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ATTITUDE DETERMINATION BASED ON GEOMETRIC RELATIONS

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Abstract—The algorithms of attitude determination based on geometric relations and least-squares method is suggested. Comparison with algorithm QUEST is fulfilled.

Index Terms—Attitude determination; algorithm.

I. INTRODUCTION

The problem of attitude determination on the basis of information about non-parallel vectors is analyzed. In the known matrix algorithms [1], [2] the angles of orientation are determined from the matrix of orientation. At the using of quaternion the axis and angle of turn of the body are determined from a quaternion of turn [3], [4]. The proposed algorithms use geometric relations, which take place when the body rotates [5].

II. PROBLEM FORMULATION

Suppose we know normalized initial vector $\vec{r}_o$ and this vector after rotation $\vec{r}$. The problem consists of determining the axis of rotation and rotation angle of the body (Fig. 1). The end of the vector trajectory is a circle centered on the axis of rotation.

Figure 1 shows the angle of rotation of the body σ, the axis of rotation of the body and the unit vector $\vec{e}$ of this axis.

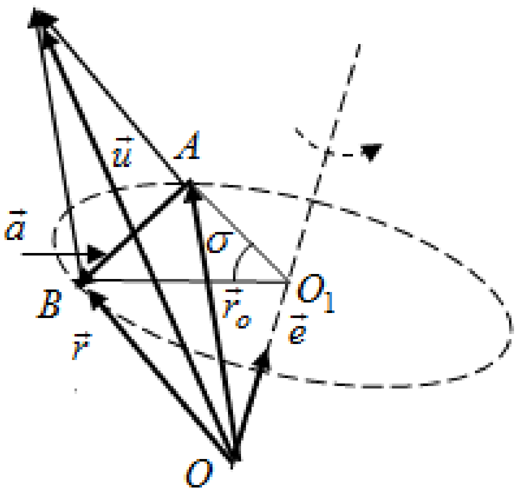

Fig. 1. Vectors

Find relations for body attitude determination on the base of geometrical approach and least-squares estimation.

III. SOLUTION OF THE PROBLEM

Using quaternion of rotation $\boldsymbol{q}$ we can right

$$\boldsymbol{r} = \boldsymbol{q} \circ \boldsymbol{r}_o \circ \tilde{\boldsymbol{q}} \quad \text{or} \quad \boldsymbol{r} \circ \boldsymbol{q} = \boldsymbol{q} \circ \boldsymbol{r}_o . \tag{1}$$

For two quaternions $\boldsymbol{\Lambda} = \lambda_0 + \vec{\lambda}$ and $\boldsymbol{M} = \mu_0 + \vec{\mu}$ we have

$$\boldsymbol{\Lambda} \circ \boldsymbol{M} = \lambda_0\mu_0 - \vec{\lambda} \cdot \vec{\mu} + \lambda_0\vec{\mu} + \mu_0\vec{\lambda} + \vec{\lambda} \times \vec{\mu} .$$

Specify $\boldsymbol{\Lambda} = \boldsymbol{q} = q_0 + \vec{q}$, $\boldsymbol{M} = \vec{\mu} = \vec{r}$. Then

$$(\vec{r} - \vec{r}_o)q_0 + (\vec{r} + \vec{r}_o) \times \vec{q} - (\vec{r} - \vec{r}_o) \cdot \vec{q} = 0 . \tag{2}$$

This expression equals zero if equals zero scalar part

$$(\vec{r} - \vec{r}_o) \cdot \vec{q} = 0 \tag{3}$$

and if equals zero vector part

$$(\vec{r} - \vec{r}_o)q_0 + (\vec{r} + \vec{r}_o) \times \vec{q} = 0 . \tag{4}$$

Denoting $\vec{a} = \vec{r} - \vec{r}_o$ write equation (3) as

$$\vec{a} \cdot \vec{q} = 0 . \tag{5}$$

It means that vector $\vec{a}$ is perpendicular to axis of rotation.

Denoting $\vec{u} = \vec{r} + \vec{r}_o$ write equation (4) as

$$\vec{a}q_0 = \vec{q} \times \vec{u} . \tag{6}$$

From equation (6) we see again that vector $\vec{a}$ is perpendicular to the axis of rotation.

When $\sigma = 180°$ we get $q_0 = \cos\frac{\sigma}{2} = 0$ and $\vec{q} \times \vec{u} = 0$ because of $\vec{q} \parallel \vec{u}$ or $\vec{u} = 0$.

The relation $\vec{q} \parallel \vec{u}$ takes places when the vectors $\vec{r}_o, \vec{r}$ are not perpendicular to the axis of rotation (and $\sigma = 180°$). As vector $\vec{u}$ is perpendicular to the vector $\vec{a}$ we conclude that the vector $\vec{q}$ is perpendicular to the vector $\vec{a}$.

The relation $\vec{u}=0$ takes places when the vectors $\vec{r}_o, \vec{r}$ are perpendicular to the axis of rotation (and $\sigma = 180°$). In this case using of the equation (6) for axis determination is impossible and we have to use the equation (5).

Analyze the equation (6) which may be written as

$$\vec{a} = \vec{g} \times \vec{u}\,, \tag{7}$$

where $\vec{g} = \dfrac{\vec{q}}{q_0} = \vec{e}\,\mathrm{tg}\dfrac{\sigma}{2}$ is the Gibbs vector.

Rewrite the equation (7) in the matrix form

$$\boldsymbol{a} = -U\boldsymbol{g}\,, \tag{8}$$

where $U = \begin{vmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{vmatrix}$, $\boldsymbol{g} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$.

For several vectors

$$\boldsymbol{a}_i = -U_i\boldsymbol{g} \quad (i = 1...n), \tag{9}$$

where n is the number of vectors.

Determine the vector $\boldsymbol{g}$ minimizing function

$$\begin{aligned} f &= \frac{1}{2}\sum_1^n \mu_i \left\|\boldsymbol{a}_i + U_i\boldsymbol{g}\right\|^2 \\ &= \frac{1}{2}\sum_1^n \mu_i \left(\boldsymbol{a}_i + U_i\boldsymbol{g}\right)^T\left(\boldsymbol{a}_i + U_i\boldsymbol{g}\right) \\ &= \frac{1}{2}\sum_1^n \mu_i \left(\boldsymbol{a}_i^T + \boldsymbol{g}^T U_i^T\right)\left(\boldsymbol{a}_i + U_i\boldsymbol{g}\right) \\ &= \frac{1}{2}\sum_1^n \mu_i \left(\boldsymbol{a}_i^T\boldsymbol{a}_i + \boldsymbol{g}^T U_i^T\boldsymbol{a}_i + \boldsymbol{a}_i^T U_i\boldsymbol{g} + \boldsymbol{g}^T U_i^T U_i\boldsymbol{g}\right), \end{aligned} \tag{10}$$

where μ_i are non-negative weights.

Write the condition of extreme

$$\begin{aligned} \frac{\partial f}{\partial \boldsymbol{g}} &= \frac{1}{2}\sum_1^n \mu_i \left[U_i^T\boldsymbol{a}_i + U_i^T\boldsymbol{a}_i + \left(U_i^T U_i + U_i^T U_i\right)\boldsymbol{g}\right] \\ &= \sum_1^n \mu_i \left(U_i^T\boldsymbol{a}_i + U_i^T U_i\boldsymbol{g}\right) = \boldsymbol{k} + G\boldsymbol{g} = 0, \end{aligned} \tag{11}$$

where

$$\boldsymbol{k} = \sum_1^n \mu_i U_i^T\boldsymbol{a}_i\,, \quad G = \sum_1^n \mu_i U_i^T U_i\,.$$

From equation (11) we find

$$\boldsymbol{g} = -G^{-1}\boldsymbol{k}\,; \tag{12}$$

$$\boldsymbol{e} = \frac{\boldsymbol{g}}{\left\|\boldsymbol{g}\right\|}\,; \tag{13}$$

$$\sigma = 2\,\mathrm{arctg}\left(\left\|\boldsymbol{g}\right\|\right). \tag{14}$$

The solution (12) exists if determinant of matrix G doesn't equal zero. If $\sigma = 180°$, the determinant of matrix G equals zero.

For example for two vectors we have

$$\det\left(G\right) = \left(u_1^2 + u_2^2\right)\left\|\vec{u}_1 \times \vec{u}_2\right\|^2. \tag{15}$$

From (15) it follows that if $\sigma = 180°$ we have $\det(G) = 0$.

For three vectors similar to (15) equation

$$\det\left(C\right) = \left(\left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \left\|\vec{u}_3\right\|^2\right) \times\left(\left\|\vec{u}_1 \times \vec{u}_2\right\|^2 + \left\|\vec{u}_1 \times \vec{u}_3\right\|^2 + \left\|\vec{u}_2 \times \vec{u}_3\right\|^2\right)$$

takes place if three vector are situated in one plane.

Explain the last equation. The expression $\vec{a} = \vec{b} \times \vec{u} = -\vec{u} \times \vec{b}$ in matrix form can be written as $\boldsymbol{a} = -U\boldsymbol{b} = U^T\boldsymbol{b}$ (matrix U is skew-symmetric $U^T = -U$). Therefore $\vec{g} = \vec{u} \times \left(\vec{b} \times \vec{u}\right) = -\vec{u} \times \left(\vec{u} \times \vec{b}\right)$ can be written as $\boldsymbol{g} = -U\left(U\boldsymbol{b}\right) = U^T U\boldsymbol{b}$.

Thus, expression $U^T\boldsymbol{a} + U^T U\boldsymbol{b} = 0$ in vector form looks like

$$\vec{u} \times \vec{a} = \vec{u} \times \left(\vec{b} \times \vec{u}\right).$$

Notice that vector $\vec{g}$ can be written as

$$\boldsymbol{g} = \boldsymbol{b}\boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{u}\boldsymbol{u}^T\boldsymbol{b} = \boldsymbol{u}^T\boldsymbol{u}\boldsymbol{b} - \boldsymbol{u}\boldsymbol{u}^T\boldsymbol{b} = \left(\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^T\right)\boldsymbol{b}$$

Assume $n = 2$. Will estimate the influence of error of the vector $\vec{r}_{02}$ on the accuracy of algorithm if

$$\psi = 10°;\ \theta = 20°;\ \varphi = 30°;$$

$$\vec{r}_{o1} = [0.5547\ 0\ 0.8321]';\ \vec{r}_{o2} = [0\ 0.995\ 0.0995]'.$$

The vector of error, by a size 0.001, will form perpendicular to the vector $\vec{r}_{o2}$ with variable direction with a step 10° in a plane, perpendicular to the vector $\vec{r}_{o2}$. The new value of vector r_{o2} is normalized. The results of simulation of errors of angle determination are presented in Fig. 2.

For comparison, the differences of errors of angles determination calculated by means of the proposed algorithm and algorithm QUEST are presented in Fig. 3.

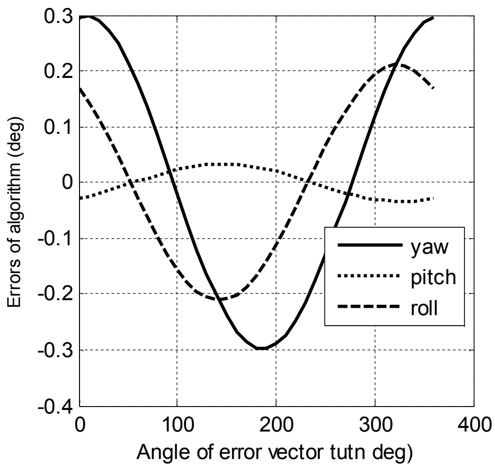

Fig. 2. Errors of angles determination

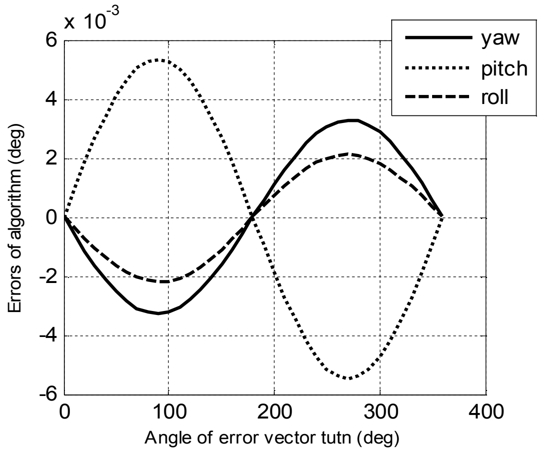

Fig. 3. Differences errors of angles determination

In Figures 4 and 5 are represented results of computations for another angles and vectors

$$\psi = 30°;\ \theta = 20°;\ \varphi = 10°$$

$\vec{r}_{o1} = [0.5547\ 0\ 0.8321]'$

$\vec{r}_{o2} = [0.9759\ 0.0976\ 0.1952]'$

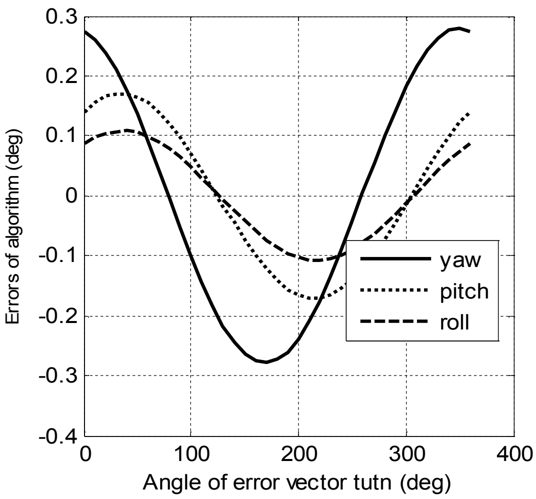

Fig. 4. Differences errors of angles determination

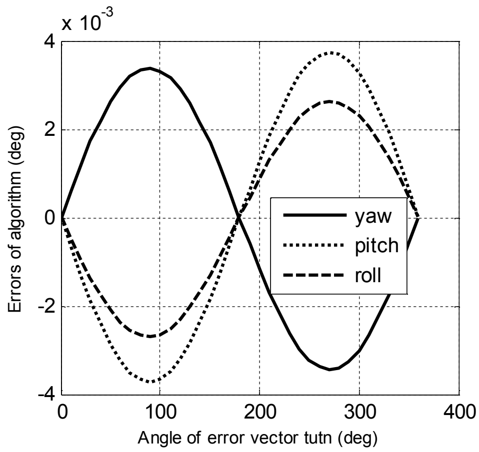

Fig. 5. Differences errors of angles determination

These results are similar to the results presented in Figs 2 and 3.

Thus if $\sigma \neq 180°$, the accuracy of proposed algorithm is equivalent to the accuracy of algorithm QUEST.

If $\sigma = 180°$ (determinant of matrix $\mathbf{G}$ equals zero), we may use the scalar form (5) and such way of vector $\vec{e}$ finding [5]

$$\vec{e} = \frac{\vec{b}}{\|\vec{b}\|}, \qquad (16)$$

where

$$\vec{b} = \vec{a}_1 \times \vec{a}_2 + \vec{a}_2 \times \vec{a}_3 + \vec{a}_3 \times \vec{a}_1;\ \ \vec{a}_3 = \vec{c} - \vec{c}_o;$$

$$\vec{c}_o = \frac{\vec{r}_{o1} \times \vec{r}_{o2}}{\|\vec{r}_{o1} \times \vec{r}_{o2}\|};\ \ \vec{c} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|}.$$

Now show that using vectors $\boldsymbol{a}_i$, we can determine vector $\vec{e}$ and angle σ separately.

Suppose we know vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$. Find vector $\vec{e}$ which is perpendicular to these vectors. In this case

$$\boldsymbol{a}_i^T \boldsymbol{e} = 0\ \left(i = 1...3\right). \qquad (17)$$

Using equation (17), write $\boldsymbol{a}_i \boldsymbol{a}_i^T \boldsymbol{e} = 0$. Summing these expressions, we obtain

$$A\boldsymbol{e} = 0, \qquad (18)$$

where

$$A = \boldsymbol{a}_1 \boldsymbol{a}_1^T + \boldsymbol{a}_2 \boldsymbol{a}_2^T + \boldsymbol{a}_3 \boldsymbol{a}_3^T.$$

Rewrite this expression in the form

$$A\boldsymbol{e} = \lambda \boldsymbol{e}, \qquad (19)$$

where $\lambda = 0$.

That is, vector $\boldsymbol{e}$ is an eigenvector $\boldsymbol{v}$ of the matrix $\mathbf{A}$ that corresponds to the eigenvalue $\lambda = 0$.

As an example, assume the parameters which correspond to Figs 4 and 5. The vectors $\vec{c}_o$ and $\vec{c}$ are calculated in accordance with equation (16).

The quaternion of rotation and vector $\boldsymbol{e}$ equal

$$q = [0.9515\ \ 0.0381\ \ 0.1893\ \ 0.2393],$$
$$\boldsymbol{e} = [0.1240\ \ 0.6156\ \ 0.7782]^T.$$

Using equation (19), we get

$$A = \begin{bmatrix} 0.4037 & 0.1006 & -0.1439 \\ 0.1006 & 0.1828 & -0.1606 \\ -0.1439 & -0,1606 & 0.1500 \end{bmatrix};\ \lambda = 4.7714\text{e-}17.$$

and the same unit vector.

After this we can find the angle of rotation. Suppose we know normalized vector $\boldsymbol{r}_n$ in body frame and the vector $\boldsymbol{e}$ (Fig. 6). Vector $\vec{m}$ which is perpendicular to the vector $\boldsymbol{e}$, equals. $\vec{m} = \vec{r}_n - (\vec{r}_n \cdot \vec{e})\vec{e}$. In matrix form we have

$$\begin{aligned} \boldsymbol{m} &= \boldsymbol{r}_n - \boldsymbol{e}(\boldsymbol{e}^T\boldsymbol{r}_n) = \boldsymbol{r}_n - \boldsymbol{e}\boldsymbol{e}^T\boldsymbol{r}_n \\ &= \boldsymbol{r}_n - E\boldsymbol{r}_n = (I - E)\boldsymbol{r}_n = E_1\boldsymbol{r}_n \end{aligned} \tag{20}$$

where $E = \boldsymbol{e}\boldsymbol{e}^T$; $E_1 = I - E$.

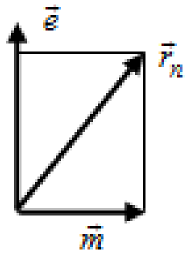

Fig. 6. Vectors

Similar equation takes place for the vector in reference frame

$$\boldsymbol{m}_o = E_1\boldsymbol{r}_{on}. \tag{21}$$

As $E^T = E$; $E_1^T = E_1$; $\boldsymbol{e}^T\boldsymbol{e} = 1$ we find

$$\begin{aligned} \vec{m}\cdot\vec{m}_o &= \boldsymbol{m}^T\boldsymbol{m}_o = (E_1\boldsymbol{r}_n)^T E_1\boldsymbol{r}_{on} = \boldsymbol{r}_n^T E_1^T E_1\boldsymbol{r}_{on} \\ &= \boldsymbol{r}_n^T E_1^2\boldsymbol{r}_{on}(I - E)^2\boldsymbol{r}_{on} = \boldsymbol{r}_n^T(I - 2E + E^2)\boldsymbol{r}_{on} \\ &= \boldsymbol{r}_n^T(I - 2E + \boldsymbol{e}\boldsymbol{e}^T\boldsymbol{e}\boldsymbol{e}^T)\boldsymbol{r}_{on} = \boldsymbol{r}_n^T(I - 2E + \boldsymbol{e}\boldsymbol{e}^T)\boldsymbol{r}_{on} \\ &= \boldsymbol{r}_n^T(I - 2E + E)\boldsymbol{r}_{on} = \gamma, \end{aligned}$$

where $\gamma = \boldsymbol{r}_n^T E_1\boldsymbol{r}_{on}$.

Thus

$$\cos\sigma = \frac{\gamma}{\|\boldsymbol{m}\|\cdot\|\boldsymbol{m}_0\|} = \frac{\gamma}{\gamma_o},$$

where $\gamma_o = \boldsymbol{r}_{on}^T E_1\boldsymbol{r}_{on}$.

Use least-square estimation and specify the lost function for three vectors

$$f = \sum_{i-1}^{3}(\gamma_i - \gamma_{oi}\mu)^2,$$

where $\mu = \cos\sigma$.

In accordance with condition $\frac{\partial f}{\partial\mu} = 0$ we find

$$\mu = \cos\sigma = \sum_{i=1}^{3}\frac{\gamma_i\gamma_{oi}}{\gamma_{oi}^2}. \tag{22}$$

For conceded above example using quaternion, we get $\sigma = 0.6251$.

Using equations (21), (22), we get

$$E_1 = \begin{bmatrix} 0.9846 & -0.0763 & -0.0965 \\ -0.0763 & 0.6210 & -0.4791 \\ -0.0965 & -0.4791 & 0.3944 \end{bmatrix};$$

$$\gamma_1 = 0.3948;\quad \gamma_{o1} = 0.4869;$$
$$\gamma_2 = 0.7210;\quad \gamma_{o2} = 0.8891;$$
$$\gamma_3 = 0.4631;\quad \gamma_{o3} = 0.5711$$

and the same angle σ.

The procedure of the angle σ sign determination is described in [5].

IV. Conclusion

The algorithm of attitude determination based on geometric relations and least-square method is proposed. If angle of body's turn doesn't equal 180° the accuracy of proposed algorithm is equivalent to the accuracy of algorithm QUEST.

The simple algorithm of separate determination of unite vector and angle of rotation is proposed.

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Л. М. Рижков. Визначення орієнтації на основі геометричних співвідношень

Запропоновано алгоритми визначення орієнтації твердого тіла, які базуються на геометричних співвідношеннях та методі найменших квадратів.

Ключові слова: визначення орієнтації; алгоритм.

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Л. М. Рыжков. Определение ориентации на основе геометрических соотношений

Предложены алгоритмы определения ориентации твердого тела, базирующиеся на геометрических соотношениях и методе наименьших квадратов.

Ключевые слова: определение ориентации; алгоритм.

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