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Reexamining the likelihood of extreme returns in international stock markets

Abstract

The authors set out in this study to reexamine the probability of extreme returns in various international stock markets using the modified Hill estimator, a procedure designed to obtain unbiased estimates of the tail index with no prior selection of the number of tail observations. The empirical results find that the probability of extreme returns is considerably higher than that proposed in Vilasuso and Katz (2000), thereby implying that extreme risk is increasing over time. The authors also find strong evidence of the increasing likelihood of extreme single-day returns, especially in the upper tail, and that this is particularly prevalent after periods of financial crisis. The authors' results suggest that market participants need to adapt their portfolios to this additional unexpected risk.

Keywords: extreme value theory, likelihood of extreme returns, test of tail index consistency.
JEL Classification: G10.

Introduction

When setting out to evaluate their investment performance, financial risk measures have become an issue of major concern for financial market participants, with a considerable body of literature having been dedicated to the study of expected returns and volatility, as well as the correlation with financial assets. However, much less attention has been paid to the exploration of extreme price movements, particularly from a standpoint of the statistical properties of such extreme movements; this is despite the fact that extreme price falls resulting from market crashes, such as the 2008 sub-prime mortgage crisis, clearly have significant impacts on both investors and the economy as a whole.

One phenomenon which is particularly prevalent during all periods of financial crisis is the significant declines found in both stock prices and regional currency values, with most of the available evidence indicating that these declines originate from deteriorations in regional economic factors, such as malfunctioning of the banking system or excessive external borrowing. However, not every financial market crash can be interpreted purely in terms of specific economic factors; for example, the October 1987 stock market collapse, even now, remains something of a puzzle (Roll, 1988; Cutler et al., 1989). Thus, it is argued that severe price falls are better viewed as extreme movements (Longin, 1996; Vilasuso and Katz, 2000), with the assessment of the likelihood of such events having now become a key issue in risk measurement.

Growing interest in extreme price movements has become obvious in the popular press over recent years (Story and Bowley, 2011), since such extreme movements are clearly more common than they used to be[1] however, despite this, there remains a distinct lack of any direct evidence of the greater likelihood of occurrences of such extreme price movements. Therefore, the primary aim of the present study is to try to determine whether there has indeed been any significant change, over time, in the likelihood of any substantial swings in single-day returns.

A sound understanding of the likelihood of extreme price movements will necessarily involve a close examination of the probability density function tails of the returns. Although it has become widely recognized over recent years that returns distributions present fatter tails than those of normal distributions, no consensus has yet been reached within empirical circles with regard to the specific types of distribution into which the observations may appropriately fit[2].

Three specific fat-tailed alternatives are worthy of note for the considerable attention received by these approaches over the years; these are heavy-tailed stable distribution, Student-t distribution and the autoregressive conditional heteroskedastic (ARCH) process. Any comparison between these models is of course hampered by the fact that some of them are non-nested and also exhibit infinite variance (Jansen and De Vries, 1991); however, if we focus instead on limiting the distribution of the extreme values, then the models are regarded as being nested.

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[1] For example, during the 30-year period from early 1980 to August 2011, ten of the 20 largest daily upswings and 11 of the 20 largest daily falls occurred in just the last three years.
[2] See Boothe and Glassman (1987), Hols and de Vries (1991) and Vilasuso and Katz (2000).

Within this context, the application of such a 'limit law' has been justified as being more feasible than placing reliance on any particular type of distribution (Jansen and De Vries, 1991; Vilasuso and Katz, 2000). As such, particular focus should be placed on the distribution of the extremes, which will necessarily involve an examination of the probability density function tails, a direct measure of which is provided by the tail index. Given the value of the tail index, we should then be able to calculate the probability of any substantial single-day movements.

The estimation of the tail index can be undertaken through the use of 'extreme value theory' (EVT), within which both parametric and non-parametric approaches can be adopted. The advantage of a non-parametric approach over any parametric estimation is that the tail index can be determined by characterizing the limiting distribution of the extreme values without any detailed knowledge of the underlying density function. As a result of its ease of implementation and the lack of any asymptotic bias, the Hill (1975) estimator is the best known and most often applied non-parametric methodology in general use. Furthermore, as noted by Jansen and De Vries (1991), this particular estimator may be more efficient than the maximum likelihood (ML) estimators typically used for parametric estimation.

Although the Hill estimator has become the benchmark within the extant literature (Koedijk et al., 1992; Danielsson and de Vries, 1997; Dewachter and Gielens, 1999; McNeil and Frey, 2000; Longin and Solnik, 2001; Cotter, 2001; Assaf, 2009), it is known to have a specific setback in the case of small samples, insofar as this can result in severe bias. An important element of this bias stems from the selection of the appropriate number of tail observations to be included within the estimation process, since this selection will ultimately result in a tradeoff between the reduction in such bias and the variance in the tail-index estimates; that is, when too many observations are included, the variance in the tail-index estimates will be reduced at the expense of bias in the tail estimates, since too many of these observations will be found in the central range. Conversely, when there are too few observations, bias is reduced but the variance in the estimates becomes too large.

Huisman et al. (2001) proposed an alternative methodology aimed at correcting for the bias in the Hill estimator whilst achieving the advantage of producing virtually unbiased estimates for relatively small samples; Huisman and colleagues referred to this as a 'modified Hill estimator', a method under which the tail estimates are not conditioned on one specific number of tail observations, as in the Hill estimator, but instead, exploits information obtained from a set of Hill estimations, each of which is conditioned on a different number of tail observations. The modified version of the Hill estimator is a weighted average of a set of conventional Hill estimations, with weights obtained by means of simple least squares techniques (Huisman et al., 2001). Recently, a number of empirical studies have applied the modified Hill estimator to estimate the tail index of financial return distribution, such as Werner and Upper (2004), Chapelle et al. (2008), Tursunalieva and Silvapulle (2014), Karimi and Voia (2015).

Therefore, using this modified version of the Hill estimator, as proposed by Huisman et al. (2001), we set out in the present study to reexamine the likelihood of extreme returns in international stock markets, and find that the probability of an extremely large single-day movement in returns is much higher than that proposed in the prior study of Vilasuso and Katz (2000). Furthermore, our structural change tests provide evidence to support the notion that the tail behavior of stock market returns is time-varying. We also find evidence of the likelihood of extreme single-day returns being much greater after periods of financial crisis, particularly in the upper tail.

The remainder of this paper is organized as follows. A description of the methodology adopted in this study is provided in Section 1, along with a brief introduction to our approach to the calculation of the likelihood of extreme returns. This is followed in Section 2 by an introduction to the data and our preliminary analysis of the series. Section 3 presents our empirical results and analysis, followed in Final Section by a summary of the conclusions drawn from this study.

1. Methodology

This section begins with a brief introduction to the statistical behavior of extreme returns before going on to illustrate the modified Hill tail index estimator (Huisman et al., 2001). Finally, we explain the method of gauging the likelihood of any substantial single-day movements in equity prices and the test of tail shape consistency.

1.1. The theory of extremes. Consider a random variable X with the probability density function, f_x, with support (l, u); let F_x represent the cumulative distribution function of X. Consider also a sample of observations $X_1, X_2, \ldots, X_n$ where n represents the sample size; let Y_n denote the highest daily return (the maximum) observed over n trading days. If the variables X_i are statistically independent, and drawn from the same distribution, then the exact distribution of the maximum return, Y_n, can be immediately written as a function of the parent distribution F_x and the length of the selection period n:

$$F_{Y_n}(x) = \left[F_x(x)\right]^n \quad (1)$$

In practice, the exact distribution is not precisely known; therefore, in the present study, we focus on the asymptotic behavior of the extremes. In order to identify the limiting distribution of interest, the maximum variable Y_n is reduced with a location parameter β_n and a scale parameter α_n (assumed to be positive), such that the distribution of the standardized extremes $(Y_n - \beta_n)/\alpha_n$ is non-degenerative. 'Extreme value theory', proven by Gnedenko (1943), specifies three possible limiting extreme value distributions:

Gumbel distribution:

$$F_Y(y) = \exp(-e^{-y}) \quad \text{for } y \in \Re \quad (2)$$

Frechet distribution:

$$F_Y(y) = \exp(-y^{\alpha}) \quad \text{for } y > 0 \ (\alpha > 0) \quad (3)$$
$$= 0 \quad \text{for } y \le 0$$

and *Weibull* distribution:

$$F_Y(y) = \exp\left(-(-y)^{-\alpha}\right) \quad \text{for } y < 0 \ (\alpha < 0) \quad (4)$$
$$= 1 \quad \text{for } y \ge 0$$

A generalized formula, proposed by Jenkinson (1955), grouped these three types of extreme value distributions, as follows:

$$F_Y(y) = \exp\left[-(1 + \gamma \cdot y)^{-\frac{1}{\gamma}}\right]$$

$$\text{for } 1 + \gamma y > 0 \text{ and } \gamma \ne 0 \quad (5)$$

$$= \exp(-\exp(-y)) \quad \text{for } \gamma = 0$$

where γ is the tail index, and is related to the shape parameter α, which determines the type of distribution, and which is equal to $1/\gamma$.

Equation (5) provides us with the relationship between the limiting extreme value distribution and the tail index; in other words, we can use the tail index to infer the limiting extreme value distribution. For example, $\gamma > 0$ ($\alpha > 0$) equates to *Frecht* distribution, $\gamma < 0$ ($\alpha < 0$) equates to *Weibull* distribution, and the intermediate case ($\gamma = 0$), equates to *Gumbel* distribution.

1.2. The modified Hill estimator. In contrast to other tail index estimators, the modified Hill estimator developed by Huisman et al. (2001) provides almost totally unbiased estimates for relatively small samples, with no prior selection of the number of tail observations. This provides us with a superior estimator which allows us to obtain robust tail index estimates from a relatively small sample.

Let us suppose that a sample of n positive independent observations is drawn from some unknown fat-tailed distribution. Let $x(i)$ be the i^{th} order statistic, such that $x(i) \ge x(i-1)$ for $i = 2,\ldots,n$. Let us also suppose that we choose to include m observations from the right tail in our estimate.

Hill (1975) proposed the following estimator, γ, for the tail index:

$$\gamma(m) = \frac{1}{m}\sum_{j=1}^{m}\ln\left(x_{n-j+1}\right) - \ln\left(x_{n-m}\right) \quad (6)$$

which is a maximum likelihood (ML) estimator for a conditional Pareto distribution, taking the $(m + 1)^{th}$ observation as the threshold.

The greatest challenge in the use of the Hill estimator is the non-trivial choice of m, essentially as a result of the trade-off between bias and precision in the selection of m (Hall, 1990). In order to correct for the bias in the Hill estimator with regard to small samples, Huisman et al. (2001) provided a modified version of the original Hill estimator which he obtained by observing the bias in the Hill estimator with an increase in the number of tail observations up until m, where m is equal to half of the sample size:

$$\gamma(m) = \beta_0 + \beta_1 m + \varepsilon(m) \quad m = 1,\ldots,n/2 \quad (7)$$

Huisman et al. (2001) demonstrated that the optimal estimation for the tail index is the intercept β_0, with the estimation of the shape parameter α being equal to $1/\beta_0$.

1.3. The test of tail shape consistency. In order to carry out a test of the hypothesis that the tail index of the distribution remains constant over time, we apply the P-test developed by Quintos et al. (2001), as follows:

$$P = \frac{m_1\hat{\alpha}_2^2\left(\hat{\alpha}_1/\hat{\alpha}_2 - 1\right)^2}{\left(\hat{\alpha}_1^2 + (m_1/m_2)\hat{\alpha}_2^2\right)} \xrightarrow{d} x^2(1) \quad (8)$$

where m_1 and m_2 correspond with the number of extreme observations considered in the two sub-samples; and $\hat{\alpha}_1$ and $\hat{\alpha}_2$ represent the estimates of the shape parameters for the two sub-samples.

2. Data description and preliminary analysis

2.1. Data description. The data for this study, which include daily aggregate stock-market index prices for Australia, Canada, Germany, Hong Kong, Japan, Singapore, Spain and the US, were collected from the Taiwan Economic Journal (TEJ), with the returns being calculated as the first differences in the natural logarithms of the daily closing prices.

The data period runs from 1 January 1980 to 30 September 2011, thereby providing approximately 8,000 observations for each series, with the exception of the Nikkei 225 index, for which data first became available only on 2 January 1981. The indices examined in this study, all of which are market-value weighted, are listed in Table 1, along with the summary statistics. As we can see from the Table, the daily returns are found to have a positive mean (ranging from 0.0027 to 0.0381%) and a high standard deviation (ranging from 1.0011 to 1.8124%). The distributions of the returns are also found to be negatively skewed and presenting excess kurtosis, thereby suggesting departure from normal distribution; this is consistent with the results of the Jarque-Bera (JB) test, which follows chi-squared distribution with two degrees of freedom.

Table 1. Summary statistics of daily stock-return data

Market	Index	Mean (%)	Std. dev (%)	Skewness	Kurtosis	JB	Ljung-Box Q (15)	Ljung-Box Q^2 (15)	ARCH χ^2_5
Australia	All-ordinaries	0.0264	1.0337	-3.18	85.45	2,263,825***	85.19***	1847.90***	6.14***
Canada	TSE	0.0233	1.0011	-0.88	16.52	61,815***	66.37***	1728.50***	2.27*
Germany	DAX	0.0302	1.3824	-0.33	9.77	15,385***	40.95***	1894.80***	4.29***
Hong Kong	Hang Seng	0.0381	1.8124	-1.93	45.27	588,041***	54.26***	1886.30***	5.08***
Japan	Nikkei 225	0.0027	1.4190	-0.32	11.94	25,245***	51.88***	1835.10***	5.21***
	Tokyo	0.0065	1.2855	-0.17	17.62	69,178***	66.07***	1686.10***	7.77***
Singapore	SAS All-share	0.0228	1.3276	-1.04	25.19	165,192***	162.90***	1650.60***	1.18
Spain	Madrid General	0.0381	1.2307	-0.11	10.74	19,818***	120.71***	1600.70***	2.73*
US	Dow Jones	0.0319	1.1287	-1.56	42.90	539,775***	44.05***	2006.30***	4.29***
	S&P 500	0.0295	1.1523	-1.21	30.11	248,297***	58.00***	2039.10***	4.35***

Notes: Daily stock return data cover the period 1 January 1980 to 30 September 2011, with the exception of the Nikkei 225 index, for which data first became available only on 2 January 1981. * significance at the 5% level; ** significance at the 1% level; *** significance at the 0.1% level.

Table 1 also summarizes the test results on autocorrelation in the stock returns. The Ljung-Box (1978) test, which presents portmanteau statistics for serial correlation up to the fifteenth-order, rejects the null hypothesis of non-correlation at the 0.1% significance level for each market index, thereby indicating that the returns are auto-correlated. As regards the squared returns, the Ljung-Box (1978) test also rejects the null hypothesis of no auto-correlation in each of the markets.

Furthermore, significant auto-correlation is revealed among the squared returns as a result of the application of the ARCH test for heteroskedasticity. Since we employ the non-parametric estimates of the tail index, the estimation results will be unaffected by auto-correlation, a finding which is commonly detected in stock market returns data (Vilasuso and Katz, 2000).

2.2. The behavior of extreme observations. In order to focus on the extreme observations in our sample, we first of all define the total number (absolute value) of extreme observations as the case where the observed returns (absolute value) are higher than 10% and 20%. Furthermore, in order to observe the clustering of extreme behavior, a single extreme observation is defined as one which is neither preceded nor followed by another extreme observation in the 5, 120 and 240 trading-day periods. That is to say, if more than one observed extreme return is found to occur in the above specified period, the number of extreme observations will be viewed as only one.

We accumulate the total number of single extreme observations as the frequency at which the extreme observations occurred, with Table 2 providing a summary of these extreme observations. As we can see from the table, all of the markets experienced single-day changes in the index in excess of 10%, with the Hong Kong market index exhibiting a relatively higher total number of extreme observations than any of the other market indices.

Table 2. Extreme return observations

Market	Larger than 10% Total	Larger than 10% Single (5)	Larger than 10% Single (120)	Larger than 10% Single (240)	Larger than 20% Total	Larger than 20% Single (5)	Min 1	Min 2	Min 1%	Max 1%	Max 2	Max 1
Australia	1	0	0	0	1	0	-28.78%	-8.55%	-2.76%	2.35%	7.06%	8.81%
Canada	1	0	0	0	0	0	-12.01%	-9.79%	-3.10%	2.50%	8.65%	9.37%

Table 2 (cont.). Extreme return observations

	Larger than 10%				Larger than 20%		Min 1	Min 2	Min 1%	Max 1%	Max 2	Max 1
	Total	Single (5)	Single (120)	Single (240)	Total	Single (5)						
Germany	3	2	1	1	0	0	-13.71%	-9.87%	-4.12%	3.48%	10.69%	10.80%
Hong Kong	14	8	6	4	2	1	-40.54%	-24.52%	-5.09%	4.57%	13.41%	17.25%
Japan: Nikkei 225	7	4	3	3	0	0	-16.14%	-12.11%	-3.94%	3.64%	12.43%	13.23%
Japan: Tokyo	5	2	2	2	0	0	-15.81%	-12.80%	-3.56%	3.36%	12.86%	16.89%
Singapore	7	4	3	3	1	0	-23.40%	-12.96%	-3.58%	3.42%	11.06%	12.87%
Spain	1	0	0	0	0	0	-9.73%	-9.68%	-3.51%	3.24%	9.87%	13.74%
US: Dow Jones	3	2	1	1	1	0	-25.63%	-8.38%	-3.00%	2.91%	10.33%	10.51%
US: S&P 500	3	2	1	1	1	0	-22.90%	-9.47%	-3.07%	2.99%	10.25%	10.96%

Notes: Total: total number of observations larger (in absolute value) than 10% and 20%; Single (5): the number of extreme observations neither followed nor proceeded by another extreme observation in 5 days; Single (120): the number of extreme observations neither followed nor proceeded by another extreme observation in 120 days, Single (240): the number of extreme observations neither followed nor proceeded by another extreme observation in a year; Max 1: the largest observation; Max 2: the second largest observation; Max 1%: the first percentile of the largest observations; Min 1: the smallest observation; Min 2: the second smallest observation; Min 1%: the first percentile of the smallest observations.

Furthermore, we find that the single extreme observations in the five-day intervals account for almost half of the total extreme observations, thereby demonstrating the clustering behavior of these extreme observations. We also find that absolute values of the smallest returns tend to be larger than those of the largest returns, with the exceptions of Japan (Tokyo) and Spain. For instance, the smallest returns in the Hong Kong market index are −40.54%; however, in terms of absolute value, they are higher than the largest returns, at 17.25%.

We also find that the two largest returns tend to be closer than the two smallest returns. For instance, the absolute value of the difference between the two smallest returns in the Hong Kong market index is 16.02%[3], which is greater than that for the two largest returns, at 3.84%[4]. Based upon these findings, we have identified a discernible and consistent phenomenon among all of the observed market indices, which is that the extreme downside impacts on the market are more striking than the extreme upside impacts.

3. Empirical results and analysis

3.1. Asymptotic distribution of extreme returns. Given an estimation of the tail index (β_0), we can establish extreme return levels, which are rarely exceeded, by extrapolating the empirical distribution (potential) outside its observed domain. For instance, consider an extreme value, $\hat{x}_p$, where our interest lies in determining the likelihood (p) that an observation over a certain time period (k) will not exceed $\hat{x}_p$. This probability can be expressed as:

$$P\{X_1 \le \hat{x}_p, ..., X_k \le \hat{x}_p\} = F_X^k(\hat{x}_p) = 1 - p \qquad (9)$$

Hols and de Vries (1991) and Jansen and de Vries (1991) further demonstrated an estimator of the excess levels, as follows:

$$\hat{x}_p = \frac{(kr / pn)^{\beta_0} - 1}{1 - 2^{-\beta_0}}[X_{(n-r)} - X_{(n-2r)}] + X_{(n-r)} \qquad (10)$$

where n is the number of observations, $r = m/2$.

Proof of the consistency of $\hat{x}_p$ was provided by Dekkers and de Haan (1989); as such, given the value of $\hat{x}_p$ together with an estimate of the tail index, Equation (10) can be used to determine the likelihood of an extreme event from the sample order statistics. De Haan et al. (1994) demonstrated that EVT enables us to calculate the probability of extreme events, even if an event has never been observed during an in-sample period. This theory is therefore very useful for investors with concerns with regard to the possible outcomes of extreme events.

Table 3 reports the estimates of the tail index, β_0, and the shape parameter, α, for the upper (positive) and lower (negative) tails using the modified Hill estimator developed by Huisman et al. (2001). As noted above, since the tail index has proven to be a good indicator of the mass in the tails, it therefore provides us with a direct measure of the type of asymptotic distribution of the extremes. As shown in Table 3, all of the estimates of the tail index are found to be greater than 0, thereby indicating that the distributions of the upper and lower tails are Frechet distributions.

Furthermore, the estimates of the shape parameter for both tails are found to be greater than 2 for each of the market indices, thereby suggesting that there is little evidence of stock returns following a heavy-tailed stable distribution, and instead, that a Student-t distribution or an ARCH process is consistent with the estimates of the shape parameter.

[3] -40.54% − (-24.52%) = -16.02%.

[4] 17.25% − 13.41% = 3.84%.

Table 3. Estimates of the tail index and daily tail probabilities

Market	Lower tail					Upper tail				
	Tail index	$\hat{\alpha}^-$	Tail probabilities			Tail index	$\hat{\alpha}^+$	Tail probabilities		
			$\hat{x}_p$ = -10%	$\hat{x}_p$ = -20%	$\hat{x}_p$ = -30%			$\hat{x}_p$ = 10%	$\hat{x}_p$ = 20%	$\hat{x}_p$ = 30%
Australia	0.321613	3.11	0.196533	0.027228	0.008211	0.261799	3.82	0.083868	0.007671	0.001782
Canada	0.339317	2.95	0.198571	0.030023	0.009583	0.308208	3.24	0.101635	0.012522	0.003544
Germany	0.282062	3.55	0.394559	0.047523	0.012744	0.281222	3.56	0.296773	0.034065	0.008957
Hong Kong	0.308060	3.25	0.942267	0.143149	0.043787	0.275107	3.63	0.583439	0.069653	0.018378
Japan: Nikkei 225	0.250604	3.99	0.381684	0.038761	0.009141	0.259213	3.86	0.310449	0.032093	0.007763
Japan: Tokyo	0.286512	3.49	0.294189	0.035480	0.009596	0.284051	3.52	0.270277	0.031801	0.008483
Singapore	0.331734	3.01	0.371589	0.056340	0.017814	0.297806	3.36	0.298893	0.037789	0.010609
Spain	0.269174	3.72	0.262862	0.028433	0.007142	0.255145	3.92	0.195203	0.018561	0.004310
US: Dow Jones	0.302340	3.31	0.224706	0.028690	0.008145	0.268347	3.73	0.139996	0.014023	0.003417
US: S&P 500	0.304154	3.29	0.247121	0.032107	0.009201	0.275642	3.63	0.156655	0.016608	0.004194

Notes: The tail index is estimated using a modified version of the Hill estimator. $\hat{\alpha}^-$: the shape parameter of lower tail. $\hat{\alpha}^+$: the shape parameter of upper tail. Table entries of the tail probabilities are the probability that a single-day return during a given period (120 days) exceeds or falls below $\hat{x}_p$. For the lower tail, $\hat{x}_p$ is set to -10%, -20%, and -30%. For the upper tail $\hat{x}_p$ is set to 10%, 20%, and 30%.

3.2. The probability of an extreme event. The likelihood of occurrences of extreme observations greater (less) than $\hat{x}_p$ which are equal to plus (minus) 10%, 20% or 30% at some time during a given 120 trading-day period is reported in Table 3. As compared to the findings of Vilasuso and Katz (2000), using the Hill estimator to estimate the tail index between 1 January 1980 and 31 December 1997, our results indicate that a substantial single day change in returns would be more likely to occur from 1 January 1980 to 30 September 2011, thereby implying that extreme risk is increasing over time.

For example, our results indicate that during a given 120 trading-day period, the probability of a single-day decline (increase) in price exceeding 10% in the Hong Kong index is approximately 0.94 (0.58). In other words, a single-day decline (increase) exceeding the 10% threshold would occur, on average, once every 0.5 (0.9) years, as compared to the 1.3 (3.7) years indicated by Vilasuso and Katz (2000).

Furthermore, the probability of a single-day decline (or increase) exceeding 10% is found to be substantially higher in the Hong Kong market index than in any of the other market indices observed in the sample; this is consistent with the findings of Vilasuso and Katz (2000). It is also worth noting that in all of the markets, without exception, the probability of a single-day decline going below −10% is above 0.19, as compared to the 0.02 indicated by Vilasuso and Katz (2000), thereby indicating extremely high downside risk in each of the markets especially from 1 January 1998 to 30 September 2011.

Table 3 provides an alternative indicator for portfolio selection based upon a comparison of the probability of extreme returns in the various international stock markets; that is, we present an indicator to show that the higher the probability of extreme returns, the more extreme the risk involved. For example, the Hong Kong market is found to have the highest probability of a single-day decline in stock prices going below −10%, −20% or −30%.

Thus, considering the requirement of investment managers to select portfolios capable of minimizing the probability of extreme losses, such managers may choose to invest in those markets with the lowest probability of extreme returns, thereby fixing their extreme losses at specific levels, or lower.

3.3. Testing the structural stability of the tail estimates. The test results on whether the occurrence of market turbulence would change the tail shape of the return distributions are presented in Table 4, based upon the tests of the *P* statistic, as shown in Equation (8). Within our examination of all stock market shocks over the past 31 years, the October 1987 stock market crash, the 1997 Asian financial crisis and the 2008 financial crisis are considered to be breaking points.

Table 4. Test of stability

Market	Lower tail			Upper tail		
	Pre	Post	*P*	Pre	Post	*P*
	α_1^-	α_2^-		α_1^+	α_2^+	
Panel A: Pre and Post of the October 1987 stock market crash						
Australia	4.06	3.81	1.00	3.97	4.90	12.10***
Canada	3.61	3.49	0.33	3.97	5.24	21.35***

Table 4 (cont.). Test of stability

Market	Lower tail			Upper tail		
	Pre	Post	P	Pre	Post	P
	α_1^-	α_2^-		α_1^+	α_2^+	
Germany	5.03	3.03	55.31***	5.42	3.58	41.52***
Hong Kong	3.40	2.73	11.43***	4.36	4.12	0.82
Japan: Nikkei 225	3.31	4.66	27.32***	3.69	3.54	0.40
Japan: Tokyo	2.72	3.32	10.17***	3.38	2.92	5.26*
Singapore	2.99	2.96	0.04	3.84	3.96	0.25
Spain	5.16	3.54	29.21***	5.42	4.10	17.24***
US: Dow Jones	5.90	3.36	69.06***	5.14	4.38	6.83**
US: S&P 500	5.23	3.80	24.16***	5.80	5.23	2.82
Panel B: Pre and Post of the Asian financial crisis of 1997						
Australia	5.04	4.20	5.33*	6.12	6.64	1.15
Canada	3.64	3.72	0.07	6.52	4.36	5.81*
Germany	4.62	4.92	0.65	5.77	4.80	13.73***
Hong Kong	3.61	4.04	2.07	5.65	3.97	19.90***
Japan: Nikkei 225	5.20	5.59	0.87	3.92	5.02	8.46***
Japan: Tokyo	3.76	4.59	6.72**	3.14	4.01	9.85***
Singapore	3.81	3.16	6.19*	5.56	3.24	43.06***
Spain	5.00	4.85	0.15	6.37	5.47	18.15***
US: Dow Jones	4.28	4.29	0.00	5.98	4.49	9.39***
US: S&P 500	4.34	6.23	20.70***	6.86	4.91	21.57***
Panel C: Pre and Post of the financial crisis of 2008						
Australia	4.00	5.58	7.62**	4.43	6.99	14.68***
Canada	5.64	5.53	0.04	6.38	4.93	9.70***
Germany	8.71	5.35	39.59***	8.36	5.53	30.00***
Hong Kong	5.29	7.55	8.96***	4.51	4.31	0.24
Japan: Nikkei 225	5.39	3.79	17.70***	9.73	5.07	90.20***
Japan: Tokyo	3.38	3.60	0.45	3.52	3.97	1.66
Singapore	4.58	5.36	2.06	4.81	3.00	40.26***
Spain	4.03	4.48	0.99	6.23	4.13	29.43***
US: Dow Jones	5.47	4.63	3.15	6.27	4.43	19.91***
US: S&P 500	5.16	5.02	0.08	5.95	3.97	28.31***

Notes: The subsample labeled as pre-1987 covers the period from 1 January 1980 to 30 September 1987, and the other subsample denoted as post-1987 covers the period from 1 November 1987 to 30 June 1997. The subsample labeled as pre-1997 ranges from 1 January 1992 to 30 June 1997, and the other subsample denoted as post-1997 ranges from 1 January 1998 to 30 June 2003. The subsample labeled as pre-2008 ranges from 1 July 2003 to 31 December 2007, and the other subsample denoted as post-2008 ranges from 1 January 2009 to 30 September 2011. For the lower tail, the *P* statistic is given by equation (8) for testing the null hypothesis that $\hat{\alpha}_1 = \hat{\alpha}_2$, where $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the estimates of the inverse of tail index for the pre and post subperiods respectively. * statistically significant at the 5% level; ** statistically significant at the 1% level; *** statistically significant at the 0.5% level.

In the case of the October 1987 stock market crash, the first sub-sample, denoted as "pre-1987", runs from 1 January 1980 to 30 September 1987, whilst the second, denoted as "post-1987", runs from 1 November 1987 to 30 June 1997. In the case of the 1997 Asian financial crisis, the first sub-sample, denoted as "pre-1997" runs from 1 January 1992 to 30 June 1997, whilst the second, denoted as "post-1997", runs from 1 January 1998 to 30 June 2003. In the case of the 2008 financial crisis, the first sub-sample, denoted as "pre-2008", runs from 1 July 2003 to 31 December 2007, whilst the second, denoted as 'post-2008' runs from 1 January 2009 to 30 September 2011.

Based upon the results of the *P* statistics in Table 4, it is clear that the only market with no significant changes in the lower tail index arising from the impacts of stock market turbulence is Canada. In all cases, without exception, we can see that there is at least one statistically significant change occurring in the upper tail index for each of the market indices. In contrast to the findings of Vilasuso and Katz (2000), the results of the tests applied in the present study demonstrate that for most of the markets, the probability of a substantial single-day return movement has been changed significantly ever since the 1987 stock-market crash.

Furthermore, we have also found that for most of the markets, the probability of a substantial increase in single-day returns has been changed significantly by the Asian and global financial crisis periods of 1997 and 2008. For example, with the exceptions of the Hong Kong and Japan (Tokyo) indices, all of the other upper tail indices are found to have undergone significant changes as a result of the impacts of the 2008 financial crisis. As for the lower tail indices, statistically significant changes are discernible in the cases of the Australian, German, Hong Kong and Japan (Nikkei 225) indices during the 2008 financial crisis period. These results suggest that tail behavior is time-varying and that stock markets tend to become more risky after a financial crisis, particularly in the upper tails.

The related probabilities of extreme changes as a result of the impacts of the 2008 financial crisis are reported in Table 5, from which we can see that the changes in the likelihood of observing an extreme single-day return are consistent with the stability test results reported in Table 4. In the case of the upper tail, we find that eight of the ten markets have a higher ex-post probability of single-day extreme rises in excess of 10%, as compared to the corresponding ex-ante probabilities of extreme rises.

Table 5. Subsample tail probabilities

Market	Sample	Lower tail			Upper tail		
		$\hat{x}_p$ = -10%	$\hat{x}_p$ = -20%	$\hat{x}_p$ = -30%	$\hat{x}_p$ = 10%	$\hat{x}_p$ = 20%	$\hat{x}_p$ = 30%
Australia	Pre-2008	0.052076	0.004338	0.000948	0.022023	0.001362	0.000250
	Post-2008	0.137590	0.006781	0.000970	0.025777	0.000579	0.000050
Canada	Pre-2008				0.002497	0.000051	0.000005
	Post-2008				0.096775	0.005567	0.000923
Germany	Pre-2008	0.049206	0.000859	0.000055	0.022776	0.000344	0.000021
	Post-2008	0.586481	0.042562	0.007382	0.254263	0.014259	0.002166
Hong Kong	Pre-2008	0.160519	0.009366	0.001490			
	Post-2008	0.420239	0.017028	0.001806			
Japan: Nikkei 225	Pre-2008	0.114717	0.005889	0.000878	0.054949	0.000746	0.000038
	Post-2008	0.632053	0.072715	0.018506	0.321856	0.021470	0.003712
Japan: Tokyo	Pre-2008						
	Post-2008						
Singapore	Pre-2008				0.052372	0.002966	0.000497
	Post-2008				0.354860	0.053180	0.016794
Spain	Pre-2008				0.009917	0.000261	0.000027
	Post-2008				0.573426	0.057491	0.013239
US: Dow Jones	Pre-2008				0.007973	0.000203	0.000020
	Post-2008				0.224805	0.018259	0.003715
US: S&P 500	Pre-2008				0.011232	0.000341	0.000038
	Post-2008				0.341072	0.034466	0.008129

As regards the lower tail, the Australian, German, Hong Kong and Japan (Nikkei 225) indices are each found to have a higher ex-post probability of single-day extreme declines than the corresponding ex-ante probabilities of extreme declines below −10%, −20% or −30%. These results add support to the evidence demonstrating that in the post-2008 financial crisis period, extreme price movements are now more common than they used to be (Story and Bowley, 2011).

Conclusions and implications

This study focuses on a re-examination of the probability of extreme returns in various international stock markets using the modified Hill estimator, an approach which provides unbiased estimations for relatively small samples with no prior selection of the number of tail observations. Our approach is a first step towards providing a new window into the examination and measurement of the likelihood of extreme single-day movements in equity prices.

Our major findings are summarized as follows. Firstly, in all of the markets, the asymptotic distributions of the extreme returns are found to be Frechet distributions, which are consistent with the prior findings. Furthermore, the estimates of the shape parameters for both the lower and upper tails are found to be greater than 2 in each of the markets; hence, heavy-tailed alternative models, such as a Student-t distribution or an ARCH process, are consistent with our estimates of the shape parameters.

Secondly, in each of the market indices, the probability of an extreme single-day movement, particularly in the lower tail, is found to be considerably higher than that reported by Vilasuso and Katz (2000). Thirdly, in contrast to the findings of Vilasuso and Katz (2000), our empirical results suggest that for most international stock markets, all tail behavior is time-varying.

Stock markets clearly tend to become more risky, particularly in the upper tails, after periods of financial crisis, with extreme movements in financial prices having tremendous impacts on investment performance; however, the probability of such occurrences has remained something of an enigma for some considerable period of time. We have set out in the present study to re-examine tail behavior in an attempt to highlight the path towards this 'black box' of extreme probabilities.

Our findings, based upon a comparison of the probability of extreme returns in various international stock markets, can be seen as providing an additional device for effective portfolio selection which can enable market participants to adapt their efficient portfolios, not only on the basis of a general risk-return tradeoff, but also with careful consideration of the devastation which can potentially arise from the occurrence of extreme returns.

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