series $\sum_{k=1}^{\infty} c_k e_k$, where $c_k = \langle f, e_k \rangle$, is called the *Fourier series* of the element f, and the numbers c_k the *Fourier coefficients* of f. For any $f \in \Phi'$, the Fourier series of f converges in Φ' to f, and conversely, every series $\sum_{k=1}^{\infty} c_k e_k$ which converges in Φ' to some elements $f \in \Phi'$ coincides with the Fourier series of f [18]. Hence, the space Φ' can be considered as the space of formal series of the form $\sum_{k=1}^{\infty} c_k e_k$.

We introduce some classes of elements connected to the operator A. Consider an increasing sequence $(m_n)_{n \in \mathbb{Z}_+}$, $m_0 = 1$ of positive numbers possessing the following properties [18]:

1) $\forall \alpha > 0 \ \exists c_{\alpha} > 0 \ \forall n \in \mathbb{Z}_+$: $m_n \geq c_{\alpha} \cdot \alpha^n$;

2) $\exists M > 0 \ \exists h > 0 \ \forall n \in \mathbb{Z}_+$: $m_{n+1} \leq M h^n m_n$.

Examples of such sequences are Gevrey

sequences of the form $m_n = (n!)^\beta$, $m_n = n^{n\beta}$, $n \in \mathbb{Z}_+$, where $\beta > 0$ is a fixed parameter.

Set

$$H_\infty(A) := \lim_{\alpha\to\infty} \operatorname{pr} H_\alpha(A), \quad H_\alpha(A) = \mathcal{D}(A^\alpha) =$$

$$\Big\{\varphi \in H \,\Big|\, \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |c_k(\varphi)|^2 < \infty\Big\},$$

$$A^\alpha \varphi = \sum_{k=1}^{\infty} \lambda_k^\alpha c_k(\varphi) e_k, \varphi \in \mathcal{D}(A^\alpha), \alpha > 0,$$

$$H_\alpha\langle m_n\rangle := \{\varphi \in H_\infty(A) \,\Big|\, \exists c > 0 :$$

$$\|A^n \varphi\| \le c\alpha^n m_n, n \in \mathbb{Z}_+\}, \alpha > 0.$$

Then the space $H_\alpha\langle m_n\rangle \supset \Phi$ is a Banach space with respect to the norm $\|\varphi\|_{H_\alpha\langle m_n\rangle} = \sup_{n\in\mathbb{Z}_+} (\|A^n\varphi\|/(\alpha^n m_n))$. We denote $H_\infty\langle m_n\rangle := \lim_{\alpha\to\infty} \operatorname{ind} H_\alpha\langle m_n\rangle$. Then $\Phi \subset H_\infty\langle m_n\rangle \subset H_\infty(A) \subset H$, and all the embeddings are dense and continuous [18]. If by $H'_\infty(A)$ and $H'_\infty\langle m_n\rangle$ one denotes the spaces of antilinear continuous functionals on $H_\infty(A)$ and $H_\infty\langle m_n\rangle$ respectively then by [18], we obtain the chain of dense continuous embeddings $H \subset H'_\infty(A) \subset H'_\infty\langle m_n\rangle \subset \Phi'$. Moreover $H'_\infty\langle m_n\rangle = \lim_{\alpha\to\infty} \operatorname{pr} H'_\alpha\langle m_n\rangle$.

The spaces $G_{\{\beta\}}(A) := H_\infty\langle n^{n\beta}\rangle$, $\beta > 0$, are called the Gevrey spaces of order β, generated by the operator A; $G_{\{1\}}(A)$ equals the set of analytic vectors of A [18].

Let

$$\rho_0(\lambda) = \sup_{n\in\mathbb{Z}_+} (\lambda^n/m_n), \quad \lambda \in [1, +\infty);$$

$$\rho(\lambda) = \begin{cases} 1, & \lambda \in [0, 1), \\ \rho_0(\lambda), & \lambda \in [1, +\infty). \end{cases}$$

By the properties of the sequence $(m_n)_{n\in\mathbb{Z}_+}$, the function ρ is continuous on $[0, \infty)$, increases on $[1, +\infty)$ faster than any power of λ [18].

The space $H_\infty\langle m_n\rangle$ coincides (see [18]) with the inductive limit of the Hilbert spaces

$$H_{\{\alpha\}} = \Big\{f \in \Phi' \,\Big|\, \sum_{k=1}^{\infty} |c_k(f)|^2 \rho^2\Big(\frac{\lambda_k}{\alpha}\Big) < \infty,$$

$$c_k(f) = \langle f, e_k\rangle\Big\}, \alpha > 0,$$

with the inner product

$$(f, g)_{H_{\{\alpha\}}} = \sum_{k=1}^{\infty} c_k(f)\overline{c_k(g)}\rho^2\Big(\frac{\lambda_k}{\alpha}\Big),$$

$$\{f, g\} \subset H_{\{\alpha\}}.$$

In view of the behavior of the Fourier coefficients of the elements, the spaces $H_\infty\langle m_n\rangle$ and $H'_\infty\langle m_n\rangle$ are described as follows [18]:

$$(f \in H_\infty\langle m_n\rangle) \Leftrightarrow (\exists\mu > 0\, \exists c > 0\, \forall k \in \mathbb{N} : \quad |c_k(f)| \le c\rho^{-1}(\mu\lambda_k)); \tag{A}$$

$$(f \in H'_\infty\langle m_n\rangle) \Leftrightarrow (\forall\mu > 0\, \exists c = c(\mu) > 0 \quad \forall k \in \mathbb{N} : \ |c_k(f)| \le c\rho(\mu\lambda_k)). \tag{B}$$

If $m_n = n^{n\beta}$, $\beta > 0$ then $\rho_0(\lambda) \sim \exp\{\lambda^{1/\beta}\}$, $\lambda \in [1, +\infty)$, so in this case for any $f \in \Phi'$ one has

$$(f \in G_{\{\beta\}}(A)) \Leftrightarrow (\exists\mu > 0\, \exists c > 0\, \forall k \in \mathbb{N} :$$

$$|c_k(f)| \le c \exp\{-\mu\lambda_k^{1/\beta}\});$$

$$(f \in G'_{\{\beta\}}(A)) \Leftrightarrow (\forall\mu > 0\, \exists c = c(\mu) > 0\, \forall k \in \mathbb{N} :$$

$$|c_k(f)| \le c \exp\{\mu\lambda_k^{1/\beta}\}).$$

3. Nonnegative self-adjoint operators as convolutions operators

Let $\{f_1, f_2\} \subset \Phi'$, $f_1 = \sum_{k=1}^{\infty} c_k(f_1)e_k$, $f_2 = \sum_{k=1}^{\infty} c_k(f_2)e_k$.

We define an operation $*$ on the space Φ' called the *abstract convolution* or just *convolution* by setting

$$f_1 * f_2 := \sum_{k=1}^{\infty} c_k(f_1)c_k(f_2)e_k \equiv \sum_{k=1}^{\infty} c_k(f_1 * f_2)e_k,$$

so, $f_1 * f_2$ is a generalized element of Φ', the Fourier coefficients of which are connected with the ones of the generalized elements f_1 and f_2 by $c_k(f_1 * f_2) = c_k(f_1)c_k(f_2)$, $k \in \mathbb{N}$.

Consider a sequence $(m_n)_{n\in\mathbb{Z}_+}$ generating the spaces $H_\alpha\langle m_n\rangle$ and $H_\infty\langle m_n\rangle$ of a special kind $m_n = n!\rho_n$, where $(\rho_n)_{n\in\mathbb{Z}_+}$, $\rho_0 = 1$, is a sequence of nonnegative numbers which is: a) decreasing; b) $\exists\omega > 1\ \forall n \geq 1$: $\rho_{n-1}/\rho_n \leq \omega^n$; c) $\lim\limits_{n\to\infty} \sqrt[n]{\rho_n} = 0$. In [19] it was proved that the sequence $(n!\rho_n)$ has properties 1), 2). As an example of a sequence (ρ_n) with properties a) – c) one can take the sequence $\rho_n = (n\beta)^{-n\beta}$, $\rho_0 = \rho_1 = \rho_2 = 1$, where $\beta \in (0,1)$ is a fixed parameter ([19, p. 57]).

If $m_n = n!\rho_n$ then the function ρ defined by that sequence is differentiable on $[0,\infty)$ [19], and moreover,

$$\forall\{x_1, x_2\} \subset [1,\infty): \quad \rho(x_1)\rho(x_2) \leq \rho(x_1 + x_2)$$

which is equivalent to

$$\forall\{x_1, x_2\} \subset [1,+\infty):$$
$$\ln\rho(x_1) + \ln\rho(x_2) \leq \ln\rho(x_1 + x_2). \qquad (2)$$

Inequality (2) we will call the convexity inequality for a function $\ln\rho$.

Notice that the sequence $(n!\rho_n)$ possesses the condition $\lim\limits_{n\to\infty} \sqrt[n]{n!\rho_n}/n = 0$ which implies (see [20]) that $\rho_0(\lambda) \geq c_0 e^{c\lambda}$, $\lambda \in [1,+\infty)$ with some constants c_0, $c > 0$.

Lemma 1.

a) If $\{f_1, f_2\} \subset H'_\infty\langle m_n\rangle$ then $f_1 * f_2 \in H'_\infty\langle m_n\rangle$.

b) For all $\varphi \in H_\infty\langle m_n\rangle$ and $f \in H'_\infty\langle m_n\rangle$ the convolution $f * \varphi$ belongs to the space $H_\infty\langle m_n\rangle$.

Proof. a) It is enough to prove that the Fourier coefficients of $c_k(f_1 * f_2)$ satisfy (B).

If $\{f_1, f_2\} \subset H'_\infty\langle m_n\rangle$, then (see (B))

$$\forall\mu_1 > 0\ \exists c_1 = c_1(\mu_1) > 0\ \forall k \in \mathbb{N}:$$
$$|c_k(f_1)| \leq c_1\rho(\mu_1\lambda_k),$$
$$\forall\mu_2 > 0\ \exists c_2 = c_2(\mu_2) > 0\ \forall k \in \mathbb{N}:$$
$$|c_k(f_2)| \leq c_2\rho(\mu_2\lambda_k).$$

Then, taking into account convexity inequality (2) for $\ln\rho$ one has

$$|c_k(f_1 * f_2)| = |c_k(f_1)| \cdot |c_k(f_2)| \leq$$
$$\leq c_1c_2\rho(\mu_1\lambda_k)\rho(\mu_2\lambda_k) =$$
$$= c_1c_2e^{\ln\rho(\mu_1\lambda_k)+\ln\rho(\mu_2\lambda_k)} \leq c_1c_2e^{\ln\rho((\mu_1+\mu_2)\lambda_k)} =$$
$$= c\rho(\mu\lambda_k), \quad c = c_1c_2, \quad \mu = \mu_1 + \mu_2.$$

Thus, $f_1 * f_2 \in H'_\infty\langle m_n\rangle$.

b) Since $\varphi \in H_\infty\langle m_n\rangle$, we obtain (see (A))

$$\exists\mu > 0\ \exists c > 0\ \forall k \in \mathbb{N}: \quad |c_k(\varphi)| \leq c\rho^{-1}(\mu\lambda_k).$$

By (B) for $\mu_1 = \mu/2$ there is $c_1 = c_1(\mu_1) > 0$ such that $|c_k(f)| \leq c_1\rho(\mu_1\lambda_k)$, $k \in \mathbb{N}$. Then convexity inequality (2) yields

$$\ln\rho(\mu_1\lambda_k) - \ln\rho(\mu\lambda_k) \leq -\ln\rho((\mu - \mu_1)\lambda_k) \equiv$$
$$\equiv -\ln\rho\Big(\frac{\mu}{2}\lambda_k\Big), k \in \mathbb{N}.$$

Hence,

$$\rho(\mu_1\lambda_k)\rho^{-1}(\mu\lambda_k) = e^{\ln\rho(\mu_1\lambda_k)-\ln\rho(\mu\lambda_k)} \leq$$
$$\leq e^{-\ln\rho\left(\frac{\mu}{2}\lambda_k\right)} = \rho^{-1}\Big(\frac{\mu}{2}\lambda_k\Big), \quad k \in \mathbb{N}.$$

Thus, for the Fourier coefficients of the convolution $f * \varphi$ the following estimates hold

$$|c_k(f * \varphi)| \leq \tilde{c}\rho^{-1}(\tilde{\mu}\lambda_k),$$
$$\tilde{c} = c_1c_2, \tilde{\mu} = \mu/2, k \in \mathbb{N},$$

that is, $f * \varphi \in H_\infty\langle m_n\rangle$.

Let f: $[0,\infty) \to [0,\infty)$ be an increasing continuous on $[0,\infty)$ function, $\lim\limits_{\lambda\to+\infty} f(\lambda) = +\infty$. Then using the function f and the operator A we define the operator $f(A)$ by

$$f(A)\varphi = \int_0^\infty f(\lambda)dE_\lambda\varphi, \quad \varphi \in \mathcal{D}(f(A)), \quad (3)$$

where E_λ, $\lambda \geq 0$ is the spectral function of the operator A with the domain

$$\mathcal{D}(f(A)) = \Big\{\varphi \in H \,\Big|\, \int_0^\infty f^2(\lambda)d(E_\lambda\varphi, \varphi) < \infty\Big\},$$
$$\overline{\mathcal{D}(f(A))} = H.$$

The operator $f(A)$ is nonnegative self-adjoint on H as well. The integral in the right hand side of (3) actually is taken just over the spectrum $\sigma(A)$ of A, which in our case is discrete and has a unique limited point at infinity: $\sigma(A) = \{\lambda_k, k \in \mathbb{N}\}$. The spectral function E_λ, $\lambda \geq 0$ is piece-wise constant and is discontinuous only at the points λ_k, $k \in \mathbb{N}$. Moreover, $E_{\lambda_k+0} - E_{\lambda_k}$ is the projection onto the subspace spanned by the eigenvalues of A that correspond to the eigenvalue λ_k. The corresponding eigenvalues e_k, $k \in \mathbb{N}$, form an orthonormal basis of H, and hence

$$(E_{\lambda_k+0} - E_{\lambda_k})\varphi = (\varphi, e_k)e_k \equiv c_k(\varphi)e_k, \varphi \in H.$$

The spectral function E_λ, $\lambda \geq 0$ in this case has the form

$$E_\lambda\varphi = \sum_{\lambda_k<\lambda} c_k(\varphi)e_k, \quad \varphi \in H,$$

and integral (3) equals

$$f(A)\varphi = \sum_{k=1}^{\infty} f(\lambda_k)c_k(\varphi)e_k, \quad \varphi \in \mathcal{D}(f(A)),$$

where $f(\lambda_k)$, $k \in \mathbb{N}$ are the eigenvalues of $f(A)$.

In the sequel, we shall use the notation $f(A) := A_f$.

The operator $f(A)$ we extend to the continuous operator $\hat{f}(A)$ on Φ' by

$$\hat{f}(A)\varphi = F^{-1}(f(\lambda_k)c_k(\varphi))_{k=1}^{\infty},$$

$$\Phi' \ni \varphi = \sum_{k=1}^{\infty} c_k(\varphi)e_k.$$

Consider the generalized element $G_f = \sum_{k=1}^{\infty} f(\lambda_k)e_k$ of the space Φ' constructed by a function f. Then $\hat{f}(A)$ is a convolution operator which acts in the space Φ' by setting

$$\hat{f}(A)\varphi = G_f * \varphi = \sum_{k=1}^{\infty} f(\lambda_k)c_k(\varphi)e_k.$$

We define the operator $f(A) \equiv A_f$ to be the restriction of the operator $\hat{f}(A)$ to the subspace $H_\infty\langle m_n\rangle$.

Lemma 2. *An operator A_f is continuous on the space $H_\infty\langle m_n\rangle$ if and only if $G_f \in H'_\infty\langle m_n\rangle$.*

Proof. Assume $G_f \in H'_\infty\langle m_n\rangle$. Then

$$\forall\mu > 0\, \exists c = c(\mu) > 0\, \forall k \in \mathbb{N} :$$

$$f(\lambda_k) \leq c\rho(\mu\lambda_k). \tag{4}$$

By Lemma 1, $G_f * \varphi \in H_\infty\langle m_n\rangle$ for all $\varphi \in H_\infty\langle m_n\rangle$. Hence, in this case the operator A_f is mapped the space $H_\infty\langle m_n\rangle$ into itself. We show that A_f in continuous on $H_\infty\langle m_n\rangle$, that is, A_f sends bounded subsets of this subspace to bounded subsets (remark that the set of all continuous operators on $H_\infty\langle m_n\rangle$ coincides with the set of all bounded operators [21]).

Let L be a bounded subset of $H_\infty\langle m_n\rangle$. Since $H_\infty\langle m_n\rangle = \bigcup_{\alpha>0} H_{\{\alpha\}}$, the set L is bounded in some Hilbert space $H_{\{\alpha_0\}}$, i.e.

$$\exists b > 0\, \forall\psi \in L : \ \ \|\psi\|^2_{H_{\{\alpha_0\}}} =$$

$$\sum_{k=1}^{\infty} |c_k(\psi)|^2\rho^2\Big(\frac{\lambda_k}{\alpha_0}\Big) \leq b,$$

or equivalently

$$\exists b_1 > 0\, \forall\psi \in L : \ \ \|c_k(\psi)\| \leq b_1\rho^{-1}\Big(\frac{\lambda_k}{\alpha_0}\Big), k \in \mathbb{N}.$$

We set $\mu = (2\alpha_0)^{-1}$ to inequality (4). Then from (2) we deduce that

$$|c_k(A_f\psi)| = f(\lambda_k)|c_k(\psi)| \leq cb_1\rho^{-1}\Big(\Big(\frac{1}{\alpha_0} - \mu\Big)\lambda_k\Big) =$$

$$= b_2\rho^{-1}\Big(\frac{\lambda_k}{\alpha_0}\Big), \quad k \in \mathbb{N},$$

where $b_2 = cb_1$. Taking into account (2), we obtain

$$|c_k(A_f\psi)| \cdot \rho\Big(\frac{\lambda_k}{4\alpha_0}\Big) \leq b_2\rho^{-1}\Big(\frac{\lambda_k}{2\alpha_0}\Big)\rho\Big(\frac{\lambda_k}{4\alpha_0}\Big) =$$

$$= b_2 \exp\Big(-\ln\rho\Big(\frac{\lambda_k}{2\alpha_0}\Big) + \ln\rho\Big(\frac{\lambda_k}{4\alpha_0}\Big)\Big) \leq$$

$$\leq b_2 \exp\Big(-\ln\rho\Big(\frac{\lambda_k}{4\alpha_0}\Big)\Big) = b_2\rho^{-1}\Big(\frac{\lambda_k}{4\alpha_0}\Big).$$

Since the function $\rho(\lambda)$ grows on $[1, +\infty)$ faster than any power of λ, by the above and by $\sum_{k:\lambda_k\neq 0} \lambda_k^{-p} < \infty$ for some $p > 0$ the following series converges

$$\sum_{k=1}^{\infty} |c_k(A_f\psi)|^2\rho^2\Big(\frac{\lambda_k}{4\alpha_0}\Big).$$

Thus, the set $A_f L$ is bounded in $H_{\{4\alpha_0\}}$, and therefore in $H_\infty\langle m_n\rangle$.

The converse statement is proved analogously using convexity inequalities (2).

Remark 1. *The condition $G_f \in H'_\infty\langle m_n\rangle$ is equivalent to the following condition on f*

$$\forall\mu > 0\,\exists c = c(\mu) > 0 :$$

$$f(\lambda) \le c\rho(\mu\lambda), \lambda \in [0,\infty). \qquad (5)$$

4. Nonlocal multipoint problem

Consider the following differential-operator equation

$$u''(t) = A_f u(t), \quad t \in (0,T], 0 < T < \infty, \quad (6)$$

where A_f is the operator constructed in Section , which is linear and continuous on the space $H_\infty\langle m_n\rangle$. In what follows, we assume also that the function f has the following property

$$\exists\mu_0 > 0\,\exists c_0 > 0 : \ \ f(\lambda) \ge c_0 \ln^2 \rho(\mu_0\lambda), \quad (7)$$

$\lambda \in [0,\infty)$.

By a solution of equation (6) *we mean a function u: $(0,T] \to H_\infty\langle m_n\rangle$, twice strongly differentiable on H, which fulfills equation* (6).

Theorem 1. *For every $\psi = \sum_{k=1}^{\infty} c_k(\psi)e_k \in H'_\infty\langle m_n\rangle$ the function*

$$\gamma(t) = \sum_{k=1}^{\infty} \exp(-t\sqrt{f(\lambda_k)})c_k(\psi)e_k \quad (8)$$

is a solution of (6).

Proof. We prove that $\gamma(t) \in H_\infty\langle m_n\rangle$ for all $t > 0$. Since $\psi \in H'_\infty\langle m_n\rangle$,

$$\forall\mu > 0\,\exists c = c(\mu) > 0 : \ |c_k(\psi)| \le c\rho(\mu\lambda_k), \quad (9)$$
$$k \in \mathbb{N}.$$

Taking into account (7), (9) and $c_k(\gamma(t)) = c_k(\psi)\exp(-t\sqrt{f(\lambda_k)})$ we obtain

$$|c_k(\gamma(t))| \le c\rho(\mu\lambda_k)\exp(-t\sqrt{f(\lambda_k)}) \le$$
$$\le c\rho(\mu\lambda_k)\exp(-c_1 t \ln\rho(\mu_0\lambda_k)),$$

where $c_1 = \sqrt{c_0}$ and c_0 is the constant from (7). The function $\ln\rho$ is convex on $[0,\infty)$, hence if $0 < c_1 t < 1$ (for a fixed t) then the following inequality holds

$$c_1 t \ln\rho(\mu_0\lambda_k) \ge \ln\rho(c_1 t\mu_0\lambda_k) \equiv \ln\rho(\tilde{\mu}_0\lambda_k),$$

where $\tilde{\mu}_0 = c_1 t\mu_0$. If for a fixed $t \in (0,T]$ one has $c_1 t > 1$ then $c_1 t = [c_1 t] + \{c_1 t\}$. Hence

$$\exp(-c_1 t\ln\rho(\mu_0\lambda_k)) =$$
$$= \exp(-\{c_1 t\}\ln\rho(\mu_0\lambda_k) - [c_1 t]\ln\rho(\mu_0\lambda_k)) \le$$
$$\le \exp(-\{c_1 t\}\ln\rho(\mu_0\lambda_k)).$$

Since $0 < \{c_1 t\} < 1$, the present case is reduced to the previous one. Thus, for every $t \in (0,T]$

$$|c_k(\gamma(t))| \le c\exp(\ln\rho(\mu\lambda_k) - \ln\rho(\tilde{\tilde{\mu}}_0\lambda_k)),$$

where $\tilde{\tilde{\mu}}_0 = \tilde{\mu}_0$ or $\tilde{\tilde{\mu}}_0 = \{c_1 t\}\mu_0$ (here t is fixed). Take $\mu = \tilde{\tilde{\mu}}_0/2$ and use inequality (2) for $\ln\rho$. Then

$$|c_k(\gamma(t))| \le c\exp\Big(-\ln\rho\Big(\frac{\tilde{\tilde{\mu}}_0}{2}\lambda_k\Big)\Big) =$$
$$= c\rho^{-1}\Big(\frac{\tilde{\tilde{\mu}}_0}{2}\lambda_k\Big), k \in \mathbb{N}.$$

Thus, we have proved that $u(t) \in H_\infty\langle m_n\rangle$ for all $t > 0$.

Now we prove the strong differentiability of $\gamma(t)$ on $(0,T]$. To do this, it is enough to prove that

$$\Phi_t(\Delta t) = \Big\|\frac{\gamma(t+\Delta t) - \gamma(t)}{\Delta t} - \omega(t)\Big\| \to 0,$$
$$\Delta t \to 0,$$

where $t \in (0,T]$ is any fixed point and

$$\omega(t) = -\sum_{k=1}^{\infty}\sqrt{f(\lambda_k)}\exp(-t\sqrt{f(\lambda_k)})c_k e_k.$$

Direct calculations show that $\Phi_t^2(\Delta t) = \sum_{k=1}^{\infty} a_k^2(\Delta t)|c_k|^2$, where

$$|a_k(\Delta t)| = \Big|\frac{1}{\Delta t}[\exp(-(t+\Delta t)\sqrt{f(\lambda_k)}) -$$
$$-\exp(-t\sqrt{f(\lambda_k)})] + \sqrt{f(\lambda_k)}\exp(-t\sqrt{f(\lambda_k)})\Big| =$$
$$= \sqrt{f(\lambda_k)}|\exp(-t\sqrt{f(\lambda_k)}) -$$

$$-\exp(-\tau\sqrt{f(\lambda_k)})| =$$
$$= f(\lambda_k)\exp(-\tau_1\sqrt{f(\lambda_k)})\theta|\Delta t|,$$
$$\tau = t+\theta\Delta t, 0<\theta<1, \tau_1 = t+\theta_1\Delta t, 0<\theta_1<1.$$

If $\Delta t > 0$, then taking into account (5) and (7) we obtain

$$|a_k(\Delta t)| \le f(\lambda_k)\exp(-t\sqrt{f(\lambda_k)})\Delta t \le$$
$$\le c\rho(\mu\lambda_k)\exp(-tc_1\ln\rho(\mu_0\lambda_k))\Delta t,$$

where $c_1 = \sqrt{c_0}$ and $\mu > 0$ is an arbitrary fixed parameter, $c = c(\mu) > 0$.

Hence,

$$|a_k(\Delta t)| \le c\exp(\ln\rho(\mu\lambda_k) - tc_1\ln\rho(\mu_0\lambda_k))\Delta t.$$

Reasoning like above, we show that

$$\exp(-tc\ln\rho(\mu_0\lambda_k)) \le \exp(-\ln\rho(\tilde{\tilde{\mu}}_0\lambda_k)),$$

where $\tilde{\tilde{\mu}}_0 = c_1t\mu_0$ for $0 < c_1t < 1$ and $\tilde{\tilde{\mu}}_0 = \{c_1t\}\mu_0$ for $c_1t > 1$. Setting $\mu = \tilde{\tilde{\mu}}_0/2$ and using the convexity inequality for the function $\ln\rho$, we obtain the next estimates

$$|a_k(\Delta t)| \le c\exp(\ln\rho(\mu\lambda_k) - \ln\rho(\tilde{\tilde{\mu}}_0\lambda_k))\Delta t \le$$
$$\le c\exp(-\ln\rho(\tilde{\tilde{\mu}}_0 - \mu)\lambda_k)\Delta t =$$
$$= c\exp\Big(-\ln\rho\Big(\frac{\tilde{\tilde{\mu}}_0}{2}\lambda_k\Big)\Big)\Delta t =$$
$$= c\rho^{-1}\Big(\frac{\tilde{\tilde{\mu}}_0}{2}\lambda_k\Big)\Delta t, k\in\mathbb{N}. \qquad (10)$$

Since $\psi \in H'_\infty\langle m_n\rangle$, estimates (9) and (10) (using also (2)) imply

$$|a_k(\Delta t)|\cdot|c_k| \le \tilde{c}\rho^{-1}(\mu_1\lambda_k), \mu_1 = \tilde{\tilde{\mu}}_0/4, k\in\mathbb{N}.$$

Hence, $\Phi_t^2(\Delta t) \le \tilde{\tilde{c}}\Delta t$, where

$$\tilde{\tilde{c}} = \tilde{c}^2\sum_{k=1}^\infty \rho^{-1}(\mu_1\lambda_k) \le b\sum_{k=1}^\infty e^{-\alpha\lambda_k} < \infty,$$
$$b, \alpha > 0$$

(here we used that $\rho(\lambda) \ge c_0e^{c\lambda}$, $\lambda\in[1,+\infty)$). Thus we obtain the limit relation $\Phi_t(\Delta t)\to 0$, $\Delta t\to 0$ for a fixed $t\in(0,T]$.

If $\Delta t < 0$ then we choose Δt to satisfy $\tau = t+\theta\Delta t \ge t/2$. Then we show that $\Phi_t(\Delta t)\to 0$ for $\Delta t\to 0$ and a fixed $t > 0$. From here we deduce the strong differentiability of $\gamma(t)$ on $(0,T]$. Analogously we prove the strong differentiability of $\gamma'(t)$ on $(0,T]$. Hence, $\gamma(t)\in C^2((0,T], H_\infty\langle m_n\rangle)$.

Observe that $\omega(t)\in H_\infty\langle m_n\rangle$ for all $t > 0$. The proof of this property uses the properties of f, $\ln\rho$ and estimate (9) of the coefficients $c_k(\psi)$.

The function $\gamma(t)$ fulfills equation (6). Indeed,

$$A_f(\gamma(t)) \equiv f(A)\gamma(t) = \sum_{k=1}^\infty f(\lambda_k)c_k(\gamma(t))e_k =$$
$$= \sum_{k=1}^\infty f(\lambda_k)\exp(-t\sqrt{f(\lambda_k)})c_ke_k.$$

On the other hand, when proving the differentiability properties of $\gamma(t)$ we established that

$$\gamma''(t) = \sum_{k=1}^\infty f(\lambda_k)\exp(-t\sqrt{f(\lambda_k)})c_ke_k.$$

Thus, $\gamma(t)$ is a solution of (6).

Remark 2. *We introduce the following notation*

$$\tilde{G}(t) = \sum_{k=1}^\infty \exp(-t\sqrt{f(\lambda_k)})e_k.$$

By the properties of f, we have that $\tilde{G}(t)\in H_\infty\langle m_n\rangle$ for all $t > 0$. Moreover, $\gamma(t) = \tilde{G}(t)*\psi \in H_\infty\langle m_n\rangle$, $\forall\psi\in H'_\infty\langle m_n\rangle$. Hence the convolution operator $\tilde{G}(t)*\cdot$ sends every element of the space $H'_\infty\langle m_n\rangle$ (in particular, every element of the space $H_\infty\langle m_n\rangle\subset H'_\infty\langle m_n\rangle$) to a solution of (6).

Setting problem: find a solution of equation (6) of type (8) possessing the condition

$$\mu u(0) - \sum_{n=1}^m \mu_n u(t_n) = g, \quad g\in H, \qquad (11)$$

where $m\in\mathbb{N}$, $\{\mu,\mu_1,\dots,\mu_m\}\subset(0,\infty)$, $\{t_1,\dots,t_m\}\subset(0,T]$ are fixed numbers, $\mu > \sum_{n=1}^m \mu_n$, $t_1 < t_2 < \dots < t_m$. Here $u(0)$ means $\lim_{t\to+0} u(t)$, where the limit is considered in the space H (i.e. we assume that there is $u_0 \in H$ such that $\|u(t) - u_0\| \to 0,\ t \to +0;\ u_0 = u(0)$). The above problem we call the multi-point problem for equation (6). Theorem and Remark imply that problem (6), (11) can be equivalently reposed as follows: find an element ψ in the class $H'_\infty\langle m_n\rangle$ the convolution of which by $\tilde{G}(t)$ is a solution of (6) satisfying (11) (in the indicated sense). To solve this problem, one should find the coefficients $c_k \equiv c_k(\psi),\ k \in \mathbb{N}$ of such an element. In order to find $c_k,\ k \in \mathbb{N}$, we multiply (11) by $e_k,\ k \in \mathbb{N}$ in the sense of the dot product, taking into account that

$$c_k(\gamma(t)) \equiv c_k(u(t)) = c_k(\tilde{G}(t))c_k(\psi),$$
$$c_k(\tilde{G}(t)) = \exp(-t\sqrt{f(\lambda_k)}).$$

Then we obtain

$$\mu c_k(\tilde{G}(0))c_k(\psi) - \sum_{n=1}^{m} \mu_n c_k(\tilde{G}(t_n))c_k(\psi) = c_k(g),$$

$$c_k(\tilde{G}(t_n)) = \exp(-t_n\sqrt{f(\lambda_k)}), \quad c_k(\tilde{G}(0)) = 1.$$

Hence

$$c_k(\psi) = c_k(g)\Big(\mu - \sum_{n=1}^{m} \mu_n \exp(-t_n\sqrt{f(\lambda_k)})\Big)^{-1}.$$

Now we set: $Q_1(t, \lambda_k) = \exp(-t\sqrt{f(\lambda_k)})$. Then

$$c_k(\psi) = c_k(g)\Big(\mu - \sum_{n=1}^{m} \mu_n Q_1(t_n, \lambda_k)\Big)^{-1}.$$

Observe that

$$\Big(\mu - \sum_{n=1}^{m} \mu_n Q_1(t_n, \lambda_k)\Big)^{-1} \le \Big(\mu - \sum_{n=1}^{m} \mu_n\Big)^{-1} \equiv \mu_0,$$

that is, $c_k(\psi) = c_k(g)Q_2(\lambda_k)$, where $Q_2(\lambda_k) = \Big(\mu - \sum_{n=1}^{m} \mu_n Q_1(t_n, \lambda_k)\Big)^{-1}$. Then

$$\sum_{k=1}^{\infty} |c_k(\psi)|^2 \le \mu_0^2 \sum_{k=1}^{\infty} |c_k(g)|^2 = \mu_0^2\|g\|^2, g \in H,$$

that is, the element ψ generating a solution of (6), (11), belongs to H, and the corresponding solution is given by

$$u(t) = \tilde{G}(t) * \psi = \sum_{k=1}^{\infty} Q_1(t, \lambda_k)c_k(g) \times$$
$$\times \Big(\mu - \sum_{n=1}^{m} \mu_n\ Q_1(t_n, \lambda_k)\Big)^{-1} \equiv$$
$$\equiv \sum_{k=1}^{\infty} Q_1(t, \lambda_k)Q_2(\lambda_k)c_k(g) = \tilde{\tilde{G}}(t) * g,$$
$$g = \sum_{k=1}^{\infty} c_k(g)e_k \in H,$$

where $\tilde{\tilde{G}}(t) = \sum_{k=1}^{\infty} Q_1(t, \lambda_k)Q_2(\lambda_k)e_k$, $\tilde{\tilde{G}}(t) \in H_\infty\langle m_n\rangle$ for all $t \in (0, T]$.

By Lemma 1, we can consider the convolution $\tilde{\tilde{G}} * g$ in the case where $g \in H'_\infty\langle m_n\rangle$. Moreover, the same lemma yields that $u(t) = \tilde{\tilde{G}} * g \in H_\infty\langle m_n\rangle$ for all $t \in (0, T]$,

$$c_k(u(t)) = c_k(\tilde{\tilde{G}}(t))c_k(g) =$$
$$= Q_1(t, \lambda_k)Q_2(\lambda_k)c_k(g) = c_k(\gamma(t))Q_2(\lambda_k),$$
$$k \in \mathbb{N}.$$

The proof of strong twice differentiability of the function $u(t) = \tilde{\tilde{G}}(t) * g,\ g \in H'_\infty\langle m_n\rangle$ is similar to the proof for the function $\gamma(t)$ (see proof of Theorem). The function $u(t)$ is a solution of (6) satisfying condition (11), where $g \in H'_\infty\langle m_n\rangle$ in the sense that

$$\mu \lim_{t\to+0} u(t) - \sum_{n=1}^{m} \mu_n \lim_{t\to t_n} u(t) = g, \qquad (12)$$
$$g \in H'_\infty\langle m_n\rangle,$$

where the limits are considered in the space $H'_\infty\langle m_n\rangle$.

To prove (12), we take any element $\psi = \sum_{k=1}^{\infty} c_k(\psi)e_k \in H_\infty\langle m_n\rangle$ and observe that, by the continuity of the inclusion of $H_\infty\langle m_n\rangle$ in the space $H'_\infty\langle m_n\rangle$ and, taking into account that $e_k,\ k \in \mathbb{N}$ in an orthonormal basis, one has

$$\langle u(t), \psi\rangle = (u(t), \psi) = \sum_{k=1}^{\infty} c_k(u(t))c_k(\psi) =$$
$$= \sum_{k=1}^{\infty} Q_1(t, \lambda_k)Q_2(\lambda_k)c_k(g)c_k(\psi).$$

Then

$$\mu \lim_{t\to+0}\langle u(t),\psi\rangle - \sum_{n=1}^{m}\mu_n \lim_{t\to t_n}\langle u(t),\psi\rangle =$$

$$= \mu \lim_{t\to+0}\sum_{k=1}^{\infty} c_k(u(t))c_k(\psi) -$$

$$- \sum_{n=1}^{m}\mu_n \lim_{t\to t_n}\sum_{k=1}^{\infty} c_k(u(t))c_k(\psi),$$

where the series $\sum_{k=1}^{\infty} c_k(u(t))c_k(\psi)$ converges uniformly on $(0,T]$. The latter fact follows from view of $c_k(u(t))$, $k\in\mathbb{N}$ and the inequality

$$|c_k(u(t))|\cdot|c_k(\psi)| \le \tilde{c}|c_k(g)|\cdot|c_k(\psi)|,$$

$$t\in(0,T], k\in\mathbb{N}.$$

Indeed, since $g\in H'_\infty\langle m_n\rangle$, we have that

$$\forall\mu>0\,\exists c=c(\mu)>0\,\forall k\in\mathbb{N}:$$

$$|c_k(g)|\le c\rho(\mu\lambda_k).$$

The condition $\psi\in H_\infty\langle m_n\rangle$ together with (A) gives that

$$\exists\mu_0>0\,\exists c_0>0\forall k\in\mathbb{N}:$$

$$|c_k(\psi)|\le c_0\rho^{-1}(\mu_0\lambda_k).$$

We set $\mu=\mu_0/2$. By convexity inequality (2),

$$|c_k(g)|\cdot|c_k(\psi)| \le cc_0\rho^{-1}(\mu_0\lambda_k)\rho\Big(\frac{\mu_0}{2}\lambda_k\Big) \le$$

$$\le cc_0\rho^{-1}\Big(\frac{\mu_0}{2}\lambda_k\Big), \quad k\in\mathbb{N},$$

which implies the desired property.

Thus,

$$\lim_{t\to t_n}\sum_{k=1}^{\infty} c_k(u(t))c_k(\psi) = \sum_{k=1}^{\infty} c_k(u(t_n))c_k(\psi) \equiv$$

$$\equiv \sum_{k=1}^{\infty} Q_1(t_n,\lambda_k)Q_2(\lambda_k)c_k(g)c_k(\psi), \qquad (13)$$

$$\lim_{t\to+0}\sum_{k=1}^{\infty} c_k(u(t))c_k(\psi) = \sum_{k=1}^{\infty} c_k(u(0))c_k(\psi) \equiv$$

$$\equiv \sum_{k=1}^{\infty} Q_2(\lambda_k)c_k(g)c_k(\psi). \qquad (14)$$

Taking into account (13) and (14) we obtain

$$\mu \lim_{t\to+0}\langle u(t),\psi\rangle - \sum_{n=1}^{m}\mu_n \lim_{t\to t_n}\langle u(t),\psi\rangle =$$

$$= \sum_{k=1}^{\infty}\Big[\Big(\mu - \sum_{n=1}^{m}\mu_n Q_1(t_n,\lambda_k)\Big)Q_2(\lambda_k)\Big]c_k(g)c_k(\psi) \equiv$$

$$\equiv \sum_{k=1}^{\infty} Q_2^{-1}(\lambda_k)Q_2(\lambda_k)c_k(g)c_k(\psi) =$$

$$= \sum_{k=1}^{\infty} c_k(g)c_k(\psi) = \langle g,\psi\rangle, \psi\in H_\infty\langle m_n\rangle,$$

and the proof is completed.

Since $u(t)=\tilde{\tilde{G}}(t)$, $t\in(0,T]$, if $g=\tilde{\delta}=\sum_{k=1}^{\infty} e_k\in H'_\infty\langle m_n\rangle$, then (12) implies that the function $\tilde{\tilde{G}}(t)$ satisfies in the space $H'_\infty\langle m_n\rangle$ the following limit relation

$$\mu \lim_{t\to+0}\tilde{\tilde{G}} - \sum_{n=1}^{m}\mu_n \lim_{t\to t_n}\tilde{\tilde{G}}(t) = \tilde{\delta}.$$

The function $\tilde{\tilde{G}}(t)$ is called a *fundamental solution* of the nonlocal multipoint problem for equation (6).

We summarize the above obtained results in the following statement.

Theorem 2. *Let* (5) *and* (7) *be satisfied. Then multipoint problem* (6), (12) *is solvable and a solution is given by* $u(t)=\tilde{\tilde{G}}(t)*g$, $t\in(0,T]$, $u(t)\in H_\infty\langle m_n\rangle$ *for all* $t\in(0,T]$.

Consider for example the self-adjoint operator A^2 in the Hilbert space $H=L_2[0,2\pi]$ generated by the differential expression $-d^2/dx^2$ and conditions $u(0)=u(2\pi)$, $u'(0)=u'(2\pi)$, $A=\sqrt{A^2}\equiv|D|$, where $|D|$ is the modulus of the differentiation operator, $|D|\ge 0$. The spectrum of the operator $A=|D|$ is discrete: $\sigma(A)=\{|k|,k\in\mathbb{Z}\}$ with a unique limit point at infinity, $e_k=e^{ikx}$, $k\in\mathbb{Z}$, $x\in\mathbb{R}$, are its eigenfunctions. In this case one has

$$\Phi_m = \Big\{\varphi\in H \,\Big|\, \varphi = \sum_{k=-m}^{m} c_{k,\varphi}e^{ikx}, c_{k,\varphi}\in\mathbb{C},$$

$$x\in\mathbb{R}\Big\}, m\in\mathbb{Z}_+,$$

that is, ever element of Φ_m is a trigonometric polynomial of degree m, Φ' is the space of all formal Fourier series which are identified with generalized 2π-periodic functions as antilinear continuous functionals on the space of trigonometric polynomials [21].

The convolution of two generalized periodic functions $\{f, g\} \subset \Phi'$ is defined as follows [21]:

$$\langle f * g, \varphi\rangle = \langle f_x, \langle g_y, \varphi(x+y)\rangle\rangle, \quad \forall \varphi \in \Phi.$$

It make sense, because

$$\langle g_y, \varphi(x+y)\rangle = \Big\langle g_y, \sum_{k=-m}^{m} c_{k,\varphi} e^{ik(x+y)} \Big\rangle =$$

$$= \sum_{k=-m}^{m} c_{k,\varphi} \langle g, e^{iky}\rangle e^{ikx} \in \Phi.$$

The mapping

$$F: \ \Phi' \ni f \to \{c_k(f) = \langle f, e^{-ikx}\rangle, k \in \mathbb{Z}\} \in S$$

sends the space $L_2[0, 2\pi]$ onto ℓ_2, the operator $|D|$ is transformed into the multiplication operator by $|k|$, and the convolution is transformed into the coordinate-wise multiplication:

$$c_k(f * g) = \langle f * g, e^{-ikx}\rangle = \langle f, \langle g_y, e^{-ik(x+y)}\rangle\rangle =$$
$$= \langle f, \langle g, e^{-iky}\rangle e^{-ikx}\rangle = c_k(f) c_k(g),$$
$$\forall \{f, g\} \subset \Phi'.$$

Hence we obtain the commutativity and associativity of the convolution on Φ', that is, in this case Φ' is a ring (with respect to the convolution) with unit which is the Dirac delta-function. Thus, $f * g$ is a generalized 2π-periodic function from Φ' which is identified with the Fourier series $\sum\limits_{k=-\infty}^{\infty} c_k(f*g) e^{ikx} = \sum\limits_{k=-\infty}^{\infty} c_k(f) c_k(g) e^{ikx}$, and the convolution on Φ' coincides with the abstract convolution introduces in Section 3.

Remark that in this case $G_{\{\beta\}}(|D|) = H_\infty\langle n^{n\beta}\rangle$, $\beta > 0$, consists of all 2π-periodic infinitely differentiable on $\mathbb{R}$ functions φ having the following property: there exist constants c, $\alpha > 0$ (depending only on the function φ) such that

$$|\varphi^{(n)}(x)| \le c\alpha^n n^{n\beta}, n \in Z_+, x \in \mathbb{R}.$$

If $0 < \beta < 1$ then the function $\varphi \in G_{\{\beta\}}(|D|)$ admits an analytic extension on the complex plane to an entire function $\varphi(z)$, $z = x + iy \in \mathbb{C}$ such that [22, p. 35–39]

$$\exists c = c(\varphi) > 0 \, \exists b = b(\varphi) > 0 :$$
$$|\varphi(x+iy)| \le c \exp(b|y|^{1/(1-\beta)}).$$

We define the sequence (ρ_n) to be equal the sequence $((n(1-\beta))^{-n(1-\beta)} e^{n(1-\beta)})$, where $\beta \in (0; 1)$ is a fixed parameter. As noticed above, the sequence ρ_n satisfies conditions a) – c), and the corresponding sequence $(m_n = n!\rho_n)$ has properties 1), 2), $\rho_0(\lambda) \sim \exp(\lambda^{1/\beta})$, $\lambda \in [1, +\infty)$. We take the function f generating the operator $f(|D|) \equiv f(A)$ to be equal λ^ν, $\nu > 0$. One can directly verify that if $2/\beta \le \nu$, $0 < \beta < 1$ then the function $f(\lambda) = \lambda^\nu$ possesses conditions (5), (7). For instance, if one set $\beta = 1/2$ then $f(\lambda) = \lambda^4$, $A_f \equiv f(|D|) = d^4/dx^4$ and equation (6) has the form

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^4 u(t,x)}{\partial x^4}, \quad x \in \mathbb{R}, t \in (0, T]. \tag{15}$$

So, the nonlocal multipoint problem for equation (15) with condition (12), where $g \in H'_\infty\langle n^{n/2}\rangle \equiv G'_{\{1/2\}}(|D|)$, is solved (in the sense indicated above: the corresponding limits in (12) are considered in the space $G'_{\{1/2\}}(|D|)$). The following function is a 2π-periodic infinitely differentiable in x solution

$$u(t,x) = \sum_{k=-\infty}^{+\infty} c_k(g) \exp(-tk^2 + ikx) \times$$
$$\times \Big(\mu - \sum_{n=1}^{m} \mu_n \exp(-t_n k^2)\Big)^{-1},$$
$$u(t, \cdot) \in G_{\{1/2\}}(|D|),$$
$$g = \sum_{k=-\infty}^{+\infty} c_k(g) e^{ikx} \in G'_{\{1/2\}}(|D|).$$

Thus, generalized 2π-periodic functions from the space $G'_{\{1/2\}}(|D|)$ can be used to pose of nonlocal condition (12). In this case the solution $u(t,x)$ of the corresponding problem preserves properties of a smooth solution of the "classical" nonlocal problem for equation (15) with condition (11), where $g \in L_2[0, 2\pi]$.

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