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WIMAN'S TYPE INEQUALITY FOR SOME DOUBLE POWER SERIES

By $\mathcal{A}^2$ denote the class of analytic functions of the form $f(z) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m$, with the domain of convergence $\mathbb{T} = \{z = (z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1,\ |z_2| < +\infty\} = \mathbb{D} \times \mathbb{C}$ and $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ in $\mathbb{T}$. In this paper we prove some analogue of Wiman's inequality for analytic functions $f \in \mathcal{A}^2$. Let a function $h \colon \mathbb{R}_+^2 \to \mathbb{R}_+$ be such that h is nondecreasing with respect to each variables and $h(r) \geq 10$ for all $r \in T := (0,1) \times (0, +\infty)$ and $\iint_{\Delta_\varepsilon} \frac{h(r)dr_1dr_2}{(1-r_1)r_2} = +\infty$ for some $\varepsilon \in (0,1)$, where $\Delta_\varepsilon = \{(t_1, t_2) \in T \colon t_1 > \varepsilon,\ t_2 > \varepsilon\}$. We say that $E \subset T$ is a set of asymptotically finite h-measure on T if $\nu_h(E) := \iint\limits_{E \cap \Delta_\varepsilon} \frac{h(r)dr_1dr_2}{(1-r_1)r_2} < +\infty$ for some $\varepsilon > 0$. For $r = (r_1, r_2) \in T$ and a function $f \in \mathcal{A}^2$ denote

$$M_f(r) = \max\{|f(z)| \colon |z_1| \leq r_1, |z_2| \leq r_2\},\ \mu_f(r) = \max\{|a_{nm}| r_1^n r_2^m \colon (n,m) \in \mathbb{Z}_+^2\}.$$

We prove the following theorem: *Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f)$ of asymptotically finite h-measure on T such that for all $r \in (T \cap \Delta_\varepsilon) \backslash E$ we have*

$$M_f(r) \leq \frac{h^{3/2}(r)\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta}\Big(\frac{h(r)\mu_f(r)}{1-r_1}\Big) \cdot \ln^{1/2+\delta} \frac{er_2}{\varepsilon}.$$

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1 Introduction

Let $\mathcal{E}_R$ be the class of analytic functions f represented by power series of the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \tag{1}$$

with the radius of convergence $R := R(f) \in (0; +\infty]$. For $r \in [0, R)$ denote $M_f(r) = \max\{|f(z)| \colon\ |z| = r\}$ and $\mu_f(r) = \max\{|a_n| r^n \colon n \geq 0\}$ the maximum modulus and maximal term of series, respectively. We also denote by $\mathcal{H}_R$ the class of continuous positive increasing to $+\infty$ on $[0; R), R \leq +\infty$, functions such that $h(r) \geq 2$ $(\forall r \in (0, R))$ and $\int_{r_0}^{R} h(r) d\ln r =$

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$+\infty$ for some $r_0 \in (0, R)$. In paper [1] the following statements are proved:
1^0. *If $h \in \mathcal{H}_R$ and $f \in \mathcal{E}_R$, then for any $\delta > 0$ there exist $E(\delta, f, h) := E \subset (0, R)$, $r_0 \in (0, R)$ such that*

$$(\forall\ r \in (r_0, R) \setminus E)\colon\ \ M_f(r) \leq h(r)\mu_f(r)\{\ln h(r) \ln(h(r)\mu_f(r))\}^{1/2+\delta} \text{ and } \int\limits_E h(r) d\ln r < +\infty.$$

2^0. *If we additionally assume that the function $f \in \mathcal{E}_R$ is unbounded, then*

$$\ln M_f(r) \leq (1 + o(1)) \ln(h(r)\mu_f(r))$$

holds as $r \to R$ $(r \notin E)$. Remark, that assertion 1^0 at $h(r) \equiv \text{const}$ implies the classical Wiman-Valiron theorem for entire functions (see [2, 3, 4, 5, 6, 8, 7]) and at $h(r) \equiv 1/(1 - r)$ theorem about the Kővari-type inequality for analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ ([10, 9, 11]). From statement 2^0 in the case where $\ln h(r) = o(\ln \mu_f(r))$ $(r \to R)$ and from Cauchy inequality $\mu_f(r) \leq M_f(r)$ it follows that $\ln M_f(r) = (1 + o(1)) \ln \mu_f(r)$ holds as $r \to R$ $(r \notin E)$.

In paper [12] it is proved some analogues of Wiman's type inequality for analytic functions represented by the series of the form

$$f(z) = f(z_1, \ldots, z_m) = \sum\nolimits_{\|n\|=0}^{+\infty} a_n z^n \tag{2}$$

with the domain of convergence

$$\mathbb{D}^p \times \mathbb{C}^{m-p} = \{z = (z_1, \ldots, z_m) \in \mathbb{C}^m\colon (z_1, \ldots, z_p) \in \mathbb{D}^p, (z_{p+1}, \ldots, z_m) \in \mathbb{C}^{m-p}\}.$$

Papers [13, 14] deal with the same for analytic ([13]) and random analytic ([14]) functions in the case $m = 2, p = 1$, i.e. when $\mathbb{T} = \mathbb{D} \times \mathbb{C} = \{z = (z_1, z_2) \in \mathbb{C}^2\colon |z_1| < 1, z_2 \in \mathbb{C}\}$ is the domain of convergence.

By $\mathcal{A}_0^2$ we denote the class of analytic functions of form (2) with the domain of convergence $\mathbb{D} \times \mathbb{C}$ and $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ in $\mathbb{D} \times \mathbb{C}$,

$$r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) + \ln r_1 > 1 \quad \big(\forall\ r = (r_1, r_2) \in (r_1^0, 1) \times (r_2^0, +\infty)\big).$$

We say that $E \subset T = (0, 1) \times \mathbb{R}_+$ is a set of asymptotically finite logarithmic measure on T_1 if there exists $r_0 \in T$ such that

$$\nu_{\ln}(E \cap \Delta_{r_0}) := \iint\limits_{E \cap \Delta_{r_0}} \frac{dr_1 dr_2}{(1 - r_1) r_2} < +\infty, \quad \Delta_{r_0} = \{r = (r_1, r_2)\colon r_1^0 \leq r_1 < 1, r_2 \geq r_2^0\}$$

i.e. the set $E \cap \Delta_{r_0}$ is a set of finite logarithmic measure on T.

The following theorem is proved in [13].

Theorem 1 ([13]). *Let $f \in \mathcal{A}_0^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T \backslash E$ we obtain*

$$M_f(r) \leq \frac{\mu_f(r)}{(1 - r_1)^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r)}{1 - r_1} \cdot \ln^{1/2+\delta} r_2. \tag{3}$$

Another result of [13] asserts that, for some function $f \in \mathcal{A}^2$ the set

$$E = \Big\{ r \in T \colon M_f(r) > \frac{\mu_f(r)}{1-r_1} \ln \frac{\mu_f(r)}{1-r_1} \Big\}$$

has infinity logarithmic measure on T.

Remark, that regarding the statement about the classical Wiman inequality, Prof. I.V. Ostrovskii in 1995 formulated the following problem: *what is the best possible description of the value of an exceptional set* E? Later, the same issue was considered in a number of articles (for example, see [7, 15, 16, 17, 18, 19, 20, 21, 22]) concerning many other relations obtained in the Wiman-Valiron theory. In this regard, each time the question arises of finding the most general possible description of the magnitude of the exceptional set in each specific case. In particular, in the case of analogs of the Wiman-Valiron inequality. The present article is devoted to obtaining a very general description of the exceptional set in some analogs of the mentioned inequality for analytic functions in $\mathbb{D} \times \mathbb{C}$.

Let $\mathcal{H}$ be the class of functions $h \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ such that h is nondecreasing with respect to each variables and $h(r) > 10$ for all $r \in T$ and some $\varepsilon \in (0,1)$

$$\int\limits_{\varepsilon}^{1} \int\limits_{\varepsilon}^{+\infty} \frac{h(r)dr_1 dr_2}{(1-r_1)r_2} = +\infty.$$

We say that $E \subset T$ is a set of asymptotically finite h-measure on T if $h \in \mathcal{H}$ and

$$\nu_h(E) := \iint\limits_{E \cap \Delta_{r_0}} \frac{h(r)dr_1 dr_2}{(1-r_1)r_2} < +\infty$$

for some $r_0 \in T$. The collection of such sets we denote by $\mathcal{C}_h$.

Some analogs of Wiemann's inequality for entire functions of one and several complex variables can be found in [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33].

2 Wiman's type inequality for analytic functions on $\mathbb{T}$

By $\mathcal{A}^2$ we denote the class of analytic functions of the form

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m \tag{4}$$

with the domain of convergence $\mathbb{T}$ and $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ on $\mathbb{T}$.

For $\varepsilon \in (0,1)$, $r = (r_1, r_2) \in T := [0,1) \times [0, +\infty)$ and function $f \in \mathcal{A}^2$ we denote

$$\Delta_\varepsilon = \{(t_1, t_2) \in T \colon t_1 > \varepsilon,\ t_2 > \varepsilon\},\ M_f(r) = \max\{|f(z)| \colon |z_1| \le r_1, |z_2| \le r_2\},$$

$$\mu_f(r) = \max\{|a_{nm}| r_1^n r_2^m \colon (n,m) \in \mathbb{Z}^2_+\},\ \mathfrak{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m.$$

Let $D_f(r)=(D_{ij})$ be a 2×2 matrix such that

$$D_{ij}=r_i\frac{\partial}{\partial r_i}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)\Big)=\partial_i\partial_j\ln\mathfrak{M}_f(r),\quad \partial_i=r_i\frac{\partial}{\partial r_i},\quad i,j\in\{1,2\}.$$

The following statemen for entire function of several variables we find in paper [27], and for analytic functions on the domain in $\mathbb{D}\times\mathbb{C}$ [13].

Theorem 2 ([13, 27])**.** *Let $f\in\mathcal{A}^2$. There exists an absolute constant C_0 such that*

$$\mathfrak{M}_f(r)\le C_0\mu_f(r)(\det(D_f(r)+I))^{1/2},$$

where I is the identity 2×2 matrix.

Lemma 1. *Let $\delta>0$. Then there exists a set $E\in\mathcal{C}_h$ such that for all $r\in T\backslash E$ the inequalities*

$$\det(D_f(r)+I)\le\frac{h(r)}{1-r_1}\cdot\Big(\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)\Big(r_2\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)+\ln\frac{er_2}{\varepsilon}\Big)\Big)^{1+\delta},\tag{5}$$

$$\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)\le\frac{h(r)}{1-r_1}\cdot\Big(\ln\mathfrak{M}_f(r)\cdot\ln\frac{er_2}{\varepsilon}\Big)^{1+\delta},\tag{6}$$

$$\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)\le\frac{h(r)}{r_2(1-r_1)^{\delta}}(\ln\mathfrak{M}_f(r))^{1+\delta}\tag{7}$$

hold.

Proof. Let $E_1\subset T$ be a set for which inequality (5) does not hold, and $\Delta_\varepsilon:=\Delta_{(\varepsilon,\varepsilon)}$ for $\varepsilon\in(0,1)$. Now we prove that E_1 is a set of asymptotically finite h-measure. Since $r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)>0$, $j\in\{1,2\}$, for any $r\in T\cap\Delta_\varepsilon$ we have

$$r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln\frac{er_j}{\varepsilon}>1.$$

Then

$$\nu_h(E_1\cap\Delta_\varepsilon)=\iint\limits_{E_1\cap\Delta_\varepsilon}\frac{h(r)dr_1dr_2}{(1-r_1)r_2}\le$$

$$\le\iint\limits_{E_1\cap\Delta_\varepsilon}\frac{\det(D_f(r)+I)(1-r_1)dr_1dr_2}{\Big(\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)\Big(r_2\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)+\ln\frac{er_2}{\varepsilon}\Big)\Big)^{1+\delta}(1-r_1)r_2}\le$$

$$\le\iint\limits_{E_1\cap\Delta_\varepsilon}\frac{1}{r_1r_2}\cdot\frac{\det(D_f(r)+I)dr_1dr_2}{\Big(r_1\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)+\ln\frac{er_1}{\varepsilon}\Big)^{1+\delta}\Big(r_2\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)+\ln\frac{er_2}{\varepsilon}\Big)^{1+\delta}}.$$

Let $U\colon T\to\mathbb{R}^2_+$ be a mapping such that $U=(u_1(r),u_2(r))$ and $u_j(r)=r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln\frac{er_j}{\varepsilon}$, $j\in\{1,2\}$, $r=(r_1,r_2)$. Then for $i,j\in\{1,2\}$ we obtain

$$\frac{\partial u_i}{\partial r_i}=\frac{\partial}{\partial r_i}\Big(r_i\frac{\partial}{\partial r_i}\ln\mathfrak{M}_f(r)+\ln\frac{er_i}{\varepsilon}\Big)=\frac{1}{r_i}\partial_i\partial_i\ln\mathfrak{M}_f(r)+\frac{1}{r_i},$$

$$\frac{\partial u_i}{\partial r_j}=\frac{\partial}{\partial r_j}\Big(r_i\frac{\partial}{\partial r_i}\ln\mathfrak{M}_f(r)+\ln\frac{er_i}{\varepsilon}\Big)=\frac{1}{r_j}\partial_i\partial_j\ln\mathfrak{M}_f(r),\ i\ne j.$$

So, the Jacobian

$$J_1 := \frac{D(u_1,u_2)}{D(r_1,r_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial r_1} & \frac{\partial u_1}{\partial r_2} \\ \frac{\partial u_2}{\partial r_1} & \frac{\partial u_2}{\partial r_2} \end{vmatrix} = \det(D_f(r)+I)\frac{1}{r_1 r_2}.$$

Therefore,

$$\nu_h(E_1\cap\Delta_\varepsilon) \le \iint\limits_{U(E_1\cap\Delta_\varepsilon)} \frac{du_1du_2}{u_1^{1+\delta}u_2^{1+\delta}} \le \int\limits_1^{+\infty}\int\limits_1^{+\infty} \frac{du_1du_2}{u_1^{1+\delta}u_2^{1+\delta}} < +\infty.$$

Suppose that $E_2\subset T$ is a set for which inequality (6) does not hold. Then

$$\nu_h(E_2\cap\Delta_\varepsilon) = \iint\limits_{E_2\cap\Delta_\varepsilon} \frac{h(r)dr_1dr_2}{(1-r_1)r_2} \le \iint\limits_{E_2\cap\Delta_\varepsilon} \frac{\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)\cdot(1-r_1)dr_1dr_2}{(\ln\mathfrak{M}_f(r)\cdot\ln\frac{er_2}{\varepsilon})^{1+\delta}(1-r_1)r_2}.$$

Consider the mapping $V\colon T\to\mathbb{R}^2_+$, where $V=(v_1(r),v_2(r))$ and $v_1=\ln\mathfrak{M}_f(r),\ v_2=\ln\frac{er_2}{\varepsilon},\ r=(r_1,r_2)$. So,

$$J_2 := \frac{D(v_1,v_2)}{D(r_1,r_2)} = \begin{vmatrix} \frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r) & \frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r) \\ 0 & \frac{1}{r_2} \end{vmatrix} = \frac{1}{r_2}\cdot\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r).$$

Therefore

$$\nu_h(E_2\cap\Delta_\varepsilon) \le \iint\limits_{E_2\cap\Delta_\varepsilon} \frac{\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)dr_1dr_2}{(\ln\mathfrak{M}_f(r)\cdot\ln\frac{er_2}{\varepsilon})^{1+\delta}r_2} =$$

$$= \iint\limits_{V(E_2\cap\Delta_\varepsilon)} \frac{du_1du_2}{(u_1\cdot u_2)^{1+\delta}} \le \int\limits_1^{+\infty}\int\limits_1^{+\infty} \frac{du_1du_2}{(u_1\cdot u_2)^{1+\delta}} < +\infty.$$

Let $E_3\subset T$ be a set for which inequality (7) does not hold. Then

$$\nu_h(E_3\cap\Delta_\varepsilon) \le \iint\limits_{E_3\cap\Delta_\varepsilon} \frac{\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)r_2dr_1dr_2}{\frac{1}{(1-r_1)^\delta}\ln^{1+\delta}\mathfrak{M}_f(r)(1-r_1)r_2}.$$

Define the mapping $W\colon T\to T$, where $W=(w_1(r),w_2(r))$ and $w_1=r_1,\ w_2=\ln\mathfrak{M}_f(r),\ r=(r_1,r_2)$. So,

$$J_3 := \frac{D(w_1,w_2)}{D(r_1,r_2)} = \begin{vmatrix} 1 & 0 \\ \frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r) & \frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r) \end{vmatrix} = \frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r).$$

Therefore,

$$\nu_h(E_3\cap\Delta_\varepsilon) \le \iint\limits_{W(E_3\cap\Delta_\varepsilon)} \frac{du_1du_2}{(1-u_1)^{1-\delta}u_2^{1+\delta}} \le \int\limits_0^1 \frac{du_1}{(1-u_1)^{1-\delta}}\cdot\int\limits_1^{+\infty}\frac{du_2}{u_2^{1+\delta}} < +\infty.$$

It remains to remark that the set $E=\bigcup_{j=1}^3 E_j$ is also a set of asymptotically finite h-measure in T. $\square$

Theorem 3. *Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \in \mathcal{C}_h$ such that for all $r \in (T \cap \Delta_\varepsilon)\backslash E$ we obtain*

$$M_f(r) \le \frac{h^{3/2}(r)\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta}\Big(\frac{h(r)\mu_f(r)}{1-r_1}\Big) \cdot \ln^{1/2+\delta} \frac{er_2}{\varepsilon}.$$

Note, that Theorem 1 follows from Theorem 3 at $h(r) \equiv 10$.

Proof. Let E' and E_0 be the exceptional sets from Theorem 1 and Lemma 1, respectively. Then for $E = E' \cup E_0$ and $\delta \in (0,1)$, we get for all $r \in T\backslash E$

$$\mathfrak{M}_f(r) \le C_0\mu_f(r)(\det(D_f(r)+I))^{1/2} \le$$
$$\le C_0\sqrt{h(r)}\mu_f(r)\left(\frac{1}{1-r_1}\Big(\frac{\partial}{\partial r_1}\ln \mathfrak{M}_f(r)\Big)^{1+\delta}\Big(r_2\frac{\partial}{\partial r_2}\ln \mathfrak{M}_f(r) + \ln\frac{er_2}{\varepsilon}\Big)^{1+\delta}\right)^{1/2}.$$

Hence by Lemma 1 for all $r \in \Delta_\varepsilon\backslash E$ one can obtain

$$\mathfrak{M}_f(r) \le C_0\mu_f(r)h^{3/2}(r)\times$$
$$\times\left(\frac{1}{(1-r_1)^2}\Big(\ln \mathfrak{M}_f(r)\cdot \ln\frac{er_2}{\varepsilon}\Big)^{(1+\delta)^2}\Big(\frac{1}{(1-r_1)^\delta}(\ln \mathfrak{M}_f(r))^{1+\delta} + \ln\frac{er_2}{\varepsilon}\Big)^{1+\delta}\right)^{1/2} <$$
$$< \frac{\mu_f(r)h^{3/2}(r)}{(1-r_1)^{1+\delta}} \ln^{(1+\delta)^2}\mathfrak{M}_f(r)\ln^{1/2+2\delta} r_2. \tag{8}$$

Using inequality (8) we get

$$\ln \mathfrak{M}_f(r) \le \ln\frac{\mu_f(r)}{(1-r_1)^{1+\delta}} + \frac{3}{2}\ln h(r) + (1+\delta)^2\ln\ln\mathfrak{M}_f(r) + \Big(\frac{1}{2}+2\delta\Big)\ln\ln\frac{er_2}{\varepsilon} \le$$
$$\le 2\ln\frac{\mu_f(r)h(r)}{(1-r_1)} + 8\ln\ln\mathfrak{M}_f(r).$$

Therefore, $\ln \mathfrak{M}_f(r) \le 2\ln\frac{h(r)\mu_f(r)}{1-r_1}$. Finally for all $r \in \Delta_\varepsilon\backslash E$ we have

$$M_f(r) \le \mathfrak{M}_f(r) \le \frac{h^{3/2}(r)\mu_f(r)}{(1-r_1)^{1+\delta}}\left(2\ln\frac{h(r)\mu_f(r)}{1-r_1}\right)^{1+2\delta+\delta^2}\ln^{1/2+2\delta}\frac{er_2}{\varepsilon} <$$
$$< \frac{h^{3/2}(r)\mu_f(r)}{(1-r_1)^{1+\delta_1}}\left(\ln\frac{h(r)\mu_f(r)}{1-r_1}\right)^{1+\delta_1}\ln^{1/2+\delta_1}\frac{er_2}{\varepsilon},$$

where $\delta_1 > 2(\delta+\delta^2)$. □

References

[1] Skaskiv O.B., Kuryliak A.O. *Wiman's type inequality for analytic and entire functions and h-measure of an exceptional sets.* Carpathian Math. Publ. 2020, **12** (2), 492–498. doi: 10.15330/cmp.12.2.492-498

[2] Polya G., Szegö G. Aufgaben und Lehrsätze aus der Analysis. V.2, Berlin, Springer, 1925.

[3] Valiron G. Functions analytiques. Paris: Press Univer. de France, 1954.

[4] Wittich H. Neuere Untersuchungen über eindeutige analytische Funktionen. Berlin-Göttingen-Heidelberg: Springer-Verlag, 1955.

[5] Rosenbloom P.C. *Probability and entire functions.* Stud. Math. Anal. and Related Topics, Stanford: Calif. Univ. Press., 1962, 325–332.

[6] Goldberg A.A., Levin B.Ja., Ostrovsky I.V., Entire and meromorphic functions. Results of scientific and technical. modern. probl. mat. fundam. guide. VINITI, 1990, **85**, 5–186.

[7] Skaskiv O.B., Zrum O.V. *On an exeptional set in the Wiman inequalities for entire functions.* Mat. Stud., 2004, **21** (1), 13–24. (in Ukrainian)

[8] O.B. Skaskiv, P.V. Filevych, *On the size of an exceptional set in the Wiman theorem.* Mat. Stud. 1999, **12** (1), 31–36. (in Ukrainian)

[9] Suleymanov N.V. *An estimate of the Wiman-Valiron type for power series with a finite radius of convergence and its sharpness.* DAN USSR, 1980, **253** (4), 822–824. (in Russian)

[10] Kővari T. *On the maximum modulus and maximal term of functions analytic in the unit disc.* J. London Math. Soc. 1996, **41**, 129–137. https://doi.org/10.1112/jlms/s1-41.1.129

[11] Skaskiv O.B., Kuryliak A.O. *Direct analogues of Wiman's inequality for analytic functions in the unit disk.* Carpathian Math. Publ. 2010, bf2, (1), 109–118. (in Ukrainian)

[12] Kuryliak A.O., Tsvigun V.L. *Wiman's type inequality for multiple power series in an unbounded cylinder domain.* Mat. Stud. 2018, **49** (1), 29–51.

[13] Kuryliak A.O., Shapovalovska L.O., Skaskiv O.B. *Wiman's type inequality for some double power series.* Mat. Stud. 2013, **39** (2), 134–141.

[14] Kuryliak A., Skaskiv O., Tsvihun V. *Levy's phenomenon for analytic functions in* $\mathbb{D} \times \mathbb{C}$. Mat. Stud. 2016, **46** (2), 121–129.

[15] Skaskiv O.B., Filevych P.V. *On the size of an exceptional set in the Wiman theorem.* Mat. Stud. 1999, **12** (1), 31–36. (in Ukrainian)

[16] Salo T.M., Skaskiv O.B., Trakalo O.M. *On the best possible description of exeptional set in Wiman-Valiron theory for entire function.* Mat.Stud. 2001, **16** (2), 131–140.

[17] Skaskiv O.B., Trakalo O.M. *On exeptional set in Borel relation for multiple entire Dirichlet series.* Mat. Stud. 2001, **15** (2), 163–172. (in Ukainian)

[18] Filevych P.V. *An exact estimate for the measure of the exceptional set in the Borel relation for entire functions.* Ukrainian Math. J. 2001., **53** (2), 328–332. https://doi.org/10.1023/A:1010489609188

[19] Skaskiv O.B. *Estimates of measures of exeptional sets in the Wiman-Valiron theory.* Nonlinear. bound. probl. Collect. sc. proc. 2001. **11**, Donetsk, 186–190.

[20] Skaskiv O.B., Trakalo O.M. *Sharp estimate of exceptional set in Borel's relation for entire functions of several complex variables.* Mat. Stud. 2002, **18** (1), 53–56. (in Ukainian)

[21] Skaskiv O.B., Zikrach D. Yu. *On the best possible description of an exceptional set in asymptotic estimates for Laplace–Stieltjes integrals.* Mat. Stud. 2011. **35** (2), 131–141.

[22] Salo T.M., Skaskiv O.B. *Minimum modulus of lacunary power series and h-measure of exceptional sets.* Ufa Math. J. 2017, **9** (4), 135–144. doi:10.13108/2017-9-4-135

[23] Skaskiv O.B. *On certain relations between the maximum modulus and the maximal term of an entire Dirichlet series*, Math. Notes. 1999, **66** (2), 223–232. https://doi.org/10.1007/BF02674881 Transl. from Math. Notes, 1999. **66** (2), 282–292.

[24] Skaskiv O.B. *On the classical Wiman's inequality for entire Dirichlet series.* Visn. Lviv. un-tu, ser. mekh.-mat. 1990. **54**, 180–182. (in Ukrainian)

[25] Kuryliak A.O., Ovchar I.E., Skaskiv O.B. *Wiman's inequality for Laplace integrals.* Int. Journal of Math Analysis. 2014, **8**, 381–385. http://dx.doi.org/10.12988/ijma.2014.4232

[26] Skaskiv O., Bandura A. Asymptotic estimates of positive integrals and entire functions, Lviv, Ivano-Frankivsk, Publisher O.M. Goliney, 2015, 108 p. (in Ukrainian)

[27] Gopala Krishna J., Nagaraja Rao I.H. *Generalised inverse and probability techniques and some fundamental growth theorems in* $\mathbb{C}^k$. J. Indian Math. Soc. 1977, **41**, 203–219.

[28] Fenton P. C. *Wiman–Valiron theory in two variables.* Trans. Amer. Math. Soc. 1995, **347** (11), 4403–4412.

[29] Schumitzky A. Wiman-Valiron theory for functions of several complex variables. Ph. D. Thesis: Cornel. Univ., 1965.

[30] Skaskiv O.B., Zrum O.V. *Wiman's type inequality for entire functions of two complex variables with rapidly oscilic coefficient.* Mat. metods and fys.-mekh. polya. 2005, **48** (4), 78–87. (in Ukrainian)

[31] Skaskiv O.B., Zrum O.V. *On inprovement of Fenton's inequality for entire functions of two complex variables.* Math. Bull. Shevchenko Sci. Soc. 2006, **3**, 56–68. (in Ukrainian)

[32] Zrum O.V., Skaskiv O.B. *On Wiman's inequality for random entire functions of two variables.* Mat. Stud. 2005, **23** (2), 149–160. (in Ukrainian)

[33] Kuryliak A.O., Skaskiv O.B. *Wiman's type inequalities without exceptional sets for random entire functions of several variables.* Mat. Stud. 2012, **38** (1), 35–50.

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Через $\mathcal{A}^2$ позначимо клас аналітичних функцій вигляду $f(z)=\sum_{n+m=0}^{+\infty}a_{nm}z_1^nz_2^m$, з областю збіжності $\mathbb{T}=\{z=(z_1,z_2)\in\mathbb{C}^2\colon |z_1|<1,\ |z_2|<+\infty\}=\mathbb{D}\times\mathbb{C}$ і $\frac{\partial}{\partial z_2}f(z_1,z_2)\not\equiv 0$ в $\mathbb{T}$. У цій статті ми доведемо деякі аналоги нерівності Вімана для аналітичних функцій $f\in\mathcal{A}^2$. Нехай функція $h\colon\mathbb{R}_+^2\to\mathbb{R}_+$ така, що h неспадна по кожній змінній і $h(r)\geq 10$ для всіх $r\in T:=(0,1)\times(0,+\infty)$ і $\iint_{\Delta_\varepsilon}\frac{h(r)dr_1dr_2}{(1-r_1)r_2}=+\infty$ для деякого $\varepsilon\in(0,1)$, де $\Delta_\varepsilon=\{(t_1,t_2)\in T\colon t_1>\varepsilon,\ t_2>\varepsilon\}$. Будемо говорити, що $E\subset T$ є множиною скінченної h-міри на T, якщо $\nu_h(E){:=}\iint\limits_{E\cap\Delta_\varepsilon}\frac{h(r)dr_1dr_2}{(1-r_1)r_2}<+\infty$ для деякого $\varepsilon>0$. Для $r=(r_1,r_2)\in T$ і функції $f\in\mathcal{A}^2$ позначимо

$$M_f(r)=\max\{|f(z)|\colon |z_1|\leq r_1,|z_2|\leq r_2\},\ \mu_f(r)=\max\{|a_{nm}|r_1^nr_2^m\colon (n,m)\in\mathbb{Z}_+^2\}.$$

Доведено таку теорему: *Нехай $f\in\mathcal{A}^2$. Для кожного $\delta>0$ існує множина $E=E(\delta,f)$ асимптотично скінченної h-міри на T така, що для всіх $r\in(T\cap\Delta_\varepsilon)\backslash E$ виконується нерівність*

$$M_f(r)\leq\frac{h^{3/2}(r)\mu_f(r)}{(1-r_1)^{1+\delta}}\ln^{1+\delta}\Big(\frac{h(r)\mu_f(r)}{1-r_1}\Big)\cdot\ln^{1/2+\delta}\frac{er_2}{\varepsilon}.$$