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On a discrete time risk model with time-delayed claims and a constant dividend barrier

Abstract

In this paper a compound binomial risk model with a constant dividend barrier is considered. Two types of individual claims, main claims and by-claims, are defined, where one main claim is dependent with its associated by-claim and the settlement of the by-claim may be delayed for one time period under a certain probability. Formulae for calculating the expected present value of dividend payments up to the time of ruin are provided for discrete-type individual claims. The dividends-penalty identity is then derived. Finally explicit expressions for the corresponding results are obtained for two classes of claim size distributions.

Keywords: compound binomial model, dividend, dividends-penalty identity, main claim, by-claim, dependence.

Intoduction

In recent years, risk models with correlated claims and models with dividend payments have been of the major interests in the risk theory literature. The risk model considered in this paper is a compound binomial model with correlated individual claims and dividend payments that are ruled by a constant dividend barrier.

In this paper a framework of correlated claims is built by introducing two kinds of individual claims, namely main claims and by-claims, and allowing the dependence between each main claim and its associated by-claim. We also assume possible delays of the settlement of by-claims. Considerations of delay in claim settlement can be found in Waters and Papatriandafylou (1985), Yuen and Guo (2001), Wu and Yuen (2004), Xiao and Guo (2007) and Yuen et al (2005). Other dependence structures in terms of main claims and by-claims are studied in Yuen and Wang (2002) and Wu and Yuen (2003).

Because of the certainty of ruin for a risk model with a constant dividend barrier, the calculation of the expected discounted dividend payments is a major problem of interest in the context, instead of the ruin probability of the business. The very first risk model with dividends in the literature was proposed by de Finetti (1957), in which a discrete time model with very simple periodic gains was studied. References for the results of de Finetti's model can be found in Bühlmann (1970) and Gerber and Shiu (2004). Other discrete time risk models involving dividends include the discrete time model with a constant barrier of Claramunt, Marmol and Alegre (2003), in which the expected present value of dividends is calculated based on a system of linear equations, and the model considered in Dickson and Waters (2004) that is used to tackle certain problems in the classical continuous time model. Also, problems relating to dividends have been considered more extensively in the continuous time setting. Related works can be found in References.

The paper is organized as follows. Section 1 defines the model of interest, describes various payments, including the premiums, claims and dividends, and lists the notation. In section 2 difference equations are developed for the expected present value of dividend payments. Then a method for calculating the expected present value of dividends is proposed, using the technique of generating functions. The dividends-penalty identity is derived in section 3. Moreover, in section 4, explicit expressions for the expected present value of dividends are obtained for two classes of claim size distributions. Numerical examples are also provided to illustrate the impact of the delays of the settlement of by-claims on the expected present value of dividends.

1. The model

We consider a discrete time compound binomial risk model with two types of individual claims: main claims and by-claims. Within each time period there will originate one main claim together with an associated by-claim under a certain probability. The settlement of the main claim is right at the end of the period. While the settlement time of the by-claim is either the same as the main claim or one time period later. The factors causing the delay of the settlement are manifold, such as the length of legal procedures, the difficulty of assessing the exact size of the claim and the possibility of sequel payments. Here, the main claim can also be interpreted as the total amount of payments that can be settled within the period, and the by-claim means the amount that may have to be delayed. In this paper, a correlation is proposed between a main claim and its associated by-claim.

Let U_k be the total amount of settled claims up to the end of the kth time period, $k \in \mathbb{N}^+$ and $U_0 = 0$. Then U_k is the total amount of main claims and by-claims settled in the first k time periods.

Let X_1, X_2,... and Y_1, Y_2,... denote the sizes of the main claims and by-claims, respectively, forming two independent and identically distributed (i.i.d.) sequences of random variables. When a main claim and a by-claim have the same subscript (say X_1 and Y_1), they are associated with each other and follow a joint p.f. $f(m, n)$, $m,n = 1, 2,...$. Otherwise, they are mutually independent of each other, following the marginal p.f.'s $f_X(m), m = 1,2,...$ and $f_Y(n)$, $n = 1,2,...$, respectively, with means μ_x and μ_Y. This is a weaker assumption about the relationship between main claims and by-claims than the one in Yuen and Guo (2001), Xiao and Guo (2007), where all X_i and Y_j are assumed to be independent of each other. However, it can be seen that the dependence in this model only effects the distribution of the total claim and is, therefore, extremely weak, which enables the analytical calculations.

The probability of originating claims, one main claim and one by-claim, in each time period is p, $0 < p < 1$, and the probability of no claim is $q = 1 - p$. Each by-claim and its associated main claim will be settled simultaneously with probability $0 \le \theta \le 1$, or the settlement of the by-claim will be delayed with probability $1 - \theta$.

Assume that premiums are received at the beginning of each time period with a constant premium rate of 1 per period, and all claim payments are made only at the end of each time period. We introduce a dividend policy to the company that certain amount of dividends will be paid to the policyholder instantly, as long as the surplus of the company at time k is higher than a constant dividend barrier b ($b > 0$). It implies that the dividend payments will only possibly occur at the beginning of each period, right after receiving the premium payment. The surplus at the end of the kth time period, S_k, is then defined to be, for $k = 1, 2,...$,

$$S_k = u + k - U_k - D_k, \qquad S_0 = u. \tag{1}$$

Here the initial surplus is u, $u = 1, 2,..., b$. The positive safety loading condition holds if $p(\mu_X + \mu_Y) < 1$. We define D_k as the sum of dividend payments in the first k periods, for $k = 1, 2, ...$,

$$D_k = d_1 + d_2 + ... + d_k, \qquad D_0 = 0,$$

where d_n denotes the amount of dividend paid out in period n, for $n = 1,2,...$, with definition

$$d_n = \max\{S_{n-1} + 1 - b, 0\}. \tag{2}$$

Define $T_b = \min\{k : S_k < 0\}$ to be the time of ruin and $\Psi(u; b) = P[T_b < \infty \,|S_0 = u]$ to be the ruin probability. Let υ be a constant annual discount factor for each period. Then the expected present value of the dividend payments due until ruin is

$$V(u;b) := \mathrm{E}\left[\sum_{k=1}^{T_b} d_k \upsilon^{k-1} \middle| S_0 = u\right].$$

2. The expected present value of dividends

To study the expected present value of the dividend payments, $V(u;\ b)$, we consider the occurrences of claims for the surplus process S_k at the end of the first time period in three scenarios. In the first scenario, there is no claim, and the process is renewed at the beginning of the second time period. In the second scenario, both a main claim and its associated by-claim occur in the first period. Thus the surplus process is also renewed at the beginning of the next time period. The last scenario is simply the complement of the first two, i.e., there is a main claim, but its associated by-claim will occur one period later. Then conditioning on the occurrences of claims at the end of the first time period, we can obtain the following result for $V(u; b)$.

Theorem 1. The expected present value of the dividend payments before ruin satisfies the following difference equation:

$$V(u;b) = \upsilon q V(u+1;b) + \upsilon p \sum_{m+n\le u+1} V(u+1-m-n;b) \cdot \\ \cdot f(m,n) + \upsilon^2 pq(1-\theta) \sum_{m+n=u+2} V(0;b) f(m,n), \tag{3}$$

$u = 0,1\ldots, b-1$, with boundary condition:

$$V(b;b) = 1 + V(b-1;b). \tag{4}$$

Proof. Since the first two scenarios defined above are quite straightforward, we will only consider the third one extensively, where the by-claim in the first time period, given there is one, will be delayed to the second period. For initial surplus $0 < u < b$, it is not difficult to verify that when $X_1 + Y_1 \le u + 1$, the delay in the settlement of Y_1 does not have any effects on $V(u; b)$. If $X_1 + Y_1 > u + 1$, then there is only one possibility ensuring that the ruin will not be observed within the first two periods, which is $X_1 + Y_1 = u + 2$ and there are no other claims in the second period. Here the delay of the settlement of Y_1 together with the second premium income at the beginning of the second period saves the surplus process from being ruined and the surplus process is renewed at the beginning of period 3 with an initial surplus of 0.

Then conditioning on the occurrences of claims at the end of the first time period, we have for $V(u; b)$:

$$V(u;b) = \upsilon q V(u+1;b) + \upsilon p\theta E\left[V(u+1-X_1-Y_1;b)\right] + \upsilon p(1-\theta)\left\{E\left[V(u+1-X_1-Y_1;b)I_{\{Y_1\le u+1-X_1\}}\right]+\right.$$
$$\left.+E\left[\upsilon V(u+2-X_1-Y_1;b)I_{\{Y_1=u+2-X_1,U_2-U_1=Y_1\}}\right]\right\} = \upsilon q V(u+1;b) + \upsilon p\theta \sum_{m+n\le u+1} V(u+1-m-n;b)f(m,n)+$$
$$+\upsilon p(1-\theta)\sum_{m+n\le u+1} V(u+1-m-n;b)f(m,n) + \upsilon^2 pq(1-\theta)\sum_{m+n=u+2} V(0;b)f(m,n) =$$
$$= \upsilon q V(u+1;b) + \upsilon p \sum_{m+n\le u+1} V(u+1-m-n;b)f(m,n) + \upsilon^2 pq(1-\theta)\sum_{m+n=u+2} V(0;b)f(m,n),$$

$u = 0,1\ldots, b-1,$

where I is an indicator function and event $U_2–U_1 = Y_1$ means there is only one claim Y_1 in the second time period. The boundary condition (4) holds because when the initial surplus is b, the premium received at the beginning of the first period will be paid out as a dividend immediately. Except the first dividend payment, the rest of the model is the same as that starting from an initial surplus $b-1$.

To be able to calculate $V(u; b)$ from (3), we define a new function $W(u)$ that satisfies the following equation, for $u = 0,1,...,$

$$W(u)=\upsilon qW(u+1)+\upsilon p\sum_{m+n\le u+1}W(u+1-m-n)f(m,n)+ +\upsilon^2pq(1-\theta)\sum_{m+n=u+2}W(0)f(m,n) \quad (5)$$

with $W(0) = 1$. The values of $W(u)$ for $u \ge 1$ can be calculated recursively from equation (5). Then following the theory of difference equations (see for example, Elaydi, 2005), we know that the solution to (3) with boundary condition (4) is of the form

$$V(u;b) = C(b)W(u), \quad (6)$$

where $C(b) = 1/\left[W(b) - W(b-1)\right]$.

Remark. Result (6) is the discrete counterpart of $V(u; b)$ for the classical model, which is of the form $V(u; b) = h(u)/h'(b)$ for a certain function h (Gerber, 1979).

Instead of the recursive calculation, the explcit expression for $V(u; b)$ is of interest. For that purpose, we will derive the probability generating function for $W(u)$. Let the generating function of $W(u)$ be:

$$\tilde{W}(z) := \sum_{u=0}^{\infty} W(u)z^u, -1 < \Re(z) < 1.$$

Similarly, $\tilde{f}(z_1 z_2) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)z_1^m z_2^n$ is the generating function of $\{f(m,n)\}_{m,n=1}^{\infty}$. It is not difficult to verify that $\tilde{f}(z,z)$ is actually the probability generating function (p.g.f.) of $X_1 + Y_1$. Furthermore, we construct a new generating function $h(z) := q + p\tilde{f}(z,z)$. Let $h(i)$ be such that $\tilde{h}(z) = \sum_{i=0}^{\infty} z^i h(i)$ with $h(i)$ being the probability function of the total claims in a single time periods in the ordinary compound binomial model with individual claim amount $X_1 + Y_1$, where the independence of X_1 and Y_1 is not necessary.

Using the technique of generating functions, multiplying both sides of (5) by z^u and summing over u from 0 to ∞, we get

$$\sum_{u=0}^{\infty} W(u)z^u = \upsilon q \sum_{u=0}^{\infty} W(u+1)z^u +$$
$$+\upsilon p \sum_{u=0}^{\infty}\sum_{m+n\le u+1} W(u+1-m-n)f(m,n)z^u +$$
$$+\upsilon^2 pq(1-\theta)W(0)\sum_{u=0}^{\infty}\sum_{m+n=u+2} f(m,n)z^u.$$

Rewriting both sides of the above equation in terms of $\tilde{W}(z)$ yields

$$\tilde{W}(z) = \upsilon qz^{-1}[\tilde{W}(z) - W(0)] + \upsilon pz^{-1}\tilde{W}(z)\tilde{f}(z,z) +$$
$$+\upsilon^2 pq(1-\theta)W(0)z^{-2}\tilde{f}(z,z).$$

From the fact that $W(0) = 1$, the above equation simplifies to

$$\tilde{W}(z)\left\{1 - \upsilon\left[q + p\tilde{f}(z,z)\right]z^{-1}\right\} = = \upsilon qz^{-1}\left[\upsilon p(1-\theta)\tilde{f}(z,z)z^{-1} - 1\right] \quad (7)$$

From (7) and the definition of $\tilde{h}(z)$ we obtain a final expression for $\tilde{W}(z)$:

$$\tilde{W}(z) = \frac{\upsilon qz^{-1}\left[\upsilon(1-\theta)\tilde{h}(z)z^{-1} - \upsilon q(1-\theta)z^{-1} - 1\right]}{1 - \upsilon\tilde{h}(z)z^{-1}}. \quad (8)$$

3. The dividends-penalty identity

In this section, as in Gerber et al. (2006), we derive an identity for the expected present value of the dividend payment and the expected discounted penalty functions.

First, define τ_b to be the first time that the surplus S_k reaches the barrier b and for $0 < \upsilon \le 1$ define

$$L(u;b) = \mathrm{E}\left[\upsilon^{\tau_b} I(\tau_b \le T_b \mid S_0 = u\right], 0 \le u \le b.$$

$L(u;b)$ can be interpreted as the expected present value of an arbitrary money unit payable at the time of the reaching barrier b from the initial surplus $u \le b$. Alternatively, it can be viewed as the probability generating function of the time to reach barrier b without ruin occurring. Using the same arguments as in section 2, we can show that $L(u;\, b)$ satisfies the following recursive formula:

$$L(u;b) = \upsilon q L(u+1;b) + \\ + \upsilon p \sum_{m+n \le u+1} L(u+1-m-n;b) f(m,n) + \\ \upsilon^2 pq(1-\theta) \sum_{m+n=u+2} L(0;b) f(m,n) \\ u = 0,1,...,b-1, \quad (9)$$

with boundary condition $L(b;b) = 1$. The solution of equation (9) is the following:

$$L(u;b) = A(b)W(u),$$

where $A(b)$ can be determined by the boundary condition $L(b;\, b) = 1$ so that $A(b) = 1/W(b)$.

Let $w(x, y)$, $x, y = 0,1, 2,...$, be a non-negative penalty function. For $0 < \upsilon < 1$, define

$$\phi\,(u;b) = \mathrm{E}\left[\upsilon^{T_b} w\left(S_{T_{b-1}}, |S_{T_b}|,\right) I(T_b \le \infty) \mid S_0 = u\right],$$

$0 \le u \le b$, to be expected discounted penalty function for the surplus before ruin and deficit at ruin. In particular, when $\upsilon = 1$, $\phi\,(u;\, b)$ simplifies to the probability of ruin, $\Psi(u;\, b)$, defined in section 1. For simplicity, denote $\phi\,(u) = \lim_{b\to\infty} \phi\,(u;\, b)$ as the expected discounted penalty function and $\Psi(u) = \lim_{b\to\infty} \Psi(u;\, b)$ as the probability of ruin for the corresponding risk model without a dividend barrier.

Using the same arguments as in Gerber et al. (2006), we have for $0 < u < b$ that

$$\phi(u;b) - \phi(u) = L(u;b)[\phi(b;b) - \phi(b)]. \quad (10)$$

The formula (10) can be obtained by considering a particular sample path of the surplus starting at $u < b$: the probability generating function of the time of ruin with and without a dividend barrier can be different only if the surplus reaches the barrier b before ruin. Furthermore, we have $\phi(b;b) = \phi(b-1;b)$, since when the process starts with $u = b-1$, premium of one money unit is received immediately so the surplus will stay at b until a claim occurs and, therefore, T_{b-1} and T_b have the same distribution.

Setting $u = b - 1$ in (10) and using the fact of $\phi(b;\, b) = \phi(b-1;\, b)$, we have

$$\phi(b;b) - \phi(b) = \frac{1}{1 - L(b-1;b)}\left[\phi(b-1) - \phi(b)\right].$$

Then,

$$\begin{aligned} \phi(u;b) &= \phi(u) + \frac{L(u;b)}{1-L(b-1;b)}[\phi(b-1)-\phi(b)] = \\ &= \phi(u) + \frac{W(u)/W(b)}{1-W(b-1)/W(b)}[\phi(b-1)-\phi(b)] = \\ &= \phi(u) + \frac{W(u)}{W(b)-W(b-1)}[\phi(b-1)-\phi(b)] = \\ &= \phi(u) + V(u;b)[\phi(b-1)-\phi(b)]. \end{aligned} \quad (11)$$

In particular, when $\upsilon = 1$, the dividends-penalty identity simplifies to

$$\Psi(u;b) = \Psi(u) + V(u;b)\left[\Psi(b-1) - \Psi(b)\right]. \quad (12)$$

The identity (12) can be used to confirm that ruin is certain for our risk model under a barrier dividend strategy as follows. Following the same procedure as in section 2, we can show that $\phi(u) = 1 - \Psi(u)$ is a particular solution of the difference equation (5) with $\upsilon = 1$. Then $1 - \Psi(u) = W(u)\phi(0)$ and therefore equation (12) simplifies to

$$\begin{aligned} \Psi(u;b) &= 1 - W(u)\Psi(0) + \\ &+ \frac{W(u)}{W(b)-W(b-1)}[W(b)\Psi(0) - W(b-1]\Psi(0) = \\ &= 1 - W(u)\Psi(0) + W(u)\Psi(0) = 1. \end{aligned}$$

4. Two classes of claim size distributions

In this section, we consider two special cases for the distribution of $X_1 + Y_1$ such that $W(u)$ has a rational generating function which can be easily inverted. One case is that the probability function of $X_1 + Y_1$ has finite support such that its p.g.f. is a polynomial, and the other case is that $X_1 + Y_1$ has a discrete K_n distribution, i.e., the p.g.f. of $X_1 + Y_1$ is a ratio of two polynomials with certain conditions.

4.1. Claim amount distributions with finite support. Now assume that the distribution of $X_1 + Y_1$ has finite support, e.g., for $N = 2, 3,...,$

$$\mathrm{P}\left(X_1 + Y_1 = x\right) = \pi_x, \qquad x = 2,3\ldots, N.$$

Then $$D_N(z) := \tilde{h}(z) = q + p\sum_{x=2}^{N} z^x \pi_x, \quad (13)$$

$-1 < \Re(z) < 1$, is a polynomial of degree N. Then $W(z)$ in (8) simplifies to

$$z\tilde{W}(z) = \frac{\upsilon^2 q(1-\theta)D_N(z) - \upsilon^2 q^2(1-\theta) - \upsilon qz}{z - \upsilon D_N(z)} = \frac{\upsilon^2 q^2(1-\theta) + \upsilon q\theta z}{\upsilon D_N(z) - z} - \upsilon q(1-\theta) =$$

$$= \frac{1}{p\pi_N}\frac{\upsilon q^2(1-\theta) + q\theta z}{(z-R_1)(z-R_2)\ldots(z-R_N)} - \upsilon q(1-\theta),$$

where R_1, R_2,..., R_N are the N roots of the equation of $\upsilon D_N(z) - z = 0$ in the whole complex plane. Further, if $R_1, R_2, \ldots, R_N$ are distinct, then by partial fractions, we have

$$z\tilde{W}(z) = \frac{1}{p\pi_N}\sum_{i=1}^{N}\frac{a_i}{R_i - z} - \upsilon q(1-\theta),$$

where $a_i = -\frac{q\theta R_i + \upsilon q^2(1-\theta)}{\prod_{j=1,j\neq i}^{N}(R_i - R_j)}$, $i = 1, 2, \ldots, N$.

Setting $z = 0$ gives $\sum_{i=1}^{N}\frac{a_i}{R_i} = \upsilon pq\pi_N(1-\theta)$.

Inverting the p.g.f. $\tilde{W}(z)$ yields

$$W(u) = \sum_{i=1}^{N}\frac{a_i}{p\pi_N}R_i^{-(u+2)}, \quad u = 0,1,\ldots. \quad (14)$$

Now $V(u;b) = C(b)W(u)$, for u = 0,1,..., b – 1, and as $V(b; b) = 1 + V(b-1; b)$, then

$$V(u;b) = \frac{W(u)}{W(b) - W(b-1)} = \frac{\sum_{i=1}^{N}a_i R_i^{-(u+2)}}{\sum_{i=1}^{N}a_i(1-R_i)R_i^{-(b+2)}}, \quad (15)$$

$u = 0,1,\ldots,b$.

Example 1. In this example, we assume $f(1,1) = 1$. Then $S_k - S_{k-1}$ can only take four possible values: 1, 0, -1 or -2. This generalizes de Finetti's original model where periodic gains are +1 or -1. The p.g.f. of $W(u)$ in (8), has a simplified expression

$$\tilde{W}(z) = \frac{\upsilon q[1 - \upsilon p(1-\theta)z]}{\upsilon pz^2 - z + \upsilon q}. \quad (16)$$

Let $0 < R_1 < 1 < R_2$ be the solutions of the equation $\upsilon pz^2 - z + \upsilon q = 0$. Then by partial fractions, (16) can be rewritten as

$$z\tilde{W}(z) = \frac{1}{p}\left(\frac{a_1}{R_1 - z} + \frac{a_2}{R_2 - z}\right) - \upsilon q(1-\theta),$$

where

$$a_1 = \frac{q\theta R_1 + \upsilon q^2(1-\theta)}{R_2 - R_1},\; a_2 = \frac{q\theta R_2 + \upsilon q^2(1-\theta)}{R_1 - R_2}.$$

Substituting them into (15) gives, for u = 0,1,..., b,

$$V(u;b) = \frac{a_1 R_1^{-(u+2)} + a_2 R_2^{-(u+2)}}{a_1(1-R_1)R_1^{-(b+2)} + a_2(1-R_2)R_2^{-(b+2)}} = \frac{(R_2 + \theta R_1)R_1^{-(u+1)} - (R_1 + \theta R_2)R_2^{-(u+1)}}{R_1^{-(b+1)}(1-R_1)(R_2+\theta R_1) - R_2^{-(b+1)}(1-R_2)(R_1+\theta R_2)}. \quad (17)$$

The last equality holds because of the properties of roots R_1 and R_2 giving $\frac{R_1 R_2}{R_1 + R_2}$. Two extreme cases of equation (17) are

$$V(u;b) = \frac{R_1^{-(u+1)} - R_2^{-(u+1)}}{R_1^{-(b+1)}(1-R_1) - R_2^{-(b+1)}(1-R_2)} \quad \text{for } \theta = 1,$$

and

$$V(u;b) = \frac{R_1^{-(u+2)} - R_2^{-(u+2)}}{R_1^{-(b+2)}(1-R_1) - R_2^{-(b+2)}(1-R_2)} \quad \text{for } \theta = 0.$$

Another value of interest in Example 1 is the optimal dividend barrier b^*, which is the optimal value of b that maximizes $V(u; b)$ for a given u. From (17) we know that b^* is the solution of equation

$$\frac{d}{db}\left|\begin{matrix} R_1^{-(b+1)}(1-R_1)(R_2+\theta R_1) - \\ -R_2^{-(b+1)}(1-R_2)(R_1+\theta R_2)\end{matrix}\right| = 0,$$

since

$$\frac{d}{db^2}\left|\begin{matrix} R_1^{-(b+1)}(1-R_1)(R_2+\theta R_1) - \\ -R_2^{-(b+1)}(1-R_2)(R_1+\theta R_2)\end{matrix}\right| > 0$$

for all $b > 0$. Thus we have

$$b^* = \frac{\ln\dfrac{(R_2-1)(R_1+\theta R_2)\ln(R_2)}{(R_1-1)(R_2+\theta R_1)\ln(R_1)}}{\ln(R_2) - \ln(R_1)} - 1,$$

which does not depend on the initial surplus u. Practically, we round b^* to the closest integral value. Furthermore, we can prove the following result.

Proposition 1. For the risk model considered in Example 1, the expected present value of the dividend payments up to the time of ruin, $V(u; b)$, increases as the probability of a delay of the by-claims is increasing as well.

Proof. The proposition can be proved by the following fact:

$$\frac{d}{d\theta}\left[\frac{(R_2+\theta R_1)R_1^{-(u+1)} - (R_1+\theta R_2)R_2^{-(u+1)}}{R_1^{-(b+1)}(1-R_1)(R_2+\theta R_1) - R_2^{-(b+1)}(1-R_2)(R_1+\theta R_2)}\right]^2 =$$

$$= \frac{R_2^{-(u+1)}R_1^{-(b+1)}(1-R_1)(R_1^2 - R_2^2) + R_2^{-(b+1)}R_1^{-(u+1)}(1-R_2)(R_2^2 - R_1^2)}{[R_1^{-(b+1)}(1-R_1)(R_2+\theta R_1) - R_2^{-(b+1)}(1-R_2)(R_1+\theta R_2)]} < 0,$$

since $0 < R_1 < 1 < R_2$

In Example 1, let $p = 0.45$, $\upsilon = 0.95$, $b = 9$, then we have $R_1 = 0.78786$, $R_2 = 1.55132$. Table 1 summaries the results for $V(u;b)$ for θ = 0, 0.25, 0.5, 0.75, 1, and u = 0,1,...,9. The numbers show that the higher the initial surplus of the insurance company, the higher the expected present value of dividend payments prior to the time of ruin. They also confirm Proposition 1 that $V(u;\, b)$ is increasing as the probability of the delay of by-claims is increasing, i.e., θ is decreasing. Moreover, the impact of the delay of by-claims on $V(u;\, b)$ is getting smaller as u increases.

Table 1. Values of $V(u;\, 9)$ when $f(1;\, 1) = 1$

	$\theta = 0$	0.25	0.5	0.75	1
$u = 0$	0.40851	0.36231	0.32549	0.29547	0.27052
1	0.60719	0.57724	0.55338	0.53392	0.51775
2	0.82786	0.80834	0.79279	0.78011	0.76957
3	1.08763	1.07477	1.06453	1.05618	1.04924
4	1.40424	1.39561	1.38874	1.38313	1.37847
5	1.79767	1.79167	1.78689	1.78300	1.77976
6	2.29159	2.28717	2.28365	2.28078	2.27839
7	2.91499	2.91144	2.90862	2.90631	2.90439
8	3.70400	3.70082	3.69829	3.69623	3.69451
9	4.70400	4.70082	4.69829	4.69623	4.69451

With fixed $\theta = 0.5$, we get the optimal dividend barrier $b^* \approx 1$. In Table 2, the expected present values of dividend payments $V(u;\, b)$ for $u = 0{,}1$ and $b = 0{,}1{,}\ldots,\, 9$ are provided. The values confirm the fact that when $b = 1$, the policyholders receive the most dividends prior to ruin.

Table 2. Values of $V(u;\, b)$ when $\theta = 0.5$

	$b = 0$	1	2	3	4
$u = 0$ $u = 1$	1.35364 -	1.42832 2.42832	1.35958 2.31146	1.19780 2.03640	1.00398 1.70688
	$b = 5$	6	7	8	9
$u = 0$ $u = 1$	0.81751 1.38987	0.65524 1.11399	0.52082 0.88546	0.41219 0.70077	0.32549 0.55338

4.2. K_n claim amount distributions. Li (2005a, 2005b) studied a class of discrete Sparre Andersen risk models in which the claims inter-arrival times are K_n distributed. This class of distributions includes geometric, negative binomial, discrete phase-type, as well as linear combinations (including mixtures) of these.

In this subsection, we assume that $P(X_1 + Y_1 = x)$ is K_n distributed for $x = 2, 3, \ldots$, and $n = 1, 2, \ldots$, i.e., the p.g.f. of $X_1 + Y_1$ is given by

$$\tilde{f}(z,z) = \frac{z^2 E_{n-1}(z)}{\prod_{i=1}^{n}(1 - zq_i)},\ \Re(z) < \min\left\{\frac{1}{q_1} : 1 \le i \le n\right\},$$

where $0 < q_i < 1$, for $i = 1, 2, \ldots, n$ and $E_{n-1}(z) = \sum_{k=0}^{n-1} z^k e_k$ is a polynomial of degree $n - 1$ or less with $E_{n-1}(1) = \prod_{i=1}^{n}(1 - q_i)$. Then $\tilde{W}(z)$ can be transformed to the following rational function

$$\tilde{W}(z) = \frac{\upsilon^2 pq^2(1-\theta) z E_{n-1}(z) - \upsilon q \prod_{i=1}^{n}(1 - zq_i)}{z\prod_{i=1}^{n}(1 - zq_i) - \upsilon q \prod_{i=1}^{n}(1 - zq_i) - \upsilon p z^2 E_{n-1}(z)}.$$

Since the denominator of the above equation is a polynomial of degree $n + 1$, it can be factored as

$$\left[(-1)^n \prod_{i=1}^{n} q_i - \upsilon p e_{n-1}\right] \prod_{i=1}^{n+1}(z - R_i),$$

where $R_1, R_2, \ldots, R_{n+1}$ are the $n + 1$ zeros of the denominator. We remark that

$$(-1)^n \prod_{i=1}^{n} q_i - \upsilon p e_{n-1} = (-1)^n \upsilon q \Big/ \prod_{i=1}^{n+1} R_i.$$

Then $\tilde{W}(z)$ simplifies to

$$\tilde{W}(z) = \left[\prod_{i=1}^{n+1} R_i\right] \frac{\prod_{i=1}^{n}(1 - zq_i) - \upsilon pq(1-\theta) z E_{n-1}(z)}{\prod_{i=1}^{n+1}(R_i - z)} =$$

$$= \left[\prod_{i=1}^{n+1} R_i\right] \sum_{i=1}^{n+1} \frac{r_i}{(R_i - z)},$$

where $r_i = \dfrac{\prod_{j=1}^{n}(1 - R_i q_j) - \upsilon pq(1-\theta) R_i E_{n-1}(R_i)}{\prod_{j=1, j\ne i}^{n+1}(R_j - R_i)}$,

$i = 1, 2, \ldots, n + 1$.

Inverting $\tilde{W}(z)$ gives

$$W(0) = \left[\prod_{i=1}^{n+1} R_i\right] \sum_{i=1}^{n+1} r_i R_i^{-1} = \sum_{i=1}^{n+1} r_i \left(\prod_{j=1, j\ne i}^{n+1} R_j\right) = 1,$$

and $W(u) = \left[\prod_{i=1}^{n+1} R_i\right] \sum_{i=1}^{n+1} r_i R_i^{-(u+1)},\ u = 1, 2, \ldots.$

Not that $C(b) = 1/[W(b) - W(b - 1)]$, then finally we have

$$V(0;b) = C(b) = \frac{1}{\left[\prod_{i=1}^{n+1} R_i\right] \sum_{i=1}^{n+1} r_i (1 - R_i) R_i^{-(b+1)}}, \quad (18)$$

and for $u = 1, 2, \ldots, b$,

$$V(u;b) = C(b) W(u) = \frac{\sum_{i=1}^{n+1} r_i R_i^{-(u+1)}}{\sum_{i=1}^{n+1} r_i (1 - R_i) R_i^{-(b+1)}}. \quad (19)$$

Example 2. In this example, we assume that the random variable $X_1 + Y_1$ follows a mixture of two geometric distributions, with a p.g.f.

$$\tilde{f}(z,z) = \frac{z^2\left[(1-\beta)(1-\gamma) + \alpha(1-z)\right]}{(1-\beta z)(1-\gamma z)},$$

where $0 < \beta,\ \gamma < 1$, $\alpha = a(\gamma - \beta) + \beta(1 - \gamma)$, and $0 < a < 1$. Here $n = 2$, $q_1 = \beta$, $q_2 = \gamma$, and $E_{n-1}(z) = (1 - \beta)(1 - \gamma) + \alpha(1 - z)$. Let R_1, R_2, R_3 be the three roots of the equation

$$z(1-\beta z)(1-\gamma z)-\upsilon q(1-\beta z)(1-\gamma z)-$$
$$-\upsilon pz^2[(1-\beta)(1-\gamma)+\alpha(1-z)]=0.$$

Then inverting the Laplace transform gives

$$W(u)=\left(R_1R_2R_3\right)\sum_{i=1}^{3} r_i R_i^{-(u+1)},\ u=0,1,2\ldots, \qquad (20)$$

where

$$r_i=\frac{(1-\beta R_i)(1-\gamma R_i)-\upsilon pq(1-\theta)R_i\left[(1-\beta)(1-\gamma)+\alpha(1-R_i)\right]}{\prod_{j=1,j\neq i}^{3}\left(R_j-R_i\right)}.$$

Then (18) and (19) simplify to

$$\upsilon(0;b)=\frac{1}{\left(\prod_{i=1}^{3}R_i\right)\sum_{i=1}^{3}r_i\left(1-R_i\right)R_i^{-(b+1)}},$$

and

$$V(u;b)=\frac{\sum_{i=1}^{3}r_iR_i^{-(u+1)}}{\sum_{i=1}^{3}r_i\left(1-R_i\right)R_i^{-(b+1)}},\ u=1,2,\ldots,b.$$

As an example, let p = 0.35, υ = 0.95, b = 9, β = 0.8, γ = 0.6, a = 0.4. From the above results we get α = 0.24, R_1 = 0.72505, R_2 = 1.05964, and R_3 = 1.43574. The values of $V(u;\ 9)$ for θ = 0, 0.25, 0.5, 0.75, 1, and u = *0,*1,..., 9 are listed in Table 3. We observe the same features as in Example 1, that $V(u;b)$ is an increasing function with respect to u, and a decreasing function over θ. Also, the impact of the delay of by-claims on the expected present value of dividends is reduced for higher initial surplus of the company.

Table 3. Values of $V(u;\ 9)$ for geometric distributed claims

	$\theta = 0$	0.25	0.5	0.75	1
$u = 0$	0.14656	0.14274	0.13911	0.13566	0.13237
1	0.22721	0.22375	0.22046	0.21734	0.21437
2	0.33612	0.33294	0.32993	0.32707	0.32435
3	0.48439	0.48145	0.47867	0.47602	0.47349
4	0.68728	0.68453	0.68192	0.67944	0.67709
5	0.96571	0.96311	0.96065	0.95831	0.95608
6	1.34851	1.34603	1.34368	1.34145	1.33932
7	1.87538	1.87299	1.87072	1.86856	1.86651
8	2.60108	2.59873	2.59650	2.59438	2.59237
9	3.60108	3.59873	3.59650	3.59438	3.59237

Conclusion

This paper considered the compound binomial model with a constant dividend barrier. Further, two types of individual claims are defined, where one main claim is dependent with its associated by-claim and the settlement of the by-claim may be delayed for one time period. Both theoretical and numerical results are obtained for the expected present value of total dividends up to the time of ruin. In particular, from the two numerical examples given above we can see that the possible delay for the settlement of by-claims has a positive effect on increasing the total expected present value of dividends. With a fixed initial surplus, the higher the probability of delay, the more the total dividends increase. However, this positive effect is reducing when the initial surplus increases.

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