JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **28**, Issue **1**, June **2020**, pp. 1–42, DOI 10.15421/142001

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

ON AN INITIAL BOUNDARY-VALUE PROBLEM FOR 1D HYPERBOLIC EQUATION WITH INTERIOR DEGENERACY: SERIES SOLUTIONS WITH THE CONTINUOUSLY DIFFERENTIABLE FLUXES

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Communicated by Prof. P. Kasyanov

Abstract. A 1-parameter initial boundary value problem for the linear homogeneous degenerate wave equation $u_{tt}(t,x;\alpha) - (a(x;\alpha)\,u_x(t,x;\alpha))_x = 0$ (*JODEA*, **27**(2), 29 − 44), where: 1) $(t,x) \in [0,T]\times[-l,+l]$; 2) the weight function $a(x;\alpha)$: *a*) $a_0\left|\dfrac{x}{c}\right|^{\alpha}$, $0 \leqslant |x| \leqslant c$; *b*) a_0, $c \leqslant |x| \leqslant l$; *c*) a_0 is a constant reference value; and 3) the parameter $\alpha \in (0,+\infty)$; is considered. Using a string analogy, the IBVP can be treated as an attempt to set an initially fixed 'string' in motion, the left end of the 'string' being fixed, whereas the right end being forced to move.

It has been proved, using the methods of Frobenius and separation of variables, that: 1) there exist 6 series solutions $u(t,x;\alpha)$, $(t,x) \in [0,T]\times[-c,+c]$, of the degenerate wave equation; 2) the only series solution, having continuous and continuously differentiable flux $f(a,u) = -au_x$, reads $u(t,x;\alpha) = U_{\alpha,0}(t) + U_{\alpha,1}(t)|x|^{\theta} + U_{\alpha,2}(t)|x|^{2\theta} + \ldots$, where *a*) $\theta = 2-\alpha$ is a derived parameter; *b*) the coefficient functions obey the following linear recurrence relations: $U''_{\alpha,\mu-1}(t) = \mu\theta\,[(\mu-1)\,\theta+1]\,c^{-\alpha}a_0\,U_{\alpha,\mu}(t)$, $\mu \in \mathbb{N}$.

It has been revealed that a nonlinear change of the independent variables $(t,x) \to (\tau,\xi)$ transforms: 1) the degenerate wave equation to the wave equation $\upsilon_{\tau\tau} - \upsilon_{\xi\xi} = \xi\rho$, or rewritten as the balance law $\pi_\tau + \varphi_\xi = \rho$, where $\pi = \upsilon_\tau$, $-\varphi(\upsilon;\alpha) = \upsilon_\xi + \xi\rho$, $\rho(\upsilon;\alpha) = \dfrac{\alpha}{\theta}\,\dfrac{\upsilon}{\xi^2}$, having: *a*) no singularity in its principal part (due to inflation of the degeneracy), and *b*) the only series solution of the form $\upsilon(\tau,\xi;\alpha) = V_{\alpha,0}(\tau) + V_{\alpha,1}(\tau)\,\xi^2 + V_{\alpha,2}(\tau)\,\xi^4 + \ldots$ (out of 5 existing and found similarly to those of the degenerate wave equation), leading to the continuous and continuously differentiable regularized flux $\varphi(\mathring{\upsilon};\alpha)$ and the continuous regularized source term $\rho(\mathring{\upsilon};\alpha)$, where $\mathring{\upsilon}(\tau,\xi;\alpha) = \upsilon(\tau,\xi;\alpha) - \upsilon(\tau,0;\alpha)$; 2) the IBVP for the degenerate wave equation to the IBVP for the transformed wave equation.

It has been shown, that if $\alpha \in (0,2)$: 1) the above results are valid; 2) the state of being fixed for the 'string' is not necessary for $(t,x) \in [0,T]\times[-l,0]$, that is a traveling wave could pass the degeneracy and excite vibrations of the 'string' between its fixed end and the point of degeneracy.

Key words: degenerate wave equation, series solutions, the Frobenius method, separation of variables, inflation of singularity, exact solutions, the Bessel functions, conservation and balance laws, the flux, regularization of the flux.

2010 Mathematics Subject Classification: 35L05, 35L80.

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1. Introduction

1.1. Presentation of the problem

The current study is a continuation of that started and shortly reported in our pilot publication [4] on the subject and dealing with the following spatially 1D degenerate wave equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x;\alpha)\,\frac{\partial u(t,x)}{\partial x}\right) = 0\,, \tag{1.1}$$

where $t\in(0,T)$, $x\in(-l,+l)$ are the independent variables, $a(\cdot;\alpha)\colon[-l,+l]\to\mathbb{R}_+$ is a given 1-parameter weight function which is assumed to be continuous, piecewise smooth and to vanish in the midpoint of the segment $[-l,+l]$ following a power law inside a subsegment $[-c,+c]\subset[-l,+l]$. For instance,

$$a(x;\alpha) = \begin{cases} a_0\left(\dfrac{|x|}{c}\right)^{\alpha}, & 0\leqslant\dfrac{|x|}{l}\leqslant\dfrac{c}{l}\,, \\[2ex] a_0\,, & \dfrac{c}{l}\leqslant\dfrac{|x|}{l}\leqslant 1\,, \end{cases}$$

where $\alpha\in(0,+\infty)$ is a parameter, and $a_0>0$ is a given constant.

Using the following non-dimensional variables and quantities

$$t=\frac{l\,\underline{t}}{\sqrt{a_0}}\,,\qquad x=l\,\underline{x}\,,\qquad u=l\,\underline{u}\,,\qquad a=a_0\,\underline{a}\,,\qquad c=l\,\underline{c}\,,$$

the degenerate wave equation and the power law can be rewritten in a non-dimensional forms. To simplify notation we drop hereafter the bars under the non-dimensional variables and quantities. As a result, the non-dimensional degenerate wave equation reads exactly as (1.1), whereas the non-dimensional power law reduces to

$$a(x;\alpha) = \begin{cases} a_*\,|x|^{\alpha}, & 0\leqslant|x|\leqslant c\,, \\ 1\,, & c\leqslant|x|\leqslant 1\,, \end{cases} \tag{1.2}$$

where the derived quantity a_* is such that $a_*c^{\alpha}=1$. Hereinafter, the space-time segment $[0,T]\times\{x=0\}$, where the degeneracy is located, is referred to as the degeneracy segment.

Another form of the degenerate wave equation (1.1), used in the current study, is known as a conservation law

$$\frac{\partial p}{\partial t} + \frac{\partial f}{\partial x} = 0\,, \tag{1.3}$$

where $f:=-aq$ is the flux, and p, q are auxiliary dependent variables

$$p:=\frac{\partial u}{\partial t}\,,\qquad q:=\frac{\partial u}{\partial x}\,\cdot$$

Our concern, as in [4], relates to the study of the following initial boundary value problem

$$\begin{cases} \dfrac{\partial^2 u(t,x;\alpha)}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a(x;\alpha)\,\dfrac{\partial u(t,x;\alpha)}{\partial x}\right), & (t,x) \in (0,T]\times(-1,+1)\,, \\ u(t,-1;\alpha) = 0\,, & t \in [0,T]\,, \\ u(t,+1;\alpha) = h(t)\,, & t \in [0,T]\,, \\ u(0,x;\alpha) = 0\,, & x \in [-1,+1]\,, \\ \dfrac{\partial u(0,x;\alpha)}{\partial t} = 0\,, & x \in [-1,+1)\,, \end{cases} \tag{1.4}$$

where $h(t)$ is a given control function.

For the sake of convenience, we interpret the function $u(t,x)$ as the distributed over the segment $[-1,+1]$ displacements of a ‘string’, though properties of the weight function $a(x;\alpha)$ have little in common with those of weight functions being admissible in ‘genuine’ wave equations for string vibrations. So, we deal with a hyperbolic system subject to the action of a control $h(t)$, imposed as the Dirichlet boundary condition at the right end $x = +1$. In contrast to the widely studied standard case, for instance, see [2, 3], we assume that the string has a defect or a damage at the interior point $x=0$, where $a(0;\alpha)=0$. Loosely speaking, the loss of string elasticity at the interior point, which turns into a swivel, is accompanied by the cease of resisting rotations and flexions.

The main question that we are going to discuss in this article is how the defect at the interior point $x=0$ affects the solution of the system (1.4) and its properties. The second point that should be clarified here is about the consistency of this problem and properties of its solutions in a neighborhood of ’the damage point’ $x = 0$. It seems that such analysis for the indicated class of degenerate systems can be important for applications, with respect, for instance, the cloaking problem [10] (building of devices that lead to invisibility properties from observation), the evolution of damage in materials, optimization problems for elastic bodies arising, e.g. in contact mechanics, coupled systems, composite materials, where ’life-cycle-optimization’ appears as a challenge.

The indicated type of degeneracy raises many new and open questions related to the well-posedness of the hyperbolic equations in suitable functional spaces. It should be emphasized here that boundary value problems for degenerate elliptic and parabolic equations have received a lot of attention in the last years (see, for instance, [5, 6, 16–18]). In the meantime, as for the control issue for degenerate wave equations, we can mention only a few recent publications [2,3,11], where the authors mainly deal with weakly or strongly degenerate wave equations for which the degeneracy zone is located at a boundary point. This analysis shows that because of the rate of ’degree of degeneracy’ in the diffusion coefficient $a(x;\alpha)$, especially when $a(x;\alpha)$ degenerations too severely, the new tools are necessary for the analysis of the corresponding initial boundary value problems.

In contrast to the above mentioned results, where the authors mainly deal with the degenerate equation of the form (1.1) with the degeneracy at the boundaries $x=\pm1$, we focus on the case where the 'damaged' point is interior. From a physical point of view, if we look at the wave equation (1.1) as at the equation of vibrating string for which its the point $x=0$ works like a swivel, then we should allow for the possibility of big angles at the profile of the string around $x=0$. Needless to say that in this case the mathematical substantiation of the wave equation for vibrating string becomes nontrivial. On the other hand, the coefficient $a(x;\alpha)$ in (1.1) can be interpreted as the stiffness of the string. The fact that this coefficient vanishes at $x = 0$ means that the string is getting very weak at this point. So, our core idea is to apply the methods of Frobenius and separation of variables in order to obtain the exact representation for solutions of the original degenerate wave equation in the form of 1-parameter power series. Such analysis allows to find out what kind of compatibility conditions we should impose at the 'damaged' point in order to pass from the original initial boundary value problem (1.4) to its equivalent version in the form of some transmission problem. Therefore, the purpose of this paper is to provide a qualitative analysis of system (1.4), obtain an exact representation for its solution, and find out how the degree of degeneracy α in the principle coefficient $a(x;\alpha)$ affects the system (1.4) and its solution.

1.2. The plan of the article

The article is organized as follows.

Section 2 is devoted to the well-posedness issues for the initial boundary value problem (1.4) provided $\alpha \in (0,2)$. Admitting only two types of degeneracy for $a(x;\alpha)$, namely the so-called weak and strong degeneracy, we prove the existence and uniqueness results for the weak and strong solutions to the problem. We also discuss the transmission conditions at the damage point $x = 0$ and show that, in general, in the framework of functional setting, the continuity of the weak and strong solutions at the point of interior degeneracy and the smoothness of the corresponding fluxes remain open questions.

In Section 3 we construct 1-parameter power series solutions of the original degenerate wave equation in one-sided vicinities of the degeneracy segment. Then, in Section 4, we study continuous matching of the obtained one-sided series solutions. Finally, in Section 5 we obtain the exact solutions of the original degenerate wave equation using the separation of variables. Among all continuous series solutions obtained in Sections 4 and 5, we find those being required in some sense, and possessing the so-called property Z.

Definition 1.1. We say that a solution to the initial boundary value problem (1.4) *possesses property Z* if it vanishes in the left space-time rectangle $[0,T]\times[-1,0]$.

A more precise formulation of our concern in terms of Definition 1.1 is to divide the required solutions to the problem (1.4) into those possessing and violating property Z. Keeping in mind the string analogy, we are interested in choosing: 1) such functions $h(t)$, specifying a motion of the right end of the string, and eventually the right part of the string (segment $(0,+1]$), and 2) such values of the parameter α, specifying a sort of degeneration (in the midpoint of the segment $[-1,+1]$), to make the left part of the string (segment $[-1,0)$) vibrate.

In Section 6 we first introduce new independent variables, 'inflating' the degeneracy, and then transform the original degenerate wave equation to a wave equation (referred to as the transformed one), having no singularity in its principal part. After, we reformulate the original initial boundary value problem for the transformed wave equation. Then, in Sections 7 − 9, we apply the approaches used in Sections 3 − 5, but for the transformed wave equation and the transformed initial boundary value problem. As for continuous matching to be implemented to the transformed wave equation, it is rewritten as a balance law.

1.3. Short announce of the main results

In the current study, solving the initial boundary value problem (1.4) for the degenerate wave equation (1.1), supplemented with the power law (1.2) of degeneracy $a(x;\alpha) \sim |x|^{\alpha}$, $\alpha \in (0,+\infty)$, has been discussed as it concerns continuity of power series solutions, treated where it is necessary, by analogy, as vibrations of an initially fixed 'string'.

1. We have introduced the definitions of required solutions to the degenerate wave equation and the initial boundary value problem (1.4), both solutions having the continuous and continuously differentiable flux.

2. We have introduced the definition of property Z for solutions to the initial boundary value problem (1.4) to remain trivial between the fixed end of the 'string' and the point of degeneracy, for $t>0$.

3. We have succeeded in finding power series solutions using the methods of: 1) Frobenius and 2) separation of variables, for the parameter of degeneracy $\alpha \in (0,+\infty)$.

4. We have proved that among the series solutions obtained, there is the only one required, being a power series of the terms $|x|^{\mu\theta}$, $\theta = 2-\alpha$, $\mu \in \mathbb{Z}_+$, for the parameter of degeneracy $\alpha \in (0,2)$.

$$u(t,x;\alpha) = U_{\alpha,0}(t) + U_{\alpha,1}(t)\,|x|^{\theta} + U_{\alpha,2}(t)\,|x|^{2\theta} + \dots,$$

where the coefficient functions obey the following recurrence relations

$$U''_{\alpha,\mu-1}(t) = \mu\theta\,[(\mu-1)\,\theta+1]\,a_* U_{\alpha,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,.$$

5. We have proved that property Z is not necessary for the only required series solution. Physically, not possessing property Z means that a traveling wave could pass the degeneracy and excite vibrations of the 'string' between its fixed

end and the point of degeneracy (see a brief introductory discussion of physical formulations of the initial boundary value problem (1.4) in Section 1 of [4]).

2. Functional setting and well-posedness issues

In this Section we are going to dwell at the well-posedness of the original initial-boundary value problem (1.4) provided $\alpha \in (0,2)$. We will admit only two types of degeneracy for $a(x;\alpha)$ (1.2), namely weak and strong degeneracy. By analogy with [2], we say that the problem (1.4) is weakly degenerate (WDP) if $\alpha \in (0,1]$, and it is strongly degenerate (SDP), if $\alpha \in (1,2)$. So, each type of degeneracy is associated with the corresponding range of the exponent α.

Let us introduce some weighted Sobolev spaces naturally associated with the system (1.4). We denote by $H^1_a(-1,+1)$ the space of all functions $u \in L^2(-1,+1)$ such that

$$\begin{cases} u \text{ is locally absolutely continuous in } [-1,0) \bigcup (0,+1]\,, \\ \sqrt{a}\, u_x \in L^2(-1,+1). \end{cases} \tag{2.1}$$

It is easy to see that $H^1_a(-1,+1)$ is a Hilbert space with respect to the scalar product

$$(u,v)_{H^1_a(-1,+1)} = \int_{-1}^{1} [uv + a\, u_x v_x]\, \mathrm{d}x, \quad \forall\, u,v \in H^1_a(-1,+1)\,,$$

and associated norm

$$\|u\|_{H^1_a(-1,+1)} = \left(\int_{-1}^{1} \left[u^2 + a\, u_x^2\right] \mathrm{d}x \right)^{\frac{1}{2}}, \quad \forall\, u \in H^1_a(-1,+1)\,.$$

We also introduce the closed subspace $H^1_{a,0}(-1,+1)$ of $H^1_a(-1,+1)$ defined as

$$H^1_{a,0}(-1,+1) = \Big\{ u \in H^1_a(-1,+1) \colon\ u(-1) = 0 = u(+1) \Big\}.$$

Arguing as in [13, Theorem 3.1], it can be shown that $H^1_{a,0}(-1,+1)$ is a Banach space with respect to the norm

$$\|u\|_{H^1_{a,0}(-1,+1)} = \left(\int_{-1}^{1} a\, u_x^2\, \mathrm{d}x \right)^{\frac{1}{2}}$$

provided $\alpha \in (0,2)$.

Starting with the weak degenerate case, we have the following result (we refer to [14, Theorem 2.3] for the details).

Theorem 2.1. *Let $a(x;\alpha)\colon [-1,+1] \to \mathbb{R}$ be a weight function defined by* (1.2) *with $\alpha \in (0,1]$. Then $H^1_a(-1,1) \hookrightarrow L^1(-1,+1)$ compactly, and $H^1_a(-1,+1)$ is continuously embedded into the class of absolutely continuous functions on $[-1,+1]$, so*

$$\lim_{x\nearrow 0} u(x) = \lim_{x\searrow 0} u(x), \quad |u(0)| < +\infty, \quad \forall\, u \in H^1_a(-1,+1). \tag{2.2}$$

If, in addition, u belongs to the space

$$H^2_a(-1,+1) := \left\{ u \in H^1_a(-1,+1) \ : \ au_x \in W^{1,2}(-1,+1) \right\}, \tag{2.3}$$

then the following transmission condition

$$\lim_{x\nearrow 0} a(x)\, u_x(x) = \lim_{x\searrow 0} a(x)\, u_x(x) = L, \quad \text{with } |L| < +\infty, \tag{2.4}$$

holds true.

Remark 2.1. It is worth to emphasize that if $\alpha \in (0,1]$ and $u \in H^1_a(-1,+1)$ is an arbitrary element, then $u(x)$ is continuous at the damage point $x = 0$. However, the situation changes drastically if we deal with the strong degeneration in (1.4). Indeed, let us consider the following example. Let $\alpha = 7/4$, $c = 1$, $a_* = 1$ in (1.2), and let

$$u(x) = \begin{cases} |x|^{-\frac{1}{4}} - 1, & \text{if } x \in (-1,0), \\ |x|^{+\frac{1}{2}} - 1, & \text{if } x \in [0,+1). \end{cases}$$

Then, the function $u\colon (-1,+1) \to \mathbb{R}$ has a discontinuity of the second kind at $x = 0$, $u \in H^1_{a,0}(-1,1)$, and

$$a(x;\alpha)\, u_x(x) = \begin{cases} \frac{1}{4}\, |x|^{\frac{1}{2}}, & \text{if } x \in (-1,0), \\ \frac{1}{2}\, |x|^{\frac{5}{4}}, & \text{if } x \in [0,+1). \end{cases}$$

So, instead of the transmission condition (2.4), we have

$$\lim_{x\nearrow 0} a(x)\, u_x(x) = \lim_{x\searrow 0} a(x)\, u_x(x) = 0. \tag{2.5}$$

In fact, in the case of strong degeneration, the transmission conditions at the damage point $x = 0$ can be specified as follows (see [14, Theorem 2.4]).

Theorem 2.2. *If $\alpha \in (1,2)$ then for any element $u \in H^1_a(-1,+1)$ the following assertions hold true:*

$$\lim_{x\nearrow 0} \sqrt{a(x)}\, u(x) = 0 = \lim_{x\searrow 0} \sqrt{a(x)}\, u(x), \tag{2.6}$$

$$\lim_{x\nearrow 0} a(x)\, u_x(x) = \lim_{x\searrow 0} a(x)\, u_x(x) = 0 \quad \text{provided } u \in H^2_a(-1,+1), \tag{2.7}$$

$$\lim_{x\nearrow 0} a(x)\, \varphi_x(x)\, u(x) = 0 = \lim_{x\searrow 0} a(x)\, \varphi_x(x)\, u(x), \quad \forall\, \varphi \in H^2_a(-1,+1). \tag{2.8}$$

In order to proceed further, we recall the main results of semi-group theory concerning weak and strong of solutions for differential operator equation. With that in mind, we introduce the Hilbert space $\mathcal{H}_a := H^1_{a,0}(-1,+1) \times L^2(-1,+1)$ and endow it with the scalar product

$$\left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \int_{-1}^{1} v(x)\, \widetilde{v}(x)\, \mathrm{d}x + \int_{-1}^{1} a(x)\, u_x(x)\, \widetilde{u}_x(x)\, \mathrm{d}x.$$

We also define the unbounded operator $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H}_a \to \mathcal{H}_a$, associated with the problem (1.4), as follows

$$\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ (au_x)_x \end{bmatrix}, \tag{2.9}$$

and

Case (WDP):

$$\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A}) \quad \textbf{if} \quad \left\{ \begin{array}{c} u \in H^2_a(-1,+1),\ v \in H^1_{a,0}(-1,+1), \\ \lim\limits_{x \nearrow 0} u(x) = \lim\limits_{x \searrow 0} u(x), \\ \lim\limits_{x \nearrow 0} a(x)\, u_x(x) = \lim\limits_{x \searrow 0} a(x)\, u_x(x), \\ u(-1) = u(+1) = 0; \end{array} \right\} \tag{2.10}$$

Case (SDP):

$$\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A}) \ \textbf{if} \left\{ \begin{array}{c} u \in H^2_a(-1,+1),\ v \in H^1_{a,0}(-1,+1), \\ \lim\limits_{x \nearrow 0} a\, \varphi_x u = 0 = \lim\limits_{x \searrow 0} a\, \varphi_x u,\ \forall\, \varphi \in H^2_a(-1,+1), \\ \lim\limits_{x \nearrow 0} a(x)\, u_x(x) = 0 = \lim\limits_{x \searrow 0} a(x)\, u_x(x), \\ u(-1) = u(+1) = 0. \end{array} \right\} \tag{2.11}$$

Arguing as in [8, Section II.2], it can be shown that, in both cases, $D(\mathcal{A})$ is a dense subset of $\mathcal{H}_a$.

Lemma 2.1. *$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}_a \to \mathcal{H}_a$ is the generator of a contraction semi-group in $\mathcal{H}_a$.*

Proof. It is well-known that if H is a Hilbert space and $B : D(B) \subset H \to H$ is a densely defined linear operator such that both B and B^* are dissipative, i. e.,

$$\langle Bu, u \rangle_H \leqslant 0 \quad \text{and} \quad \langle u, B^* u \rangle_H \leqslant 0 \quad \forall\, u \in D(B),$$

then B generates a strongly continuous semi-group of contraction operators [15, p. 686]. Let us show that $\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}_a$ for all $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A})$, and this operator satisfies the above mentioned properties.

Since the inclusion $\mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}_a$ is obvious for each $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A})$, it remains to check the properties

$$\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} \leqslant 0, \quad \text{and} \quad \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, (\mathcal{A})^* \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} \leqslant 0 \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A}). \tag{2.12}$$

We do it for the case (SDP), because the case (WDP) can be considered in a similar manner. Then the first inequality in (2.12) immediately follows from the definition of the set $D(\mathcal{A})$, transmission conditions (2.8), and the relations

$$\begin{aligned}
\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} &= \left\langle \begin{bmatrix} v \\ (au_x)_x \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \int_{-1}^{1} (au_x)_x \, v \, \mathrm{d}x + \int_{-1}^{1} a v_x u_x \, \mathrm{d}x \\
&= \lim_{x \nearrow 0} \left[\int_{-1}^{x} (au_s)_s \, v \, \mathrm{d}s + \int_{-1}^{x} a v_s u_s \, \mathrm{d}s \right] \\
&\quad + \lim_{x \searrow 0} \left[\int_{x}^{1} (au_s)_s \, v \, \mathrm{d}s + \int_{x}^{1} a v_s u_s \, \mathrm{d}s \right] \\
&= \left[\lim_{x \nearrow 0} a(x)\, u_x(x) v(x) \right] - \left[\lim_{x \searrow 0} a(x)\, u_x(x) v(x) \right] = 0,
\end{aligned} \tag{2.13}$$

which hold true for all $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A})$. Taking into account the equality

$$\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \mathcal{A}^* \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a}, \quad \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \in D(\mathcal{A}),$$

we see that

$$\begin{aligned}
\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} &= \left\langle \begin{bmatrix} v \\ (au_x)_x \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \int_{-1}^{1} (au_x)_x \, \widetilde{v} \, \mathrm{d}x + \int_{-1}^{1} a v_x \widetilde{u}_x \, \mathrm{d}x \\
&= \lim_{x \nearrow 0} \left[\int_{-1}^{x} (au_s)_s \, \widetilde{v} \, \mathrm{d}s + \int_{-1}^{x} a v_s \widetilde{u}_s \, \mathrm{d}s \right] \\
&\quad + \lim_{x \searrow 0} \left[\int_{x}^{1} (au_s)_s \, \widetilde{v} \, ds + \int_{x}^{1} a v_s \widetilde{u}_s \, \mathrm{d}s \right] \\
&= \lim_{x \nearrow 0} \left[-\int_{-1}^{x} a u_s \widetilde{v}_s \, \mathrm{d}s - \int_{-1}^{x} v \, (a\widetilde{u}_s)_s \, \mathrm{d}s \right] \\
&\quad + \lim_{x \searrow 0} \left[-\int_{x}^{1} a u_s \widetilde{v}_s \, \mathrm{d}s - \int_{x}^{1} v \, (a\widetilde{u}_s)_s \, \mathrm{d}s \right] \\
&\quad + \left[\lim_{x \nearrow 0} a(x)\, u_x(x)\, \widetilde{v}(x) - \lim_{x \searrow 0} a(x)\, u_x(x)\, \widetilde{v}(x) \right] \\
&\quad + \left[\lim_{x \nearrow 0} a(x)\, \widetilde{u}_x(x)\, v(x) - \lim_{x \searrow 0} a(x)\, \widetilde{u}_x(x)\, v(x) \right] \\
&= -\int_{-1}^{1} (a\widetilde{u}_x)_x \, v \, \mathrm{d}x - \int_{-1}^{1} a \widetilde{v}_x u_x \, \mathrm{d}x = \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} -\widetilde{v} \\ -(a\widetilde{u}_x)_x \end{bmatrix} \right\rangle_{\mathcal{H}_a}.
\end{aligned}$$

Hence, $\mathcal{A}^* \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} = \begin{bmatrix} -\widetilde{v} \\ -\left(a\widetilde{u}_x\right)_x \end{bmatrix}$, and arguing as in (2.13), we see that $\mathcal{A}^*$ is a dissipative operator as well. Thus, $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H}_a \to \mathcal{H}_a$ generates a strongly continuous semi-group of contraction operators. $\square$

We denote this semi-group by $e^{\mathcal{A}t}$. Then for any $U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathcal{H}_a$, the representation $U(t) = e^{\mathcal{A}t}\, U_0$ gives the so-called mild solution of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \mathcal{A}\, U(t), \quad t > 0, \\ U(0) = U_0\,. \end{cases} \tag{2.14}$$

When $U_0 \in D(\mathcal{A})$, the solution $U(t) = e^{\mathcal{A}t}U_0$ is strong in the sense that

$$U(\cdot) \in C^1\left([0,\infty); \mathcal{H}_a\right) \bigcap C\left([0,\infty); D(\mathcal{A})\right)$$

and equation (2.14) holds everywhere in $(0,\infty)$.

In view of the above consideration, we adopt the following concept.

Definition 2.1. We say that, for a given control $h(t)$, a function $u = u(t,x;\alpha)$ is the weak solution to the problem (1.4) if

$$u \in C^1([0,\infty); L^2(-1,+1)) \bigcap C([0,\infty); H^1_{a,0}(-1,+1)), \tag{2.15}$$

$$u(t,x;\alpha) = y(t,x;\alpha) + G(t,x), \tag{2.16}$$

and $U(t) := \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$ is the mild solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \mathcal{A}U(t) + F(t), \quad t > 0, \\ U(0) = \begin{bmatrix} -G(0,\cdot) \\ -G_t(0,\cdot) \end{bmatrix}, \end{cases} \tag{2.17}$$

where

$$F(t) = \begin{bmatrix} 0 \\ (aG_x)_x - G_{tt} \end{bmatrix}, \tag{2.18}$$

and $G \in W^{2,2}(0,T; H^2_a(-1,+1)) \bigcap C^2(0,T; L^2(-1,+1))$, $\forall\, T > 0$, is an arbitrary function such that

$$\begin{gathered} G(t,-1) = 0,\ G(t,1) = h(t),\ G(0,x) \in H^1_{a,0}(-1,+1), \\ \text{and } G_t(0,x) \in L^2(-1,+1) \text{ for a.a. } t \geqslant 0 \text{ and } x \in [-1,+1]. \end{gathered} \tag{2.19}$$

Definition 2.2. We say that a function $u = u(t,x;\alpha)$ is the strong solution to the problem (1.4) if each relation in (1.4) is satisfied for all $t \in [0,\infty)$ and a.a. $x \in [-1,+1]$,

$$u \in C^2([0,\infty); L^2(-1,+1)) \bigcap C^1([0,\infty); H^1_{a,0}(-1,+1)) \cap C([0,\infty); H^2_a(-1,+1)),$$

the representation (2.16) holds with a function G that, in addition to (2.19), satisfies

$$G(0,x) \in H^2_a(-1,+1),\ G_t(0,x) \in H^1_{a,0}(-1,+1) \text{ for } t \geqslant 0 \text{ and } x \in [-1,+1], \tag{2.20}$$

and the function y such that $U(t) := \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$ is the strong solution of the problem (2.17). If, in addition, all relations in (1.4) are satisfied for for all $t \in [0,\infty)$ and $x \in [-1,+1]$, then $u = u(t,x;\alpha)$ is the classical solution of the original initial-boundary value problem (1.4).

Our further intention is to examine the well-posedness of the problem (1.4) in the weak and strong degenerate cases.

Theorem 2.3. *Let $a(x;\alpha)\colon [-1,+1] \to \mathbb{R}$ be a weight function defined by* (1.2) *with $\alpha \in (0,2)$. Assume that for a given control $h(t)$ there exists function $G = G(t,x)$ satisfying properties* (2.19)*. Then initial boundary value problem*(1.4) *admits a unique weak solution $u = u(t,x;\alpha)$ for which representation* (2.16) *holds.*

Proof. Let $G \in W^{2,2}(0,T; H^2_a(-1,+1)) \bigcap C^2(0,T; L^2(-1,+1))$ be a function with properties (2.19). Then $(aG_x)_x - G_{tt} \in C([0,T]; L^2(-1,+1))$ and $U(0) \in \mathcal{H}_a$. Hence, by the Duhamel principle, we deduce that there exists a unique mild solution of the problem (2.17) and it can be represented as follows

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = e^{\mathcal{A}t} U(0) + \int_0^t e^{\mathcal{A}(t-s)} F(s)\,\mathrm{d}x \quad \forall\, t \in [0,T]. \tag{2.21}$$

As immediately follows from (2.16), the function $u(t,x)$ satisfies both initial and boundary conditions in (1.4), and its functional properties (2.15) easily follow from the semi-group properties of $e^{\mathcal{A}t}$ and (2.21). Thus, $u(t,x;\alpha)$ is a weak solution to the problem (1.4) in the sense of Definition 2.1.

As for uniqueness of the weak solution, it is a direct consequence of representation (2.16) and formula (2.21). □

Arguing in a similar manner, it can be established the following result.

Theorem 2.4. *Let $a(x;\alpha)\colon [-1,+1] \to \mathbb{R}$ be a weight function defined by* (1.2) *with $\alpha \in (0,2)$. Assume that for a given control $h(t)$ there exists function $G = G(t,x)$ satisfying properties* (2.19)*–*(2.20)*. Then initial boundary value problem*(1.4) *admits a unique strong solution.*

To prove this assertion, it is enough to notice that due to the properties (2.19)–(2.20), the function $G(t,x)$ is sufficiently smooth and the vector of initial data $U(0)$ belongs to the set $D(\mathcal{A})$ both in the weak and strong degenerate cases.

Remark 2.2. In view of the transmission conditions (see Theorem 2.2), it is still unknown whether the strong solution to the problem (1.4) preserves its continuity at the damage point $x = 0$ in the case of strong degeneration. The second point that should be clarified, is about existence of a classical solution to the problem (1.4) for different range of parameter $\alpha > 0$ provided $h \in C([0,T])$. To the best knowledge of authors, this question remains arguably open for nowadays.

For our further analysis, in order to specify the notion of solution to the problem (1.4) possessing the continuously differentiable flux for a wide range of parameter α, we adopt the following concept.

Definition 2.3. A function $u(t,x;\alpha)$ is called a solution with the continuously differentiable flux (or, shortly the required solution) to the initial boundary value problem (1.4) if it: 1) is continuous in the space-time rectangle $[0,T]\times[-1,+1]$ and is twice continuously differentiable in the variables t and x inside the space-time rectangle, except for the degeneracy segment; 2) has the one-sided derivative in the variable x, bounded or integrable, and the flux, continuously differentiable in the variable x, both on the degeneracy segment; 3) satisfies the degenerate wave equation inside the space-time rectangle; and 4) satisfies the initial and the boundary conditions of the problem.

3. Series solutions of the original wave equation

In this Section we construct power series solutions $u^{\mp}(t,x;\alpha)$ of the original degenerate wave equation in one-sided vicinities $[0,T]\times(-\epsilon,0)$ and $[0,T]\times(0,+\epsilon)$, $\epsilon \leqslant c$, of the degeneracy segment. These solutions are further referred to as one-sided and constitute a pair for the ϵ-band $\subseteq c$-band (with the degeneracy segment removed or not removed). To construct such pairs, we introduce the set of rational numbers

$$\mathbb{Q}_o := \left\{ \frac{m}{n} :\ m \in \mathbb{Z}\,,\ \ n \in \mathbb{N}\,,\ \ m \neq 0\,,\ \ n > 1\,,\ \ n \equiv 1\ (\mathtt{mod}\,2) \right\}, \tag{3.1}$$

where m, n are coprime numbers, and those derived from subsets of $\mathbb{R}$, for instance, $(0,2)_o := (0,2) \bigcap \mathbb{Q}_o$, etc., to distinguish between the exponents σ of those power monomials x^σ, $x > 0$, extendable straightforwardly to $x < 0$ and not extendable. Let $\mathbb{Q}_{o,e}$ be the subset of $\mathbb{Q}_o$, where $m \equiv 0\ (\mathtt{mod}\,2)$, and $\mathbb{Q}_{o,o}$ be the subset of $\mathbb{Q}_o$, where $m \equiv 1\ (\mathtt{mod}\,2)$, then both subsets partition the set $\mathbb{Q}_o$: $\mathbb{Q}_{o,e} \bigcup \mathbb{Q}_{o,o} = \mathbb{Q}_o$, $\mathbb{Q}_{o,e} \bigcap \mathbb{Q}_{o,o} = \varnothing$. It is clear that monomials x^σ are extendable to $x < 0$ evenly, if $\sigma \in \mathbb{Q}_{o,e}$, and oddly, if $\sigma \in \mathbb{Q}_{o,o}$.

Initially (A), we present our attempts to find pairs of one-sided series solutions, substituting pairs of trial one-sided power series, or pairs of ansatze, into the original degenerate wave equation, and finally (B), we find the above pairs of one-sided series solutions (A) once again using the Frobenius method [7, 12].

A) Let the pair of trial one-sided power series solutions of the degenerate wave equation be of the form

$$u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\sigma^{\mp}_{\alpha,1}} + U^{\mp}_{\alpha,2}(t)\,|x|^{\sigma^{\mp}_{\alpha,2}} + \ldots,$$

where $U^{\mp}_{\alpha,0}(t)$, $U^{\mp}_{\alpha,1}(t)$, $U^{\mp}_{\alpha,2}(t)$, etc., are unknown coefficient functions of variable t; $\sigma^{\mp}_{\alpha,1}$, $\sigma^{\mp}_{\alpha,2}$, etc., are unknown real exponents. Differentiating the above pair of the ansatze with respect to t and x (and dropping for a while hereafter the argument t of the functions and some of the lower and upper indices of the functions and the exponents, to simplify the notation where this will not lead to confusion)

$$\begin{aligned}
\frac{\partial p^{\mp}}{\partial t} &= U''_{\alpha,0} + U''_{\alpha,1}\,|x|^{\sigma_1} + U''_{\alpha,2}\,|x|^{\sigma_2} + \ldots,\\
q^{\mp} &= \mp\Big(\quad \sigma_1\,U_{\alpha,1}\,|x|^{\sigma_1-1} + \quad \sigma_2\,U_{\alpha,2}\,|x|^{\sigma_2-1} + \ldots\Big),\\
-f^{\mp} &= \mp a_*\Big(\quad \sigma_1\,U_{\alpha,1}\,|x|^{\omega_1} \quad + \quad \sigma_2\,U_{\alpha,2}\,|x|^{\omega_2} \quad + \ldots\Big),\\
-\frac{\partial f^{\mp}}{\partial x} &= \quad a_*\Big(\omega_1\,\sigma_1\,U_{\alpha,1}\,|x|^{\omega_1-1} + \omega_2\,\sigma_2\,U_{\alpha,2}\,|x|^{\omega_2-1} + \ldots\Big),
\end{aligned}$$

where $\omega_1 = \sigma_1 + \alpha - 1$, $\omega_2 = \sigma_2 + \alpha - 1$, etc., and substituting the obtained pairs of the series for the derivatives into the degenerate wave equation (1.3) yields to the following pair of the one-sided series identities

$$\begin{aligned}
&\underbrace{U''_{\alpha,0}}_{1} + \underbrace{U''_{\alpha,1}\,|x|^{\sigma_1}}_{2} + \underbrace{U''_{\alpha,2}\,|x|^{\sigma_2}}_{3} + \ldots\\
&= a_*\Big(\underbrace{\sigma_1\,\omega_1\,U_{\alpha,1}\,|x|^{\omega_1-1}}_{1} + \underbrace{\sigma_2\,\omega_2\,U_{\alpha,2}\,|x|^{\omega_2-1}}_{2} + \underbrace{\sigma_3\,\omega_3\,U_{\alpha,3}\,|x|^{\omega_3-1}}_{3} + \ldots\Big),
\end{aligned} \tag{3.2}$$

being used for non-unique determining all unknown functions and exponents.

i) Let terms 1, 2, 3, etc., in the pair of the one-sided series identities (3.2) (i. e. the couples of summands marked with the same numbers) be of like powers and cancel respectively each other, then, after applying a little portion of algebra, we obtain: 1) the explicit expressions for the exponents $\sigma^{\mp}_{\alpha,\mu} = \mu\theta$, $\mu \in \mathbb{N}$, where $\theta = 2 - \alpha$ is a derived parameter, leading to the following pairs of the one-sided series solutions

$$u^{\mp}(t,x;\alpha) = \begin{cases} U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\theta} + U^{\mp}_{\alpha,2}(t)\,|x|^{2\theta} + \ldots\,,\\ U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\ \ x^{\ \theta} + U^{\mp}_{\alpha,2}(t)\ \ x^{\ 2\theta} + \ldots\,, \end{cases} \tag{3.3}$$

valid respectively for $\alpha \in (0,2) \bigcup (2,+\infty)$ and $(0,2)_o \bigcup (2,+\infty)_o$; and 2) the following pairs of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,\mu-1}(t)}{\mathrm{d}t^2} = \mu\theta\,[(\mu-1)\,\theta+1]\,a_*\,U^{\mp}_{\alpha,\mu}(t)\,, \qquad \mu\in\mathbb{N}\,. \tag{3.4}$$

ii) Let each term in the pair of the one-sided series identities (3.2) vanish separately, then: 1) the functions $U^{\mp}_{\alpha,0}(t)$, $U^{\mp}_{\alpha,\mu}(t)$, $\mu\in\mathbb{N}$, are linear; 2) the exponents are equal each other: $\sigma^{\mp}_{\alpha,\mu}=1-\alpha$; and 3) the resulting pairs of the one-sided series solutions reduce to the following pairs of the one-sided binomials

$$u^{\mp}(t,x;\alpha) = \begin{cases} U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{1-\alpha}, & \alpha\in(0,+\infty)\,,\\ U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\;\;x^{\,1-\alpha}, & \alpha\in(0,+\infty)_o. \end{cases} \tag{3.5}$$

iii) Let terms 1 in the pair of the one-sided series identities (3.2) vanish separately, then: 1) $U^{\mp}_{\alpha,0}(t)$ are linear functions; 2) $\omega^{\mp}_{\alpha,1}=0$, from where $\sigma^{\mp}_{\alpha,1}=1-\alpha$. Applying the procedure from item *i*) to terms 2, 3, 4, etc., we find: 1) the explicit expressions for the exponents $\sigma^{\mp}_{\alpha,\mu+1}=\mu\theta+1-\alpha$, $\mu\in\mathbb{N}$, leading to the following pairs of the one-sided series solutions

$$\begin{cases} u^{\mp}(t,x;\alpha)=U^{\mp}_{\alpha,0}(t)+|x|^{1-\alpha}\Big(U^{\mp}_{\alpha,1}(t)+U^{\mp}_{\alpha,2}(t)\,|x|^{\theta}+U^{\mp}_{\alpha,3}(t)\,|x|^{2\theta}+\ldots\Big),\\ u^{\mp}(t,x;\alpha)=U^{\mp}_{\alpha,0}(t)+\;x^{\,1-\alpha}\Big(U^{\mp}_{\alpha,1}(t)+U^{\mp}_{\alpha,2}(t)\;\;x^{\,\theta}+U^{\mp}_{\alpha,3}(t)\;\;x^{\,2\theta}+\ldots\Big), \end{cases} \tag{3.6}$$

valid respectively for $\alpha\in(0,+\infty)$ and $\alpha\in(0,+\infty)_o$; and 2) the following pairs of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,\mu}(t)}{\mathrm{d}t^2} = \mu\theta\,[(\mu+1)\,\theta-1]\,a_*\,U^{\mp}_{\alpha,\mu+1}(t)\,, \qquad \mu\in\mathbb{N}\,. \tag{3.7}$$

B) Let the pair of trial one-sided power series solutions of the degenerate wave equation, following the Frobenius method, be of the form

$$u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha}(t) + |x|^{\omega^{\mp}_{\alpha}}\Big(U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\sigma^{\mp}_{\alpha,1}} + U^{\mp}_{\alpha,2}(t)\,|x|^{\sigma^{\mp}_{\alpha,2}} + \ldots\Big)\,,$$

where $U^{\mp}_{\alpha}(t)$, $U^{\mp}_{\alpha,0}(t)$, $U^{\mp}_{\alpha,1}(t)$, $U^{\mp}_{\alpha,2}(t)$, etc., are unknown coefficient functions of variable t; $\omega^{\mp}_{\alpha}$, $\sigma^{\mp}_{\alpha,1}$, $\sigma^{\mp}_{\alpha,2}$, etc., are unknown real exponents. Differentiating the above pair of the ansatze with respect to t and x

$$\begin{aligned}
\frac{\partial p^{\mp}}{\partial t} &= U''_{\alpha} + |x|^{\omega}\Big(U''_{\alpha,0} + U''_{\alpha,1}\,|x|^{\sigma_1} + U''_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big),\\
q^{\mp} &= \mp\quad |x|^{\omega-1}\quad\Big(\omega\,U_{\alpha,0} + \omega_1\,U_{\alpha,1}\,|x|^{\sigma_1} + \omega_2\,U_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big),\\
-f^{\mp} &= \mp a_*|x|^{\omega-1+\alpha}\Big(\omega\,U_{\alpha,0} + \omega_1\,U_{\alpha,1}\,|x|^{\sigma_1} + \omega_2\,U_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big),\\
-\frac{\partial f^{\mp}}{\partial x} &= \quad a_*|x|^{\omega-2+\alpha}\Big(\omega\,(\omega-1+\alpha)\,U_{\alpha,0} + \omega_1\,(\omega-1+\alpha+\sigma_1)\,U_{\alpha,1}\,|x|^{\sigma_1}\\
&\qquad\qquad + \omega_2\,(\omega-1+\alpha+\sigma_2)\,U_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big),
\end{aligned}$$

where $\omega_1 = \omega + \sigma_1$, $\omega_2 = \omega + \sigma_2$, etc., and substituting the obtained pairs of the series for the derivatives into the degenerate wave equation, we obtain the following pair of the one-sided series identities

$$\begin{aligned} &\underbrace{U''_\alpha}_{1} + |x|^\omega \Big(\underbrace{U''_{\alpha,0}}_{2} + \underbrace{U''_{\alpha,1}\,|x|^{\sigma_1}}_{3} + U''_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big) \\ &= a_*|x|^{o-1}\Big(\underbrace{\omega o\, U_{\alpha,0}}_{1} + \underbrace{\omega_1\,(o+\sigma_1)\, U_{\alpha,1}\,|x|^{\sigma_1}}_{2} + \underbrace{\omega_2\,(o+\sigma_2)\, U_{\alpha,2}\,|x|^{\sigma_2}}_{3} + \ldots\Big), \end{aligned} \tag{3.8}$$

where $o = \omega - 1 + \alpha$. The identities are to be used for determining all unknown functions and exponents, but in a non-unique way.

i) Let terms 1 in (3.8) be of like powers (accounting for the multiplier $a_*|x|^{o-1}$), then we obtain: *a*) $o_\alpha^\mp = 1$, $\omega_\alpha^\mp = \theta$; and *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_\alpha^\mp(t)}{\mathrm{d}t^2} = \omega_\alpha^\mp o_\alpha^\mp\, a_* U_{\alpha,0}^\mp(t) = \theta\, a_* U_{\alpha,0}^\mp(t)\,.$$

Let now terms 2 in (3.8) be of like powers, then we obtain: *a*) $\sigma_{\alpha,1}^\mp = \omega_\alpha^\mp = \theta$; and *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_{\alpha,0}^\mp(t)}{\mathrm{d}t^2} = \omega_{\alpha,1}^\mp \big(o_\alpha^\mp + \sigma_{\alpha,1}^\mp\big)\, a_* U_{\alpha,1}^\mp(t) = 2\theta\,(\theta+1)\, a_* U_{\alpha,1}^\mp(t)\,.$$

Then, repeating the above procedure for terms 3 in (3.8), we obtain: *a*) $\sigma_{\alpha,2}^\mp = \omega_\alpha^\mp + \sigma_{\alpha,1}^\mp = 2\theta$, and *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_{\alpha,1}^\mp(t)}{\mathrm{d}t^2} = \omega_{\alpha,2}^\mp \big(o_\alpha^\mp + \sigma_{\alpha,2}^\mp\big)\, a_* U_{\alpha,2}^\mp(t) = 3\theta\,(2\theta+1)\, a_* U_{\alpha,2}^\mp(t)\,,$$

etc. That is, we obtain nothing but the pairs of the one-sided series solutions (3.3) again.

ii) Let in (3.8): *a*) the functions $U_\alpha^\mp(t)$ and $U_{\alpha,0}^\mp(t)$ be linear; *b*) $\omega_\alpha^\mp = 1-\alpha$; and *c*) $\sigma_{\alpha,\mu}^\mp = 0$, $\mu \in \mathbb{N}$, then we obtain the pairs of the one-sided binomial solutions (3.5) again.

iii) Let terms 1 in (3.8) vanish separately, then: *a*) the functions $U_\alpha^\mp(t)$ are linear; *b*) $o_\alpha^\mp = 0$, $\omega_\alpha^\mp = 1-\alpha$. Let now terms 2 in (3.8) be of like powers, then we obtain: *a*) $\sigma_{\alpha,1}^\mp = \theta$; *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_{\alpha,0}^\mp(t)}{\mathrm{d}t^2} = \omega_{\alpha,1}^\mp \big(o_\alpha^\mp + \sigma_{\alpha,1}^\mp\big)\, a_* U_{\alpha,1}^\mp(t) = \theta\,(2\theta-1)\, a_* U_{\alpha,1}^\mp(t)\,.$$

Treating terms 3 in (3.8) in the same way we obtain: *a*) $\sigma_{\alpha,2}^\mp = 2\theta$; *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,1}(t)}{\mathrm{d}t^2} = \omega^{\mp}_{\alpha,2}\big(o^{\mp}_{\alpha} + \sigma^{\mp}_{\alpha,2}\big)\, a_* U^{\mp}_{\alpha,2}(t) = 2\theta\,(3\theta-1)\, a_* U^{\mp}_{\alpha,2}(t)\,.$$

Repeating the above procedure, we eventually obtain: 1) the explicit expressions for the exponents $\sigma^{\mp}_{\alpha,\mu} = \mu\theta$, leading to the following pairs of the one-sided series solutions

$$\begin{cases} u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha}(t) + |x|^{1-\alpha}\Big(U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\theta} + U^{\mp}_{\alpha,2}(t)\,|x|^{2\theta} + \ldots\Big), \\ u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha}(t) + \ x^{\,1-\alpha}\Big(U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\ x^{\,\theta} + U^{\mp}_{\alpha,2}(t)\ x^{\,2\theta} + \ldots\Big), \end{cases} \tag{3.9}$$

valid respectively for $\alpha \in (0,+\infty)$ and $(0,+\infty)_o$; and 2) the following pairs of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,\mu-1}(t)}{\mathrm{d}t^2} = \mu\theta\,[(\mu+1)\,\theta - 1]\, a_* U^{\mp}_{\alpha,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,. \tag{3.10}$$

That is, we obtain nothing but the pairs of the the one-sided series solutions (3.6) again.

C) When constructing the pairs of the one-sided series solutions (3.3), (3.5) and (3.9) of the degenerate wave equation, we have not accounted for the following (1-parameter families of the) pairs of the binomial solutions

$$u^{\mp}(t,x;2) = \begin{cases} U^{\mp}_{2,0}(t) + U^{\mp}_{2,\sigma}(t)\,|x|^{\sigma^{\mp}}, & \sigma^{\mp} \in \mathbb{R}\,, \\ U^{\mp}_{2,0}(t) + U^{\mp}_{2,\sigma}(t)\ x^{\,\sigma^{\mp}}, & \sigma^{\mp} \in \mathbb{R}_o\,, \end{cases} \tag{3.11}$$

where $\sigma^{\mp} \in \mathbb{R}$, $U^{\mp}_{2,0}(t)$ are linear functions, and the functions $U^{\mp}_{2,\sigma}(t)$ satisfy the following ordinary linear homogeneous differential equations

$$\frac{\mathrm{d}^2 U^{\mp}_{2,\sigma}(t)}{\mathrm{d}t^2} - \sigma\,(\sigma+1)\, a_* U^{\mp}_{2,\sigma}(t) = 0\,. \tag{3.12}$$

Now we are to follow Definition 2.3 and to select the required series solutions out of those obtained in this Section.

4. Series based analysis of the original problem

In this Section we turn out to possessing or violating property Z by solutions to the initial boundary value problem (1.4) using the series solutions of the degenerate wave equation. On the one hand, the one-sided series solutions constituting a pair are independent, on the other hand, possessing property Z implies integrity of the 'string' on the degeneracy segment and, more generally, fulfilling the conditions of the Definition 2.3, except perhaps the last one.

To gain the required properties for the series solutions, we follow the three-step procedure applied on the degeneracy segment: 1) continuous matching the one-sided series solutions by setting $U_\alpha^-(t) = U_\alpha^+(t)$, $U_{\alpha,\mu}^-(t) = U_{\alpha,\mu}^+(t)$ (provided all exponents of the one-sided series solutions are non-negative for the resulting continuous, or two-sided, series solutions to be bounded); 2) verifying continuity of the flux; 3) verifying differentiability of the flux.

First, we apply the procedure to the pairs of the one-sided series solutions (3.3), (3.5), and (3.9).

1) Continuous matching the above one-sided series solutions formally yields to the following bounded and continuous, or two-sided, nontrivial series solutions (trivial solutions, valid if $\alpha \in [2, +\infty)$, are discussed in the proof of Proposition 4.2)

$$
\begin{aligned}
&u_1(t,x;\alpha) = U_{\alpha,0}(t) + \sum_{\mu=1}^{\infty} U_{\alpha,\mu}(t)\,|x|^{\mu\theta}, && \alpha \in (0,2)\,,\\
&u_2(t,x;\alpha) = U_{\alpha,0}(t) + \sum_{\mu=1}^{\infty} U_{\alpha,\mu}(t)\ \ x^{\ \mu\theta}, && \alpha \in (0,2)_o\,,\\
&u_3(t,x;\alpha) = U_{\alpha,0}(t) + U_{\alpha,1}(t)\,|x|^{1-\alpha}, && \alpha \in (0,1)\,,\\
&u_4(t,x;\alpha) = U_{\alpha,0}(t) + U_{\alpha,1}(t)\ \ x^{\ 1-\alpha}, && \alpha \in (0,1)_o\,,\\
&u_5(t,x;\alpha) = U_\alpha(t) + |x|^{1-\alpha}\sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\,|x|^{\mu\theta}, && \alpha \in (0,1)\,,\\
&u_6(t,x;\alpha) = U_\alpha(t) +\ \ x^{\ 1-\alpha}\sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\ \ x^{\ \mu\theta}, && \alpha \in (0,1)_o\,.
\end{aligned}
\tag{4.1}
$$

2) The respective one-sided values $-f^{\mp} = aq^{\mp}$ of the fluxes for the above two-sided series solutions (for $x < 0$ and $x > 0$) are given by the following expressions

$$
\begin{aligned}
-f_1^{\mp}(t,x;\alpha) &= a_*\theta\, x \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{(\mu-1)\theta},\\
-f_2^{\mp}(t,x;\alpha) &= \mp a_*\theta\, x \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\ \ x^{\ (\mu-1)\theta},\\
-f_3^{\mp}(t,x;\alpha) &= \mp a_*(1-\alpha)\, U_{\alpha,1}(t)\,,\\
-f_4^{\mp}(t,x;\alpha) &= \mp a_*(1-\alpha)\, U_{\alpha,1}(t)\,,\\
-f_5^{\mp}(t,x;\alpha) &= \mp a_* \sum_{\mu=0}^{\infty} (1-\alpha+\mu\theta)\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta},\\
-f_6^{\mp}(t,x;\alpha) &= \mp a_* \sum_{\mu=0}^{\infty} (1-\alpha+\mu\theta)\, U_{\alpha,\mu}(t)\ \ x^{\ \mu\theta}.
\end{aligned}
\tag{4.2}
$$

To complete the current step of the procedure we calculate the one-sided values of the fluxes on the degeneracy segment (by substituting in (4.2) zero value instead of x) and conclude that the series solutions 1 and 2 produce zero one-sided fluxes: $f_{1,2}(t,0-;\alpha)=0=f_{1,2}(t,0+;\alpha)$, whereas the series solutions $3-6$ produce non-zero one-sided fluxes of opposite signs: $f_{3-6}(t,0-;\alpha)=-f_{3-6}(t,0+;\alpha)$. Therefore, we retain the series solutions 1 and 2 with the continuous fluxes to implement the next step.

3) It is clear that the flux of the series solution 1 is continuously differentiable on the degeneracy segment, whereas the flux of the series solution 2 is not. Therefore, the only required series solution, we retain for further studying, reads

$$u(t,x;\alpha) = U_{\alpha,0}(t) + \sum_{\mu=1}^{\infty} U_{\alpha,\,\mu}(t)\,|x|^{\mu\theta}, \qquad \alpha \in (0,2)\,, \tag{4.3}$$

where the coefficient functions obey the following recurrence relations

$$U''_{\alpha,\mu-1}(t) = \mu\theta\,[(\mu-1)\,\theta+1]\,a_* U_{\alpha,\,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,. \tag{4.4}$$

Second, we apply the procedure to the pairs of the binomial solutions (3.11).

1) Implementing continuous matching yields to the following bounded and continuous solutions (1-parameter families of solutions)

$$\begin{cases} u_7(t,x;2,\sigma) = U_{2,0}(t) + U_{2,\sigma}(t)\,|x|^{\sigma}, & \sigma \in (0,+\infty)\,,\\ u_8(t,x;2,\sigma) = U_{2,0}(t) + U_{2,\sigma}(t)\;\,x^{\,\sigma}, & \sigma \in (0,+\infty)_o\,, \end{cases} \tag{4.5}$$

where the functions $U_{2,0}(t)$ are linear, and the functions $U_{2,\sigma}(t)$ satisfy the following ordinary linear homogeneous differential equations

$$U''_{2,\sigma}(t) - a_*\sigma\,(\sigma+1)\,U_{2,\sigma}(t) = 0\,. \tag{4.6}$$

2) The one-sided values of the fluxes for the above solutions

$$\begin{aligned} -f_7(t,x;2,\sigma) &= \mp a_*\sigma\,U_{2,\sigma}(t)\,|x|^{\sigma+1}\,,\\ -f_8(t,x;2,\sigma) &= \quad a_*\sigma\,U_{2,\sigma}(t)\;\,x^{\,\sigma+1}\,, \end{aligned}$$

vanish on the degeneracy segment, proving continuity of both fluxes.

3) Continuous differentiability of both fluxes is evident. Completing the procedure, we conclude that both binomial solutions (4.5) are required.

Proposition 4.1. *Possessing property Z is not necessary for the required solution to the 1-parameter initial boundary value problem* (1.4) *if* $\alpha \in (0,2)$.

Proof. Applying the required series solution (4.3) exactly on the degeneracy segment, we find that $u(t,0;\alpha)=U_{\alpha,0}(t)$, where the undetermined function $U_{\alpha,0}(t)$ is a solution to the Cauchy problem

$$U''_{\alpha,0}(t) = \theta\, a_* U_{\alpha,1}(t)\,, \quad t \in (0,T]\,, \qquad U_{\alpha,0}(0) = 0\,, \quad U'_{\alpha,0}(0) = 0\,,$$

assembled from the recurrence relations (4.4) and the initial conditions of the problem (1.4). Non-uniqueness of $U_{\alpha,0}(t)$ stems from the undetermined function $U_{\alpha,1}(t)$.

Assumption $U_{\alpha,1}(t) \equiv 0$ yields to a linear function $U_{\alpha,0}(t)$, and from the above initial conditions we conclude that $U_{\alpha,0}(t) \equiv 0$. But assuming that $U_{\alpha,1}(t)$ is not identically zero, we conclude that $U_{\alpha,0}(t)$ is not identically equal to zero as well, though the initial conditions it satisfies are all zero. Hence, property Z is not necessary for the required solution to the initial boundary value problem (1.4).

It should be noted, that continuity of the required solution is essentially used to prove the proposition. Indeed, let the 'string' lose its integrity, then problem (1.4) immediately splits into two quite independent subproblems referred to as the left one

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial u}{\partial x}\right), & (t,x) \in (0,T)\times(-l,0)\,, \\ u(t,-l;\alpha) = 0\,, & t \in [0,T]\,, \\ u(t,\quad 0;\alpha) = 0\,, & t \in [0,T]\,, \\ u(0,x;\alpha) = 0\,, & x \in [-l,0]\,, \\ \dfrac{\partial u(0,x;\alpha)}{\partial t} = 0\,, & x \in [-l,0]\,, \end{cases} \tag{4.7}$$

and the right one

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial u}{\partial x}\right), & (t,x) \in (0,T)\times(0,+l)\,, \\ u(t,\quad 0;\alpha) = 0\,, & t \in [0,T]\,, \\ u(t,+l;\alpha) = h(t)\,, & t \in [0,T]\,, \\ u(0,x;\alpha) = 0\,, & x \in [0,+l]\,, \\ \dfrac{\partial u(0,x;\alpha)}{\partial t} = 0\,, & x \in [0,+l)\,. \end{cases} \tag{4.8}$$

Both problems are evidently solvable, whereas the unique solution to the left subproblem is trivial and property Z necessarily holds. □

Proposition 4.2. *Possessing property Z is necessary for the required solutions to the 1-parameter initial boundary value problem* (1.4) *if* $\alpha \in [2,+\infty)$.

Proof. Let $\alpha = 2$, then the required series solutions to the problem are those given by (4.5), where the linear functions $U_{2,0}(t) \equiv 0$, due to the initial conditions of the problem. Therefore, $u(t,0;2)\equiv 0$, and the unique solution to the left subproblem (4.7) is trivial. Hence, property Z necessarily holds.

Let $\alpha\in(2,+\infty)$, then the required series solutions (3.3), (3.5), and (3.9) to the problem are those where all functions $U_{\alpha,\mu}(t)$ preceding the power monomials are identically equal to zero. Therefore, the series solutions reduce to the leading terms $U_\alpha(t)$ or $U_{\alpha,0}(t)$, being zero due to the initial conditions of the problem. The remaining part of the proof is the same as for $\alpha=2$. □

5. Separation of variables applied to the original equation

In this Section we find the series solutions to the original degenerate wave equation that can be rewritten as

$$\frac{\partial^2 u(t,x)}{\partial t^2} - a(x;\alpha)\,\frac{\partial^2 u(t,x)}{\partial x^2} = a'(x;\alpha)\,\frac{\partial u(t,x)}{\partial x}\,, \tag{5.1}$$

in a way different to that used in Section 3.

First, we assume that the independent variables in (5.1) are separable, hence the trial solution reads

$$u(t,x;\alpha) = O(t;\alpha)\,X(x;\alpha)\,, \tag{5.2}$$

and substitute the above representation into (5.1) to obtain the following system of two ordinary differential equations of the second order

$$\begin{cases} O''(t;\alpha) \mp \lambda^2\,O\,(t;\alpha) = 0\,, \\ a(x;\alpha)\,X''(x;\alpha) + a'(x;\alpha)\,X'(x;\alpha) \mp \lambda^2 X(x;\alpha) = 0\,, \end{cases} \tag{5.3}$$

where $\pm\lambda^2$ is the unknown parameter of separation of the variables ($\lambda\geqslant 0$).

Second, we: *a*) substitute the power law (1.2) into the system (5.3)

$$a_* x^\alpha\,X''(x;\alpha) + a_*\alpha\,x^{\alpha-1}\,X'(x;\alpha) \mp \lambda^2 X(x;\alpha) = 0\,,$$

assuming initially that $x>0$; *b*) introduce a new dependent variable

$$X(x;\alpha) = x^\beta w(x;\alpha)\,, \tag{5.4}$$

where β is the undetermined exponent; and *c*) find the relations between the first and the second derivatives of X and w as follows

$$X' = \beta\,x^{\beta-1}\,w + x^\beta w', \qquad X'' = \beta\,(\beta-1)\,x^{\beta-2}\,w + 2\beta\,x^{\beta-1}w' + x^\beta w'';$$

d) then, a 4-parameter $(\alpha,\beta,\lambda,\mp)$ family of ordinary differential equations of the second order for the required function w reads

$$a_* x^\alpha \, w'' + a_* x^{\alpha-1} \, (2\beta + \alpha) \, w' + a_* x^{\alpha-2} \beta \, (\beta - 1 + \alpha) \, w' \pm \lambda^2 w = 0 \,. \tag{5.5}$$

Third, we a) introduce a new independent variable: $\bar{x} = \chi(x)$, where function $\chi(x)$ is invertible and differentiable; and b) obtain the relations between the first and the second derivatives of $w(x;\alpha)$ and $\bar{w}(\bar{x};\alpha) := w(\chi^{-1}(\bar{x});\alpha)$

$$w' = \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} \frac{\mathrm{d}\chi}{\mathrm{d}x} \,, \qquad w'' = \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} \left(\frac{\mathrm{d}\chi}{\mathrm{d}x} \right)^2 + \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} \frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} \,.$$

Imposing constraint $a_* x^\alpha \left(\dfrac{\mathrm{d}\chi}{\mathrm{d}x} \right)^2 = 1$ on the required function $\chi(x)$ yields to

$$\frac{\mathrm{d}\chi}{\mathrm{d}x} = \frac{1}{\sqrt{a_* x^\alpha}} \,, \qquad a_* x^\alpha \, \frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} = -\frac{\alpha}{2} \frac{1}{x} \sqrt{a_* x^\alpha} \,,$$

then assuming that $\bar{x}=0$ when $x=0$, we obtain by integration the explicit relation between x and $\bar{x}$

$$\bar{x} = \frac{2}{\theta} \frac{x}{\sqrt{a_* x^\alpha}} = \Omega \, x^{\frac{\theta}{2}} \qquad \Leftrightarrow \qquad \frac{\sqrt{a_* x^\alpha}}{x} = \frac{2}{\theta} \frac{1}{\bar{x}} \,,$$

where an α-dependent auxiliary quantity is used

$$\Omega = \frac{2}{\theta} \frac{1}{\sqrt{a_*}} \,. \tag{5.6}$$

Replacing the couple of the variables (x, w) with that of the variables $(\bar{x}, \bar{w})$ in (5.5) yields to a new 4-parameter $(\alpha, \beta, \lambda, \mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} + \left(2\beta + \frac{\alpha}{2} \right) \frac{2}{\theta} \frac{1}{\bar{x}} \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} + \beta \, (\beta - 1 + \alpha) \left(\frac{2}{\theta} \frac{1}{\bar{x}} \right)^2 \bar{w} \mp \lambda^2 \bar{w} = 0 \,.$$

Setting the value of the first coefficient

$$\left(2\beta + \frac{\alpha}{2} \right) \frac{2}{\theta} = 1 \,,$$

we find: a) the exponent in the transformation (5.4)

$$\beta = \frac{1 - \alpha}{2} \,,$$

b) the value of the second coefficient in the above ordinary differential equation

$$\beta \, (\beta - 1 + \alpha) \left(\frac{2}{\theta} \right)^2 = -\beta^2 \left(\frac{2}{\theta} \right)^2 = -\left(\frac{1 - \alpha}{\theta} \right)^2 \equiv -\varrho^2,$$

and c) the resulting 3-parameter $(\varrho(\alpha), \lambda, \mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} + \frac{1}{\bar{x}}\,\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} - \frac{\varrho^2}{\bar{x}^2}\,\bar{w} \mp \lambda^2\bar{w} = 0\,. \tag{5.7}$$

I) Assuming that $\lambda>0$ and introducing a new couple of the independent and the dependent variables once again: $s = \lambda\bar{x}$, $W(s;\alpha) := \bar{w}(\lambda^{-1}s;\alpha)$, we eventually obtain a 2-parameter $(\varrho(\alpha),\mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\varrho^2}{s^2} \pm 1\right) W = 0\,.$$

The choice of the lower sign in the obtained family leads to the 1-parameter $(\varrho(\alpha))$ Bessel equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\varrho^2}{s^2} - 1\right) W = 0\,, \tag{5.8}$$

whereas the choice of the upper sign leads to the 1-parameter $(\beta(\alpha))$ modified Bessel equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\varrho^2}{s^2} + 1\right) W = 0\,. \tag{5.9}$$

i) A 3-parameter family of solutions of the ordinary differential equation (5.8), when $\varrho \notin \mathbb{Z}$ (i. e. $\alpha\in(0,1)\bigcup(1,2)$) , is known [9, 19] to be

$$W(s;\alpha) = A_1\,\mathtt{J}_{-\varrho}(s) + A_2\,\mathtt{J}_{+\varrho}(s)\,, \tag{5.10}$$

where $A_{1,2}$ are arbitrary constants (parameters), and $\mathtt{J}_{\mp\varrho}(s)$ are the Bessel functions of the first kind of orders $\mp\varrho$

$$\mathtt{J}_{\mp\varrho}(s) = \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{\mu!\,\Gamma(\mu\mp\varrho+1)} \left(\frac{s}{2}\right)^{2\mu\mp\varrho} = \left(\frac{s}{2}\right)^{\mp\varrho} \sum_{\mu=0}^{\infty} A_{\mp\varrho,\,\mu} \left(\frac{s}{2}\right)^{2\mu}. \tag{5.11}$$

whereas in case $\varrho \in \mathbb{Z}$ (i. e. $\alpha=1$) a 2-parameter family of solutions of the ordinary differential equation (5.8) is known [9, 19] to be

$$W(s;\alpha) = B_1\,\mathtt{J}_0(s) + B_2\,\mathtt{N}_0(s)\,, \tag{5.12}$$

where $B_{1,2}$ are arbitrary constants (parameters), $\mathtt{J}_0(s)$ and $\mathtt{N}_0(s)$ are respectively the Bessel and the Neumann functions of the first kind, both of order zero

$$\mathtt{J}_0(s) = \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{(\mu!)^2} \left(\frac{s}{2}\right)^{2\mu} = \sum_{\mu=0}^{\infty} A_{0,\,\mu} \left(\frac{s}{2}\right)^{2\mu}, \tag{5.13}$$

$$\mathtt{N}_0(s) = \frac{2}{\pi}\left(C + \mathtt{ln}\,\frac{s}{2}\right)\mathtt{J}_0(s) - \frac{1}{\pi}\sum_{\mu=0}^{-1} \frac{(-\mu-1)!}{\mu!}\left(\frac{s}{2}\right)^{2\mu} - \frac{2}{\pi}\sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}\,\Phi(\mu)}{(\mu!)^2}\left(\frac{s}{2}\right)^{2\mu},$$

where $C = 0.5772\ldots$ is the Euler constant and $\Phi(\mu) = \sum_{\rho=1}^{\mu} \frac{1}{\rho}$, $\Phi(0) = 0$.

ii) A 3-parameter family of solutions of the ordinary differential equation (5.9), when $\varrho \notin \mathbb{Z}$ (i. e. $\alpha \in (0,1) \bigcup (1,2)$), is known [9, 19] to be

$$W(s;\alpha) = C_1 \mathtt{I}_{-\varrho}(s) + C_2\, \mathtt{I}_{+\varrho}(s)\,, \tag{5.14}$$

where $C_{1,2}$ are arbitrary constants (parameters), and $\mathtt{I}_{\mp\varrho}(s)$ are the modified Bessel functions of the first kind of orders $\mp\varrho$

$$\mathtt{I}_{\mp\varrho}(s) = \sum_{\mu=0}^{\infty} \frac{1}{\mu!\,\Gamma(\mu \mp \varrho + 1)} \left(\frac{s}{2}\right)^{2\mu\mp\varrho} = \left(\frac{s}{2}\right)^{\mp\varrho} \sum_{\mu=0}^{\infty} B_{\mp\varrho,\,\mu} \left(\frac{s}{2}\right)^{2\mu}, \tag{5.15}$$

whereas in case $\varrho \in \mathbb{Z}$ (i. e. $\alpha = 1$) a 2-parameter family of solutions of the ordinary differential equation (5.9) is known [9, 19] to be

$$W(s;\alpha) = D_1 \mathtt{I}_0(s) + D_2\, \mathtt{K}_0(s)\,, \tag{5.16}$$

where $D_{1,2}$ are arbitrary constants (parameters), $\mathtt{I}_0(s)$ and $\mathtt{K}_0(s)$ are respectively the modified Bessel function of the first kind and the modified Bessel function of the second kind, both of order zero

$$\mathtt{I}_0(s) = \sum_{\mu=0}^{\infty} \frac{1}{(\mu!)^2} \left(\frac{s}{2}\right)^{2\mu} = \sum_{\mu=0}^{\infty} B_{0,\,\mu} \left(\frac{s}{2}\right)^{2\mu}, \tag{5.17}$$

i) Tracing backwards all the transformations of the independent and the dependent variables we find from (5.10), (5.12) the following family of solutions of the second equation of the system (5.3)

$$X(x;\alpha) = \begin{cases} x^{\frac{1-\alpha}{2}} \left[A_1\, \mathtt{J}_{-\varrho}\left(\lambda\Omega\, x^{\frac{\theta}{2}}\right) + A_2\, \mathtt{J}_{+\varrho}\left(\lambda\Omega\, x^{\frac{\theta}{2}}\right)\right], & \alpha \in (0,1) \bigcup (1,2)\,, \\ B_1\, \mathtt{J}_0 \left(\lambda\Omega\, x^{\frac{\theta}{2}}\right) + B_2\, \mathtt{N}_0 \left(\lambda\Omega\, x^{\frac{\theta}{2}}\right), & \alpha = 1\,. \end{cases}$$

Substituting the respective power series instead of $\mathtt{J}_{\mp\varrho}$, $\mathtt{J}_0$, $\mathtt{N}_0$ and retaining only the bounded terms we obtain

$$X(x;\alpha) = \begin{cases} \underbrace{\overbrace{A_1\, \Lambda^{-\frac{1-\alpha}{\theta}} \sum_{\mu=0}^{\infty} A_{-\varrho,\mu}\, \Lambda^{2\mu}\, x^{\mu\theta}}^{\alpha\in(0,2)} + x^{1-\alpha}\, A_2\, \Lambda^{+\frac{1-\alpha}{\theta}} \sum_{\mu=0}^{\infty} A_{+\varrho,\mu}\, \Lambda^{2\mu}\, x^{\mu\theta}}_{\alpha\in(0,1)}, \\ B_1 \sum_{\mu=0}^{\infty} A_{0,\mu}\, \Lambda^{2\mu}\, x^{\mu\theta}, \qquad \alpha = 1\,, \end{cases}$$

where $2\Lambda = \lambda\,\Omega$. Then, performing the even extension of the above function $X(x;\alpha)$ to $x<0$, leading to the continuous and continuously differentiable flux, similarly to Section 4, we obtain the following required composite solution of the second equation of the system (5.3)

$$X(x;\alpha) = \sum_{\mu=0}^{\infty} A_{-\varrho,\mu}\,\Lambda^{2\mu}\,|x|^{\mu\theta}, \tag{5.18}$$

where the coefficients $A_{-\varrho,\mu}$ are taken from the series (5.11), if $\alpha\in(0,1)$ $(\varrho>0)$ and $\alpha\in(1,2)$ $(\varrho<0)$, and are taken from the series (5.13), if $\alpha=1$ $(\varrho=0)$.

ii) Tracing backwards all the transformations of the independent and the dependent variables we find from (5.14), (5.16) the following family of solutions of the second equation of the system (5.3)

$$X(x;\alpha) = \begin{cases} x^{\frac{1-\alpha}{2}}\left[C_1\,\mathtt{I}_{+\varrho}\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right) + C_2\,\mathtt{I}_{-\varrho}\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right)\right], & \alpha\in(0,1)\bigcup(1,2)\,,\\[2ex] D_1\,\mathtt{I}_0\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right) + D_2\,\mathtt{K}_0\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right), & \alpha=1\,. \end{cases}$$

Substituting the respective power series instead of $\mathtt{I}_{\mp\varrho}$, $\mathtt{I}_0$, $\mathtt{K}_0$ and retaining only the bounded terms we obtain

$$X(x;\alpha) = \begin{cases} \underbrace{\overbrace{C_1\,\Lambda^{-\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty} B_{-\varrho,\mu}\,\Lambda^{2\mu}\,x^{\mu\theta}}^{\alpha\in(0,2)} + x^{1-\alpha}\,C_2\,\Lambda^{+\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty} B_{+\varrho,\mu}\,\Lambda^{2\mu}\,x^{\mu\theta}}_{\alpha\in(0,1)}, \\[2ex] D_1\sum_{\mu=0}^{\infty} B_{0,\mu}\,\Lambda^{2\mu}\,x^{\mu\theta}, \qquad \alpha=1\,, \end{cases}$$

Performing the even extension of the above function $X(x;\alpha)$ to $x<0$ and doing as in case of the solution (5.10), (5.12), we obtain the following required composite solution of the second equation of the system (5.3)

$$X(x;\alpha) = \sum_{\mu=0}^{\infty} B_{-\varrho,\mu}\,\Lambda^{2\mu}|x|^{\mu\theta}, \tag{5.19}$$

where the coefficients $B_{-\varrho,\mu}$ are taken from the series (5.15), if $\alpha\in(0,1)$ $(\varrho>0)$ and $\alpha\in(1,2)$ $(\varrho<0)$, and are taken from the series (5.17), if $\alpha=1$ $(\varrho=0)$.

II) Assuming that $\lambda=0$, we obtain directly from (5.7) a 1-parameter $(\varrho(\alpha))$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} + \frac{1}{\bar{x}}\,\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} - \frac{\varrho^2}{\bar{x}^2}\,\bar{w} = 0\,. \tag{5.20}$$

A 3-parameter family of solutions of equation (5.20) reads

$$\bar{w}\,(\bar{x};\alpha) = E_1\,\bar{x}^{-\varrho} + E_2\,\bar{x}^{+\varrho},$$

where $E_{1,2}$ are arbitrary constants. Applying the transformation leading from $(\bar{x},\bar{w})$ to (x,w) backwards we obtain

$$w\,(x;\alpha) = E_1\Big(\Omega\,x^{\frac{\theta}{2}}\Big)^{-\varrho} + E_2\Big(\Omega\,x^{\frac{\theta}{2}}\Big)^{+\varrho} = \ldots = \bar{E}_1\,x^{-\frac{1-\alpha}{2}} + \bar{E}_2\,x^{+\frac{1-\alpha}{2}}.$$

Eventually, applying the transformation (5.4) yields to the solution of the second equation of the system (5.3)

$$X(x;\alpha) = x^{+\frac{1-\alpha}{2}}\left(\bar{E}_1\,x^{-\frac{1-\alpha}{2}} + \bar{E}_2\,x^{+\frac{1-\alpha}{2}}\right) = \bar{E}_1 + \bar{E}_2\,x^{1-\alpha}.$$

The evident extensions of the above function to $x<0$

$$X(x;\alpha) = \begin{cases} \bar{E}_1 + \bar{E}_2\,|x|^{1-\alpha}, \\ \bar{E}_1 + \bar{E}_2\;\;x^{\;1-\alpha}, \end{cases} \tag{5.21}$$

lead, as it is known from Section 4, to the discontinuous flux f, therefore no function (5.21) must be accounted for in the representation (5.2).

Now we turn out to the first ordinary differential equation of the system (5.3); its 2-parameter $((\lambda,\mp),\ \lambda>0)$ family of solutions is known to be

$$O(t;\alpha) = \begin{cases} F_1\,\mathtt{exp}\,(-\lambda t) + F_2\,\mathtt{exp}\,(+\lambda t)\,, \\ G_1\,\mathtt{cos}\,(+\lambda t) + G_2\,\mathtt{sin}\,(+\lambda t)\,, \end{cases} \tag{5.22}$$

where $F_{1,2}$, $G_{1,2}$ are arbitrary constants.

This completes finding solutions of the original wave equation using separation of variables, but we especially note that the spatial parts $X(x;\alpha)$ (5.18), (5.19) of the solutions (5.2) obtained in this Section includes the same terms $|x|^{\mu\theta}$, $\mu\in\mathbb{Z}_+$, as those present in the required series solution (4.3).

6. Reformulation of the original problem

The power law (1.2) in the coefficient function $a(x;\alpha)$ produces the degeneracy of the wave equation (1.1). We implement stretching of the spatial independent variable x leading to 'inflation' of the degeneracy. For this we introduce a transformation of the independent variables $(t,x)\to(\tau,\xi)$ using the following system of the first order differential equations

$$\begin{cases} \dfrac{\mathrm{d}\tau}{\mathrm{d}t} = 1\,, \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}x} = \dfrac{1}{\sqrt{a(x;\alpha)}}\,, \end{cases} \tag{6.1}$$

supplemented with the evident boundary conditions: $\tau = 0$ when $t = 0$ and $\xi = 0$ when $x = 0$. The only non-trivial solution of the Cauchy problem gives the desired 1-parameter transformation of the independent variables

$$\begin{cases} \tau = \vartheta(t)\,, \\ \xi = \psi(x;\alpha)\,, \end{cases} \tag{6.2}$$

where $\vartheta(t) = t$,

$$\psi(x;\alpha) = \mathtt{sign}\,(x) \begin{cases} \Omega\,|x|^{\frac{\theta}{2}}, & 0 \leqslant |x| \leqslant c\,, \\ \xi_c + (|x| - c)\,, & c < |x| \leqslant 1\,, \end{cases} \tag{6.3}$$

and $\xi_c = \psi(c;\alpha)$. The function (6.3) is monotonic differentiable (with the only exception for the point $x{=}0$, Fig. 6.1), hence the transformation (6.4) is uniquely invertible

$$\begin{cases} t = \bar{\vartheta}(\tau)\,, \\ x = \phi(\xi;\alpha)\,, \end{cases} \tag{6.4}$$

where $\bar{\vartheta}(\tau) = \vartheta^{-1}(\tau) = \tau$,

$$\phi(\xi;\alpha) = \mathtt{sign}\,(\xi) \begin{cases} \left(\Omega^{-1}|\xi|\right)^{\frac{2}{\theta}}, & 0 \leqslant |\xi| \leqslant \xi_c\,, \\ c + (|\xi| - \xi_c)\,, & \xi_c < |\xi| \leqslant \xi_l\,, \end{cases} \tag{6.5}$$

$\xi_l = \psi(1;\alpha)$, and Ω is given by (5.6).

The transformation formulas (6.2), (6.4) are equivalent to the following operator identities

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \tau^2}\,, \quad \frac{\partial}{\partial x} = \left(\frac{1}{\sqrt{a}}\right)_{x\to\xi} \frac{\partial}{\partial \xi}\,, \quad \frac{\partial^2}{\partial x^2} = \left(\frac{1}{a}\right)_{x\to\xi} \frac{\partial^2}{\partial \xi^2} - \frac{1}{2}\left(\frac{a'}{a\sqrt{a}}\right)_{x\to\xi} \frac{\partial}{\partial \xi}\,,$$

yielding to the transformed wave equation

$$\frac{\partial^2 \upsilon(\tau,\xi;\alpha)}{\partial \tau^2} - \frac{\partial^2 \upsilon(\tau,\xi;\alpha)}{\partial \xi^2} = g(\xi;\alpha)\,\frac{\partial \upsilon(\tau,\xi;\alpha)}{\partial \xi}\,, \tag{6.6}$$

where $\upsilon(\tau,\xi;\alpha) := u\left(\vartheta(\tau), \phi(\xi;\alpha);\alpha\right)$, $g(\xi;\alpha) := \left(\sqrt{a(x;\alpha)}\,\right)'_{x\to\xi}$.

There are two ways to rewrite the transformed wave equation purely in the variables (τ,ξ). The first one is straightforward and implies two steps. The first step needs obtaining an explicit dependence of the coefficient function $g(\xi;\alpha)$ in (6.6) on the variable x

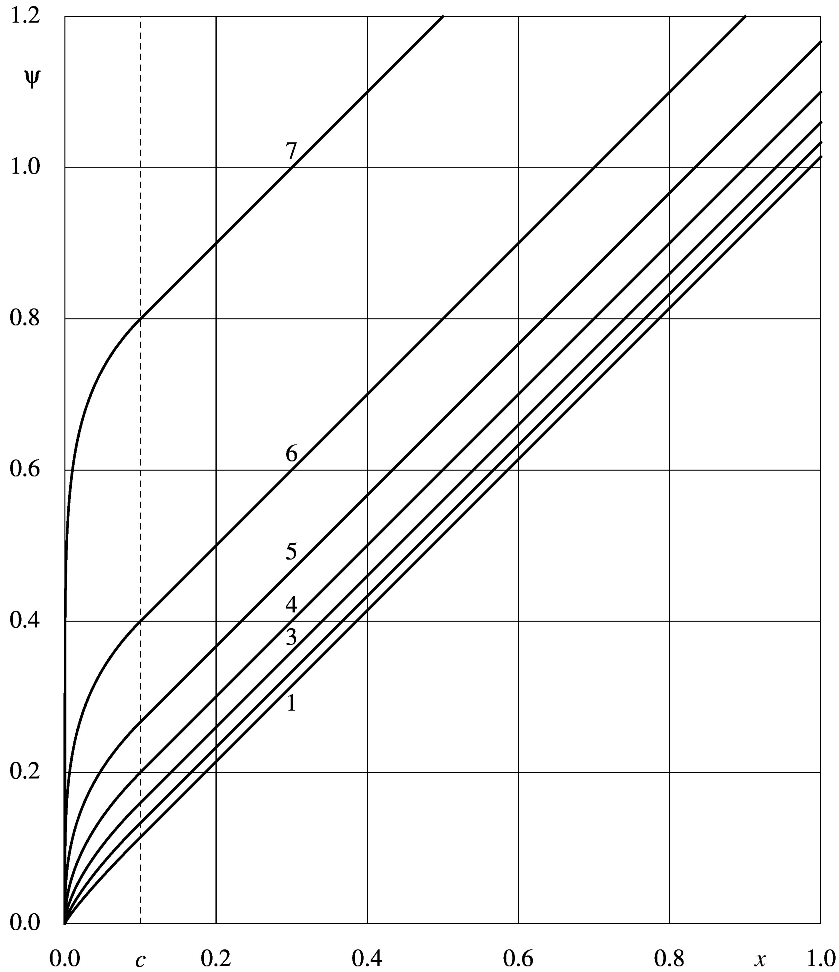

Fig. 6.1. The function $\psi(x;\alpha)$ (6.3) stretches the variable x near the degeneracy segment and 'inflates' the degeneracy of the original wave equation (1.1), (5.1): bold solid curves $1-7$ are drawn for $\alpha=0.25\,(0.25)\,1.75$ respectively ($c=0.1,\ x\geqslant 0$); the thin dashed line shows the right boundary $x=c$ of the segment $[-c,+c]$ in which the power law $a(x;\alpha)=a_*|x|^{\alpha}$ (1.2) of the degeneracy holds

$$\Big(\sqrt{a(x;\alpha)}\Big)' \equiv \frac{1}{2}\,\frac{a'(x;\alpha)}{\sqrt{a(x;\alpha)}} = \mathtt{sign}\,(x) \begin{cases} \dfrac{1}{2}\,\dfrac{a_*\alpha\,|x|^{\alpha-1}}{\sqrt{a_*}\,|x|^{\frac{\alpha}{2}}} = \dfrac{\alpha}{\theta}\,\Big(\Omega\,|x|^{\frac{\theta}{2}}\Big)^{-1}, & 0<|x|\leqslant c\,,\\[2ex] 0\,, & c<|x|\leqslant 1\,,\end{cases}$$

whereas the second step needs replacing the variable x with formula (6.5) and yields to the desired composite expression

$$g(\xi;\alpha) \equiv \left(\sqrt{a(x;\alpha)}\right)'_{x\to\xi} = \begin{cases} \dfrac{\alpha}{\theta}\,\dfrac{1}{\xi}\,, & 0\ < |\xi| \leqslant \xi_c\,, \\ 0\,, & \xi_c < |\xi| \leqslant \xi_l\,, \end{cases} \tag{6.7}$$

where the hyperbolic and identically equaled to zero branches are not continuously matched as shown in Fig. 6.2.

The second way implies two steps to be done as well. The first step needs implementing the transformation of the variable x into the variable ξ

$$\frac{1}{2}\left(\frac{a'(x;\alpha)}{\sqrt{a(x;\alpha)}}\right)_{x\to\xi} = \frac{1}{2}\,\frac{1}{\sqrt{a(\phi(\xi;\alpha);\alpha)}}\,\frac{\mathrm{d}a(\phi(\xi;\alpha);\alpha)}{\mathrm{d}\xi}\,\underbrace{\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)_{x\to\xi}}_{\frac{1}{\sqrt{a(\phi(\xi;\alpha);\alpha)}}} = \frac{1}{2}\,\frac{b'(\xi;\alpha)}{b(\xi;\alpha)}\,,$$

where $b(\xi;\alpha)$ is the coefficient function of the original wave equation expressed purely through the variable ξ

$$b(\xi;\alpha) \equiv a(\phi(\xi;\alpha);\alpha) = \begin{cases} a_*\left(\Omega^{-1}|\xi|\right)^{\frac{2\alpha}{\theta}}\,, & 0\ < |\xi| \leqslant \xi_c\,, \\ 1\,, & \xi_c < |\xi| \leqslant \xi_l\,, \end{cases} \tag{6.8}$$

whereas the second step involves 'deciphering' the coefficient function of the only first derivative of the transformed wave equation in terms of the variable ξ and yields exactly to the previously obtained composite expression

$$\frac{1}{2}\,\frac{b'(\xi;\alpha)}{b(\xi;\alpha)} = g(\xi;\alpha) \overset{(6.7)}{\equiv} \left(\sqrt{a(x;\alpha)}\right)'_{x\to\xi}\,.$$

Finally, we obtain the following transformed formulation of the original initial boundary value problem (1.4)

$$\begin{cases} \dfrac{\partial^2 \upsilon(\xi;\alpha)}{\partial\tau^2} = \dfrac{\partial^2 \upsilon(\xi;\alpha)}{\partial\xi^2} + g(\xi;\alpha)\dfrac{\partial \upsilon(\xi;\alpha)}{\partial\xi}\,, & (\tau,\xi)\in(0,T]\times(-\xi_l,+\xi_l), \\ \upsilon(\tau,-\xi_l;\alpha) = 0\,, & \tau\in[0,T]\,, \\ \upsilon(\tau,+\xi_l;\alpha) = h(\tau)\,, & \tau\in[0,T]\,, \\ \upsilon(0,\xi;\alpha) = 0\,, & \xi\in[-\xi_l,+\xi_l]\,, \\ \dfrac{\partial\upsilon(0,\xi;\alpha)}{\partial\tau} = 0\,, & \xi\in[-\xi_l,+\xi_l)\,, \end{cases} \tag{6.9}$$

referred to as the transformed initial boundary value problem.

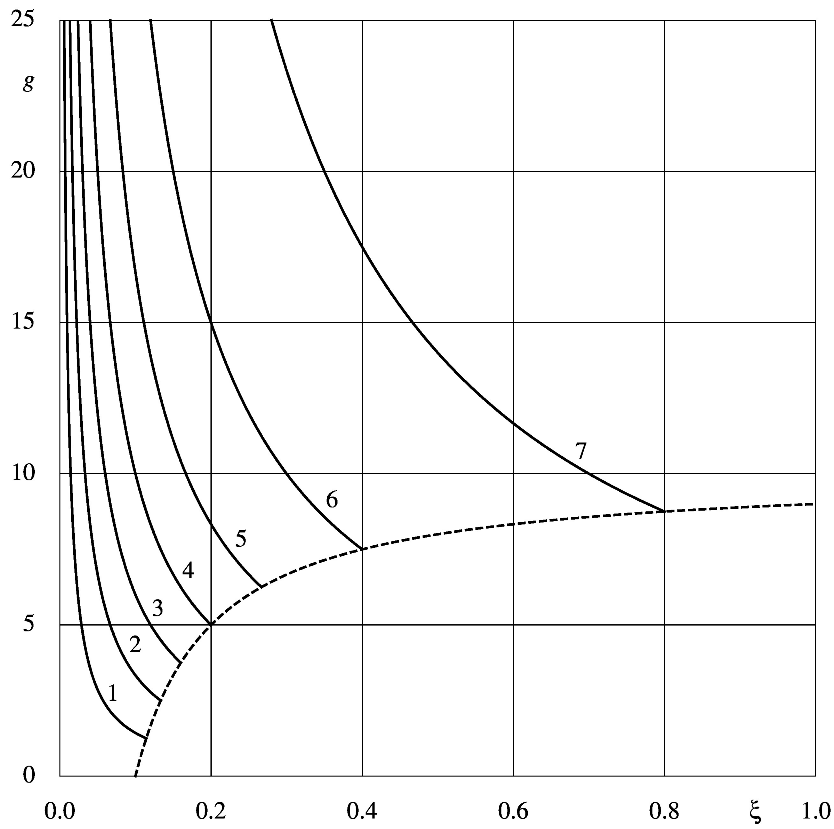

Fig. 6.2. The piece-wise continuous and differentiable coefficient function $g(\xi;\alpha)$ (6.7) of the transformed wave equation: only hyperbolic branches are drawn with bold solid curves $1-7$ for $\alpha = 0.25\,(0.25)\,1.75$ respectively; the bold dashed line joins the right ends of the hyperbolic branches

7. Series solutions of the transformed wave equation

There are two ways, to find one-sided series solution of the transformed wave equation (6.6). The first one uses the transformation of the independent variables (6.2), (6.4), applied to the pairs of the one-sided series solutions (3.3), (3.5), and (3.9) of the original wave equation (1.1), whereas the second one uses the procedures of Sections 3, but as applied to the transformed wave equation.

I) Implementing the first way is straightforward to replace the independent variables $(t,x)\to(\tau,\xi)$, where $t=\bar{\vartheta}(\tau)\equiv\tau$, whereas $|x|, x$ are respectively replaced as

$$\begin{cases} |x| = \left(\Omega^{-1}|\xi|\right)^{\frac{2}{\theta}}, & \alpha\in(0,2)\,, \\ x = \left(\Omega^{-1}\,\xi\,\right)^{\frac{2}{\theta}}, & \alpha\in(0,2)_o\,. \end{cases} \tag{7.1}$$

Initially, consider case *A*) of Section 3.

i) Let both pairs of the one-sided series solutions (3.3) be given, then using the independent variables substitution yields to the only pair

$$\upsilon^{\mp}(\tau,\xi;\alpha) \equiv u^{\mp}\big(\bar\vartheta(\tau),\phi(\xi;\alpha);\alpha\big) = \sum_{\mu=0}^{\infty} V^{\mp}_{\alpha,\mu}(\tau)\,\xi^{2\mu}, \tag{7.2}$$

where

$$V^{\mp}_{\alpha,\mu}(\tau) = \left(\frac{\theta}{2}\right)^{2\mu} a_*^{\mu}\, U^{\mp}_{\alpha,\mu}\left(\bar\vartheta(\tau)\right), \qquad \mu\in\mathbb{Z}_+\,, \tag{7.3}$$

and the following pair of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 V^{\mp}_{\alpha,\mu-1}(\tau)}{\mathrm{d}\tau^2} = 4\mu\frac{(\mu-1)\,\theta+1}{\theta}\, V^{\mp}_{\alpha,\mu}(\tau)\,, \qquad \mu\in\mathbb{N}\,. \tag{7.4}$$

ii) Let both pairs of the one-sided binomial solutions (3.5) be given, then using the independent variables substitution yields to the following two pairs

$$\upsilon^{\mp}(\tau,\xi;\alpha) = \begin{cases} V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,|\xi|^{(1-\alpha)\frac{2}{\theta}}, & \alpha\in(0,+2)\,,\\[1ex] V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\ \xi^{\,(1-\alpha)\frac{2}{\theta}}, & \alpha\in(0,+2)_o, \end{cases} \tag{7.5}$$

where functions $V^{\mp}_{\alpha,0}(\tau)$ and $V^{\mp}_{\alpha,1}(\tau)$ are linear and

$$V^{\mp}_{\alpha,0}(\tau) = U^{\mp}_{\alpha,0}\left(\bar\vartheta(\tau)\right), \qquad V^{\mp}_{\alpha,1}(\tau) = \left[\left(\frac{\theta}{2}\right)^2 a_*\right]^{\frac{1-\alpha}{\theta}} U^{\mp}_{\alpha,1}\left(\bar\vartheta(\tau)\right). \tag{7.6}$$

iii) Let both pairs of the one-sided series solutions (3.6) be given, then using the independent variables substitution yields to the following two pairs

$$\upsilon^{\mp}(\tau,\xi;\alpha) = \begin{cases} V^{\mp}_{\alpha,0}(\tau) + |\xi|^{(1-\alpha)\frac{2}{\theta}} \displaystyle\sum_{\mu=1}^{\infty} V^{\mp}_{\alpha,\mu}(\tau)\,\xi^{2(\mu-1)}, & \alpha\in(0,+2)\,,\\[1ex] V^{\mp}_{\alpha,0}(\tau) +\ \xi^{\,(1-\alpha)\frac{2}{\theta}} \displaystyle\sum_{\mu=1}^{\infty} V^{\mp}_{\alpha,\mu}(\tau)\,\xi^{2(\mu-1)}, & \alpha\in(0,+2)_o, \end{cases} \tag{7.7}$$

where

$$V^{\mp}_{\alpha,\mu}(\tau) = \left[\left(\frac{\theta}{2}\right)^2 a_*\right]^{\frac{\theta\mu-1}{\theta}} U^{\mp}_{\alpha,\mu}\big(\bar\vartheta(\tau)\big), \qquad \mu\in\mathbb{N}\,, \tag{7.8}$$

and the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 V^{\mp}_{\alpha,\mu}(\tau)}{\mathrm{d}\tau^2} = 4\mu\,\frac{\mu\theta-1}{\theta}\, V^{\mp}_{\alpha,\mu+1}(\tau)\,, \qquad \mu\in\mathbb{N}\,. \tag{7.9}$$

Considering three items of case B) of Section 3 is performed exactly in the same way, therefore we omit case B) and complete implementing the first way.

II) Now we turn out to implementing the second way.

A) Let trial one-sided series solutions of the transformed wave equation (6.6) be of the form

$$\upsilon^{\mp}(\tau,\xi;\alpha) = V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,1}} + V^{\mp}_{\alpha,2}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,2}} + \dots$$

where all undetermined functions and quantities have the same meaning, as in Section 3, then

$$\begin{aligned}
\frac{\partial^2\upsilon^{\mp}}{\partial\tau^2} &= V''_{\alpha,0} + V''_{\alpha,1}\,|\xi|^{\sigma_1} + V''_{\alpha,2}\,|\xi|^{\sigma_2} + \dots,\\
\frac{1}{\xi}\,\frac{\partial\upsilon^{\mp}}{\partial\xi} &= \mp\Big(\quad \sigma_1\,V_{\alpha,1}\,|\xi|^{\sigma_1-2} + \quad \sigma_2\,V_{\alpha,2}\,|\xi|^{\sigma_2-2} + \dots\Big),\\
\frac{\partial^2\upsilon^{\mp}}{\partial\xi^2} &= \quad \omega_1\,\sigma_1\,V_{\alpha,1}\,|\xi|^{\sigma_1-2} + \omega_2\,\sigma_2\,V_{\alpha,2}\,|\xi|^{\sigma_2-2} + \dots \quad ,
\end{aligned}$$

and substituting the above series instead of the respective terms of the transformed wave equation yields to the following pair of the one-sided series identities

$$\underbrace{V''_{\alpha,0}}_{1}+\underbrace{V''_{\alpha,1}\,|\xi|^{\sigma_1}}_{2}+V''_{\alpha,2}\,|\xi|^{\sigma_2}+\dots = \underbrace{\sigma_1\,o_1\,V_{\alpha,1}\,|\xi|^{\sigma_1-2}}_{1}+\underbrace{\sigma_2\,o_2\,V_{\alpha,2}\,|\xi|^{\sigma_2-2}}_{2}+\dots, \quad (7.10)$$

where $\omega_1=\sigma_1-1$, $\omega_2=\sigma_2-1$, etc., $o_1=\omega_1+\theta^{-1}\alpha$, $o_2=\sigma_2+\theta^{-1}\alpha$, etc.

i) Assuming that in (7.10) the terms with the same numbers have the same powers in the variable $|\xi|$, we obtain the only pair of one-sided series solutions (7.2) and the pair of the one-sided recurrence relations (7.4) again.

ii) Assuming that in (7.10) the terms must vanish separately, we obtain two pairs of the binomial solutions (7.5) again.

iii) Assuming that in (7.10) both terms 1 vanish separately, we obtain two pairs of the one-sided series solutions (7.7) and the pair of the one-sided recurrence relations (3.7) again.

B) Let trial one-sided series solutions of the transformed wave equation (6.6) be of the form

$$\upsilon^{\mp}(\tau,\xi;\alpha) = V^{\mp}_{\alpha}(\tau) + |\xi|^{\omega^{\mp}_{\alpha}}\left(V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,1}} + V^{\mp}_{\alpha,2}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,2}} + \dots\right),$$

then

$$
\begin{aligned}
\frac{\partial^2 \upsilon^{\mp}}{\partial \tau^2} &= V''_{\alpha} + |\xi|^{\omega}\Big(V''_{\alpha,0} + V''_{\alpha,1}\,|\xi|^{\sigma_1} + V''_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots\Big),\\
\frac{1}{\xi}\,\frac{\partial \upsilon^{\mp}}{\partial \xi} &= \mp|\xi|^{\omega-2}\Big(\omega\, V_{\alpha,0} + \omega_1\, V_{\alpha,1}\,|\xi|^{\sigma_1} + \omega_2\, V_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots\Big),\\
\frac{\partial^2 \upsilon^{\mp}}{\partial \xi^2} &= \quad |\xi|^{\omega-2}\Big(\omega\,(\omega-1)\, V_{\alpha,0} + \omega_1\,(\omega-1+\sigma_1)\, V_{\alpha,1}\,|\xi|^{\sigma_1}\\
&\qquad\qquad + \omega_2\,(\omega-1+\sigma_2)\, V_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots\Big),
\end{aligned}
$$

and substituting the above series instead of the respective terms of the transformed wave equation we obtain the following pair of the one-sided series identities

$$
\begin{aligned}
&\underbrace{V''_{\alpha}}_{1} + |\xi|^{\omega}\Big(\underbrace{V''_{\alpha,0}}_{2} + \underbrace{V''_{\alpha,1}\,|\xi|^{\sigma_1}}_{3} + V''_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots\Big)\\
&= |\xi|^{\omega-2}\Big(\underbrace{\omega\, o\, V_{\alpha,0}}_{1} + \underbrace{\omega_1\,(o+\sigma_1)\, V_{\alpha,1}\,|\xi|^{\sigma_1}}_{2} + \underbrace{\omega_2\,(o+\sigma_2)\, V_{\alpha,2}\,|\xi|^{\sigma_2}}_{3} + \ldots\Big),
\end{aligned}
\tag{7.11}
$$

where $o = \omega - 1 + \theta^{-1}\alpha$, $\omega_1 = \omega + \sigma_1$, $\omega_2 = \omega + \sigma_2$, etc.

i) Let the terms in (7.11) with the same numbers be of the same powers, then we obtain: 1) the explicit expressions for the exponents: $\omega_{\alpha}^{\mp} = 2$, $\sigma_{\alpha,\mu}^{\mp} = 2\mu$, leading to the only pair of the one-sided series solutions

$$
\upsilon^{\mp}(\tau,\xi;\alpha) = V_{\alpha}^{\mp}(\tau) + V_{\alpha,0}^{\mp}(\tau)\,\xi^2 + V_{\alpha,1}^{\mp}(\tau)\,\xi^4 + \ldots;
$$

and 2) the pair of the one-sided recurrence relations

$$
\frac{\mathrm{d}^2 V_{\alpha}^{\mp}(\tau)}{\mathrm{d}\tau^2} = \frac{4}{\theta}\, V_{\alpha,0}^{\mp}(\tau)\,, \qquad \frac{\mathrm{d}^2 V_{\alpha,\mu-1}^{\mp}(\tau)}{\mathrm{d}\tau^2} = 4(\mu+1)\,\frac{\mu\theta+1}{\theta}\, V_{\alpha,\mu}^{\mp}(\tau)\,, \qquad \mu \in \mathbb{N}\,.
$$

It is evident, that we obtain nothing but the series solution (7.2), (7.4) again.

ii) Let the functions $V_{\alpha}^{\mp}(\tau)$, $V_{\alpha,0}^{\mp}(\tau)$ be linear, then $\theta\,\omega_{\alpha}^{\mp} = 2\,(1-\alpha)$, $\sigma_{\alpha,\mu}^{\mp} = 0$, $\mu \in \mathbb{N}$, and we obtain the pairs of the binomial solutions (7.5) again.

iii) Let terms 1 in (7.11) vanish separately, and the other terms with the same numbers be of the same powers, then we obtain: 1) that $V_{\alpha}^{\mp}(\tau)$ are linear functions; 2) the explicit expressions for the exponents: $\theta\,\omega_{\alpha}^{\mp} = 2\,(1-\alpha)$, $\sigma_{\alpha,\mu}^{\mp} = 2\mu$, leading to the following pairs of the one-sided series solutions

$$
\begin{aligned}
\upsilon^{\mp}(\tau,\xi;\alpha) &= V_{\alpha}^{\mp}(\tau) + |\xi|^{(1-\alpha)\frac{2}{\theta}}\Big(V_{\alpha,0}^{\mp}(\tau) + V_{\alpha,1}^{\mp}(\tau)\,\xi^2 + V_{\alpha,2}^{\mp}(\tau)\,\xi^4 + \ldots\Big)\,,\\
\upsilon^{\mp}(\tau,\xi;\alpha) &= V_{\alpha}^{\mp}(\tau) + \ \xi^{\,(1-\alpha)\frac{2}{\theta}}\Big(V_{\alpha,0}^{\mp}(\tau) + V_{\alpha,1}^{\mp}(\tau)\,\xi^2 + V_{\alpha,2}^{\mp}(\tau)\,\xi^4 + \ldots\Big)\,,
\end{aligned}
\tag{7.12}
$$

valid respectively if $\alpha \in (0,2)$ and if $\alpha \in (0,2)_o$; 3) the following pair of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 V^{\mp}_{\alpha,\mu-1}(\tau)}{\mathrm{d}\tau^2} = 4\mu \, \frac{(\mu+1)\,\theta - 1}{\theta} \, V^{\mp}_{\alpha,\mu}(\tau)\,, \qquad \mu \in \mathbb{N}\,. \tag{7.13}$$

This completes implementing the second way of obtaining one-sided series solutions for the transformed wave equation.

Both ways have been proved to give identical one-sided series solutions.

8. Series based analysis of the transformed problem

There are two ways of finding the required continuous, or two-sided, series solutions of the transformed wave equation (6.6). The first way uses the transformation of the independent variables (6.2), (6.4), applied to the only required two-sided series solution (4.3) of the original wave equation (1.1), whereas the second one uses the procedure of continuous matching of one-sided series solutions, but as applied to the transformed wave equation, followed by introducing the proper definition of the required solution and keeping in mind property Z.

I) Implementing the first way immediately gives the following two-sided series solution

$$\upsilon(\tau,\xi;\alpha) = V_{\alpha,0}(\tau) + V_{\alpha,1}(\tau)\,\xi^2 + V_{\alpha,2}(\tau)\,\xi^4 + \ldots = \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, \tag{8.1}$$

where the coefficient functions obey the following recurrence relations

$$V''_{\alpha,\mu-1}(\tau) = 4\mu \, \frac{(\mu-1)\,\theta + 1}{\theta} \, V_{\alpha,\mu}(\tau)\,, \qquad \mu \in \mathbb{N}\,. \tag{8.2}$$

II) Implementing the second way follows the three-step procedure of Section 4.

1) Implementing the continuous matching of the one-sided series solutions obtained in Section 7 gives the following two-sided bounded series solutions

$$\begin{aligned}
\upsilon_1(\tau,\xi;\alpha) &= \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, && \alpha \in (0,2)\,,\\
\upsilon_3(\tau,\xi;\alpha) &= V_{\alpha,0}(\tau) + V_{\alpha,1}(\tau)\,|\xi|^{(1-\alpha)\frac{2}{\theta}}, && \alpha \in (0,1)\,,\\
\upsilon_4(\tau,\xi;\alpha) &= V_{\alpha,0}(\tau) + V_{\alpha,1}(\tau)\ \xi^{\,(1-\alpha)\frac{2}{\theta}}, && \alpha \in (0,1)_o\,,\\
\upsilon_5(\tau,\xi;\alpha) &= V_{\alpha}(\tau) + |\xi|^{(1-\alpha)\frac{2}{\theta}} \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, && \alpha \in (0,1)\,,\\
\upsilon_6(\tau,\xi;\alpha) &= V_{\alpha}(\tau) + \ \xi^{\,(1-\alpha)\frac{2}{\theta}} \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, && \alpha \in (0,1)_o\,.
\end{aligned} \tag{8.3}$$

2) Implementing continuity of the flux needs, first of all, the definition of the flux, therefore we rewrite the transformed wave equation in a flux form

$$\frac{\partial \pi}{\partial \tau} + \frac{\partial \varphi}{\partial \xi} = \rho \,, \tag{8.4}$$

usually referred to as the balance law, where $\pi(\tau, \xi; \alpha) := \dfrac{\partial \upsilon}{\partial \tau}$,

$$\begin{cases} \varphi(\tau, \xi; \alpha) := -\dfrac{\partial \upsilon}{\partial \xi} - \dfrac{\alpha}{\theta} \dfrac{\upsilon}{\xi}\,, & \rho(\tau, \xi; \alpha) := \dfrac{\alpha}{\theta} \dfrac{\upsilon}{\xi^2}\,, \qquad 0 \leqslant |\xi| \leqslant \xi_c\,, \\ \varphi(\tau, \xi; \alpha) := -\dfrac{\partial \upsilon}{\partial \xi}\,, & \rho(\tau, \xi; \alpha) := 0\,, \qquad \xi_c < |\xi| \leqslant 1\,, \end{cases} \tag{8.5}$$

φ being the desired flux and ρ being the right side, or the source term.

Calculation of the flux and the right side of the balance law (8.4) for the series solutions 1, 3 − 6 shows that both are unbounded on the degeneracy segment, due to the leading terms $V_{\alpha,0}(\tau)$ or $V_\alpha(\tau)$, therefore we rewrite the balance law (8.4) by introducing regularization

$$\frac{\partial \pi}{\partial \tau} + \frac{\partial \mathring{\varphi}}{\partial \xi} = \mathring{\rho} \,, \tag{8.6}$$

where

$$\mathring{\varphi}(\tau, \xi; \alpha) := -\frac{\partial \upsilon}{\partial \xi} - \frac{\alpha}{\theta} \frac{\mathring{\upsilon}}{\xi}\,, \qquad \mathring{\rho}(\tau, \xi; \alpha) := \frac{\alpha}{\theta} \frac{\mathring{\upsilon}}{\xi^2}\,, \qquad 0 \leqslant |\xi| \leqslant \xi_c\,, \tag{8.7}$$

and $\mathring{\upsilon}(\tau, \xi; \alpha) := \upsilon(\tau, \xi; \alpha) - \upsilon(\tau, 0; \alpha)$.

The above regularization yields to the following fluxes calculated on both sides of the degeneracy segment

$$\begin{aligned} -\mathring{\varphi}_1^{\mp}(\tau, \xi; \alpha) &= \xi \sum_{\mu=1}^{\infty} \Big(\frac{\alpha}{\theta} + 2\mu\Big) V_{\alpha,\mu}(\tau)\, \xi^{2\mu-2}, \\ -\mathring{\varphi}_3^{\mp}(\tau, \xi; \alpha) &= \mp V_{\alpha,1}(\tau)\, |\xi|^{-\frac{\alpha}{\theta}}, \\ -\mathring{\varphi}_4^{\mp}(\tau, \xi; \alpha) &= \quad V_{\alpha,1}(\tau)\ \ \xi^{\,-\frac{\alpha}{\theta}}, \\ -\mathring{\varphi}_5^{\mp}(\tau, \xi; \alpha) &= \mp |\xi|^{-\frac{\alpha}{\theta}} \sum_{\mu=0}^{\infty} (1 + 2\mu)\, V_{\alpha,\mu}(\tau)\, \xi^{2\mu}, \\ -\mathring{\varphi}_6^{\mp}(\tau, \xi; \alpha) &= \quad \xi^{\,-\frac{\alpha}{\theta}} \sum_{\mu=0}^{\infty} (1 + 2\mu)\, V_{\alpha,\mu}(\tau)\, \xi^{2\mu}, \end{aligned} \tag{8.8}$$

and we immediately find the flux $\mathring{\varphi}_1(\tau, \xi; \alpha)$ for the series solution $\mathring{\upsilon}_1(\tau, \xi; \alpha)$ to be the only continuous and even continuously differentiable on the degeneracy segment, whereas all other fluxes turn out to be unbounded.

We gather below the series solution $\upsilon_1(\tau,\xi;\alpha)$ and the derived source term and the flux

$$\begin{aligned} \upsilon(\tau,\xi;\alpha) &= \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, \\ \mathring{\rho}(\tau,\xi;\alpha) &= \sum_{\mu=1}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu-2}, \\ -\mathring{\varphi}(\tau,\xi;\alpha) &= \sum_{\mu=1}^{\infty} \left(\frac{\alpha}{\theta}+2\mu\right) V_{\alpha,\mu}(\tau)\,\xi^{2\mu-1}, \end{aligned} \tag{8.9}$$

to introduce the following

Definition 8.1. A function $\upsilon(\tau,\xi;\alpha)$ is called the solution with the continuously differentiable flux (or shortly, the required solution) to the initial boundary value problem (6.9) if inside the space-time rectangle $[0,T]\times[-\xi_l,+\xi_l]$ it: 1) is continuous and twice continuously differentiable in the variables τ and ξ; 2) satisfies the degenerate wave equation; and 3) satisfies the initial and boundary conditions of the problem.

Hence, the only required series solution of the transformed wave equation (introduced by Definition 8.1, in which the last item is removed) out of those obtained in Section 7 and presented in (8.3) is that given by (8.9). Note, that continuity of: 1) the right side of the transformed wave equation and 2) the flux $\mathring{\varphi}$ and the source term $\mathring{\rho}$ of the regularized balance law (8.6) are evident implications of Definition 8.1.

Proposition 8.1. *Possessing property Z is not necessary for the required solution to the 1-parameter initial boundary value problem* (6.9) *if* $\alpha\in(0,2)$.

Proof. Essential part of the proof is a repetition of that for Proposition 4.1 at p. 18. □

9. Separation of variables applied to the transformed equation

Assume that the trial solution of the transformed wave equation is of the form

$$\upsilon(\tau,\xi;\alpha) = \Theta\,(\tau;\alpha)\;\Xi\,(\xi;\alpha)\,, \tag{9.1}$$

where functions $\Theta\,(\tau;\alpha)$ and $\Xi\,(\xi;\alpha)$ are to be determined. Then, substituting the ansatz (9.1) into the transformed wave equation (6.6), (6.7) $(0<|\xi|\leqslant\xi_c)$

$$\frac{\partial^2\upsilon}{\partial\tau^2} - \frac{\partial^2\upsilon}{\partial\xi^2} = \frac{\alpha}{\theta}\,\frac{1}{\xi}\,\frac{\partial\upsilon}{\partial\xi}\,,$$

we find the above equation rewritten as

$$\frac{\Theta''(\tau;\alpha)}{\Theta\,(\tau;\alpha)} = \frac{1}{\Xi\,(\tau;\alpha)}\left(\Xi''(\tau;\alpha) + \frac{\alpha}{\theta}\,\frac{1}{\xi}\,\Xi'(\tau;\alpha)\right) = \pm\lambda^2 = \mathtt{const}\,,$$

where $\pm\lambda^2$ is the unknown parameter of separation of the independent variables ($\lambda\geqslant 0$), whereas the functions $\Theta\,(\tau;\alpha)$, $\Xi\,(\xi;\alpha)$ satisfy the following system of two ordinary differential equations of the second order

$$\begin{cases} \Theta''(\tau;\alpha) \mp \lambda^2\,\Theta\,(\tau;\alpha) = 0\,, \\ \Xi''(\xi;\alpha) + \dfrac{\alpha}{\theta}\,\dfrac{1}{\xi}\,\Xi'(\xi;\alpha) \mp \lambda^2\,\Xi\,(\xi;\alpha) = 0\,. \end{cases} \tag{9.2}$$

We start our solving the system (9.2) from the second equation: 1) assume that $\xi>0$; 2) introduce a new dependent variable $w(\xi;\alpha)$ using a power substitution (the dependent variable transformation)

$$\Xi\,(\xi;\alpha) = \xi^{\beta} w(\xi;\alpha)\,, \tag{9.3}$$

where the exponent β is to be determined by imposing a proper constraint; and then 3) determine uniquely the ordinary differential equation the function $w\,(\xi;\alpha)$ satisfies. For this, we apply the following three-step procedure.

First, differentiating the substitution (9.3) twice yields to the relations between the first and the second derivatives of the functions $\Xi\,(\xi;\alpha)$ and $w(\xi;\alpha)$

$$\Xi' = \beta\,\xi^{\beta-1} w + x^{\beta} w', \qquad \Xi'' = \beta\,(\beta-1)\,\xi^{\beta-2} w + 2\beta\,\xi^{\beta-1} w' + \xi^{\beta} w''.$$

Second, substituting the obtained relations into the second differential equation of the system (9.2) gives a 4-parameter $(\alpha,\beta,\lambda,\mp)$ family of ordinary differential equations of the second order

$$\xi^{\beta} w'' + \xi^{\beta-1}\left(2\beta + \frac{\alpha}{\theta}\right) w' + \xi^{\beta-2}\beta\left(\beta - 1 + \frac{\alpha}{\theta}\right) w \mp \lambda^2 w = 0\,.$$

Third, imposing the following constraint on the exponent β

$$2\beta + \frac{\alpha}{\theta} = 1 \qquad \Rightarrow \qquad \beta(\alpha) = \frac{1-\alpha}{\theta}$$

leads to a 3-parameter $(\beta(\alpha),\lambda,\mp)$ family of ordinary differential equations of the second order

$$w'' + \frac{1}{\xi}\,w' - \left(\frac{\beta^2}{\xi^2} \pm \lambda^2\right) w = 0\,. \tag{9.4}$$

I) Assuming that $\lambda \neq 0$, we change both dependent and independent variables: $s=\lambda\xi$, $W(s;\alpha):= w\left(\lambda^{-1}s;\alpha\right)$, and obtain a 2-parameter $(\beta(\alpha),\mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\beta^2}{s^2} \pm 1\right) W = 0\,.$$

The choice of the lower sign in the obtained family leads to the 1-parameter $(\beta(\alpha))$ Bessel equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\beta^2}{s^2} - 1\right) W = 0\,, \tag{9.5}$$

whereas the choice of the upper sign leads to the 1-parameter $(\beta(\alpha))$ modified Bessel equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\beta^2}{s^2} + 1\right) W = 0\,. \tag{9.6}$$

i) A 3-parameter family of solutions of (9.5), $\beta \notin \mathbb{Z}$ (i. e. $\alpha \in (0,1) \bigcup (1,2)$), is known [9, 19] to be

$$W(s;\alpha) = A_1\, \mathtt{J}_{-\beta}(s) + A_2\, \mathtt{J}_{+\beta}(s)\,, \tag{9.7}$$

where $A_{1,2}$ are arbitrary constants, and $\mathtt{J}_{\mp\beta}(s)$ are the Bessel functions of the first kind of orders $\mp\beta$ (see Section 5), whereas a 2-parameter family of solutions of the equation (9.5), $\beta\in\mathbb{Z}$, is known [9, 19] to be

$$W(s;\alpha) = B_1\, \mathtt{J}_0(s) + B_2\, \mathtt{N}_0(s)\,, \tag{9.8}$$

where $B_{1,2}$ are arbitrary constants, and $\mathtt{J}_0(s)$ and $\mathtt{N}_0(s)$ are respectively the Bessel and the Neumann functions of the first kind of order zero (see Section 5).

ii) A 3-parameter family of solutions of the equation (9.6), $\beta \notin \mathbb{Z}$, is known [9, 19] to be

$$W(s;\alpha) = C_1\, \mathtt{I}_{-\beta}(s) + C_2\, \mathtt{I}_{+\beta}(s)\,, \tag{9.9}$$

where $C_{1,2}$ are arbitrary constants, and $\mathtt{I}_{\mp\beta}(s)$ are the modified Bessel functions of the first kind of orders $\mp\beta$ (see Section 5), whereas a 2-parameter family of solutions of the equation (9.6), $\beta\in\mathbb{Z}$, is known [9, 19] to be

$$W(s;\alpha) = D_1\, \mathtt{I}_0(s) + D_2\, \mathtt{K}_0(s)\,, \tag{9.10}$$

where $D_{1,2}$ are arbitrary constants, and $\mathtt{I}_0(s)$ and $\mathtt{K}_0(s)$ are respectively the modified Bessel function of the first kind and the modified Bessel function of the second kind of order zero (see Section 5).

i) Similarly to Section 5, we find from (9.7), (9.8) the following families of solutions of the second equation of the system (9.2)

$$\Xi(\xi;\alpha) = \begin{cases} \xi^{\frac{1-\alpha}{\theta}} \Big[A_1\, \mathtt{J}_{-\beta}\,(\lambda\xi) + A_2\, \mathtt{J}_{+\beta}\,(\lambda\xi) \Big], & \alpha \in (0,1) \bigcup (1,2)\,, \\ B_1\, \mathtt{J}_0\, \ (\lambda\xi) + B_2\, \mathtt{N}_0\, \ (\lambda\xi)\ \ , & \alpha = 1\,, \end{cases}$$

and after retaining only the bounded terms the families read

$$\Xi(\xi;\alpha)=\begin{cases}\underbrace{\overbrace{A_1\,\lambda^{-\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}A_{-\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}^{\alpha\in(0,2)}+\xi^{\frac{1-\alpha}{\theta}}\,A_2\,\lambda^{+\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}A_{+\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}_{\alpha\in(0,1)}, \\ B_1\sum_{\mu=0}^{\infty}\;A_{\,0,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}, \qquad \alpha=1\,,\end{cases}$$

The straightforward extension of the above bounded families to $\xi<0$ and retaining only those terms leading to the continuous and continuously differentiable flux gives the required composite solution of the second equation of the system (9.2)

$$\Xi(\xi;\alpha)=\sum_{\mu=0}^{\infty}A_{-\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}, \qquad \alpha\in(0,2), \tag{9.11}$$

where the coefficients $A_{-\beta,\mu}$ are taken from the series (5.11), if $\alpha\in(0,1)$ $(\beta>0)$ and $\alpha\in(1,2)$ $(\beta<0)$, and are taken from the series (5.13), if $\alpha=1$ $(\beta=0)$.

ii) Again, doing similarly to Section 5, we find from (9.9), (9.10) the following families of solutions of the second equation of the system (9.2)

$$\Xi(\xi;\alpha)=\begin{cases}\xi^{\frac{1-\alpha}{\theta}}\left[C_1\,\mathtt{I}_{-\beta}\,(\lambda\xi)+C_2\,\mathtt{I}_{+\beta}\,(\lambda\xi)\right], \quad \alpha\in(0,1)\bigcup(1,2)\,, \\ D_1\,\mathtt{I}_0\;\;(\lambda\xi)+D_2\,\mathtt{K}_0\;\;(\lambda\xi)\;, \quad \alpha=1\,.\end{cases}$$

Retaining only the bounded terms in the families yields to

$$\Xi(\xi;\alpha)=\begin{cases}\underbrace{\overbrace{C_1\,\lambda^{-\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}B_{-\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}^{\alpha\in(0,2)}+\xi^{\frac{1-\alpha}{\theta}}\,C_2\,\lambda^{+\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}B_{+\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}_{\alpha\in(0,1)}, \\ D_1\sum_{\mu=0}^{\infty}\;B_{\,0,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}, \qquad \alpha=1\,.\end{cases}$$

Eventually, doing exactly as in item *i*), we obtain the following required composite solution of the second equation of the system (9.2)

$$\Xi(\xi;\alpha)=\sum_{\mu=0}^{\infty}B_{-\beta,\mu}\,\lambda^{2\mu}\xi^{2\mu}, \tag{9.12}$$

where the coefficients $B_{-\varrho,\mu}$ are taken from the series (5.15), if $\alpha\in(0,1)$ ($\varrho>0$) and $\alpha\in(1,2)$ ($\varrho<0$), and are taken from the series (5.17), if $\alpha=1$ ($\varrho=0$).

II) Assuming that $\lambda=0$, we obtain directly from (9.4) a 1-parameter ($\beta(\alpha)$) ordinary differential equation of the second order

$$\frac{\mathrm{d}^2 w_0}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}w_0}{\mathrm{d}\xi}-\frac{\beta^2 w_0}{\xi^2}=0\,. \tag{9.13}$$

A 3-parameter family of solutions of the equation (9.13) is as follows

$$w_0\,(\xi;\alpha)=E_1\,\xi^{-\beta}+E_2\,\xi^{+\beta},$$

where $E_{1,2}$ are arbitrary constants, and after applying the inverse transformation (9.3) we obtain the family of solutions of the second equation of the system (9.2)

$$\Xi\,(\xi;\alpha)=\xi^{\beta}\left(E_1\,\xi^{-\beta}+E_2\,\xi^{+\beta}\right)=E_1+E_2\,\xi^{(1-\alpha)\frac{2}{\theta}},$$

extendable to $\xi<0$ as follows

$$\Xi\,(\xi;\alpha)=\begin{cases} E_1+E_2\,|\xi|^{(1-\alpha)\frac{2}{\theta}}, & \alpha\in(0,2),\\[2ex] E_1+E_2\ \xi^{\,(1-\alpha)\frac{2}{\theta}}, & \alpha\in(0,2)_o.\end{cases} \tag{9.14}$$

Both families (9.14) leads, as it is known from Section 8, to the discontinuous flux $\mathring{\varphi}$ (8.7), therefore no family (9.14) must be accounted for in the ansatz (9.1).

Now we turn to the first ordinary differential equation of the system (9.2); its 3-parameter families of solutions are known to be

$$\Theta(\tau;\alpha)=\begin{cases} F_1\,\mathtt{exp}\,(-\lambda\tau)+F_2\,\mathtt{exp}\,(+\lambda\tau)\,,\\[2ex] G_1\,\mathtt{cos}\,(+\lambda\tau)+G_2\,\mathtt{sin}\,(+\lambda\tau)\,,\end{cases} \tag{9.15}$$

where $F_{1,2}$, $G_{1,2}$ are arbitrary constants.

Combining the families (9.15) for $\Theta(\tau;\alpha)$ and the families (9.11) and (9.12) for $\Xi\,(\xi;\alpha)$ in the ansatz (9.1), we obtain the required solutions of the transformed wave equation (6.6), (6.7). The spatial parts $\Xi\,(\xi;\alpha)$ (9.11), (9.12) of the obtained solutions include the same terms $\xi^{2\mu}$, $\mu\in\mathbb{Z}_+$, as those present in the only required series solution (8.9).

10. Functional properties of the series solutions

In this Section we turn out to our preliminary functional estimations concerning solutions of the degenerate wave equation made in Section 2.

Proposition 10.1. *All bounded series solutions* $u(t,x;\alpha)$ (4.1) *of the original degenerate wave equation* (1.1) *are elements of the functional space* $H^1_a(-c,+c)$ *for all* $t\in[0,T]$.

Proof. The underlying idea of the proof is straightforwardly based on properties of convergent power series [1, 20]. On the one hand, 1) usual convergence implies absolute convergence, then 2) absolute convergence implies uniform convergence, and eventually 3) uniform convergence implies term-by-term differentiation and integration (term-by-term differentiation was, by the way, implied when finding the one-sided series solutions in Section 3 and then matching the obtained one-sided series solutions continuously in Section 4). On the other hand, uniform convergence of two (or more) power series implies their term-by-term product, and the resulting power series is also uniformly convergent. These properties and the definition of the norm of the functional space $H^1_a(-c,+c)$ are quite sufficient to immediately complete the proof, nevertheless we perform careful calculations of the norm for some of the series solutions (4.1). The only thing we need is to assume that all series solutions are convergent in the c-band. Accounting for the exact solutions of the degenerate wave equation found in Section 9 using separation of the variables our assumption seems even more than reasonable.

First, we take the series solution $u_1(t,x;\alpha)$ (4.1), valid if $\alpha \in (0,2)$ and uniformly convergent for all $t \in [0,T]$, then we have

$$
\begin{aligned}
u_1^2 &= \left(\sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\,|x|^{\mu\theta}\right)^2 = \sum_{\mu=0}^{\infty} \bar{U}_{\alpha,\mu}(t)\,|x|^{\mu\theta},\\
q_1 &= \mp\theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta-1},\\
aq_1 &= \mp a_*\theta\,|x|^{\alpha} \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta-1} = \mp a_*\theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{(\mu-1)\theta+1},\\
aq_1^2 &= a_*\theta^2 \left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta-1}\right)\left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{(\mu-1)\theta+1}\right) = \sum_{\mu=1}^{\infty} \widetilde{U}_{\alpha,\mu}(t)\,|x|^{\mu\theta},
\end{aligned}
$$

where the coefficient functions $\bar{U}_{\alpha,\mu}(t)$, $\mu \in \mathbb{Z}_+$, and $\widetilde{U}_{\alpha,\mu}(t)$, $\mu \in \mathbb{N}$, are determined using the series product rule, the upper sign is taken for $x<0$, whereas the lower sign for $x>0$, and $t \in [0,T]$. It is clear that the derived series $u_1^2 + aq_1^2$ is uniformly convergent for all $t \in [0,T]$, therefore the norm of $u_1(t,x;\alpha)$ is bounded to be

$$
\begin{aligned}
\|u_1\|^2_{H^1_a(-c,+c)} &= \int_{-c}^{c} \left[u_1^2 + aq_1^2\right] \mathrm{d}x\\
&= \bar{U}_{\alpha,0}(t) \int_{-c}^{c} \mathrm{d}x + \sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) + \widetilde{U}_{\alpha,\mu}(t)\right) \int_{-c}^{c} |x|^{\mu\theta}\,\mathrm{d}x\\
&= 2c\,\bar{U}_{\alpha,0}(t) + 2\sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) + \widetilde{U}_{\alpha,\mu}(t)\right) \frac{2c^{\mu\theta+1}}{\mu\theta+1}\,.
\end{aligned}
$$

Second, we take the series solution $u_2(t,x;\alpha)$ (4.1), valid if $\alpha \in (0,2)_o$ and uniformly convergent for all $t \in [0,T]$, then we have

$$
\begin{aligned}
u_2^2 &= \left(\sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\, x^{\mu\theta}\right)^2 = \sum_{\mu=0}^{\infty} \bar{U}_{\alpha,\mu}(t)\, x^{\mu\theta}, \\
q_2 &= \theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{\mu\theta-1}, \\
aq_2 &= a_*\theta\, |x|^{\alpha} \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{\mu\theta-1} = \mp a_*\theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{(\mu-1)\theta+1}, \\
aq_2^2 &= \mp a_*\theta^2 \left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{\mu\theta-1}\right)\left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{(\mu-1)\theta+1}\right) = \mp \sum_{\mu=1}^{\infty} \widetilde{U}_{\alpha,\mu}(t)\, x^{\mu\theta},
\end{aligned}
$$

and the norm of $u_2(t,x;\alpha)$ reads

$$
\begin{aligned}
\|u_2\|^2_{H_a^1(-c,+c)} = \int_{-c}^{c} \left[u_2^2 + aq_2^2\right] \mathrm{d}x &= \bar{U}_{\alpha,0}(t) \int_{-c}^{c} \mathrm{d}x \\
&+ \sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) - \widetilde{U}_{\alpha,\mu}(t)\right) \int_{-c}^{0} x^{\mu\theta}\, \mathrm{d}x \\
&+ \sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) + \widetilde{U}_{\alpha,\mu}(t)\right) \int_{0}^{c} x^{\mu\theta}\, \mathrm{d}x < \infty\,.
\end{aligned}
$$

Performing the calculation gives for the norm the following bounded expressions

$$
\begin{aligned}
\|u_2\|^2_{H_a^1(-c,+c)} &= 2c\, \bar{U}_{\alpha,0}(t) \\
&+ \begin{cases}
\displaystyle\sum_{\gamma=1}^{\infty} \bar{U}_{\alpha,2\gamma}(t)\, \frac{2c^{\,2\gamma\theta+1}}{2\gamma\theta+1} + \sum_{\mu=1}^{\infty} \widetilde{U}_{\alpha,\mu}(t)\, \frac{2c^{\,\mu\theta+1}}{\mu\theta+1}\,, & \alpha \in \mathbb{Q}_{o,o}\,, \\
\displaystyle\sum_{\mu=1}^{\infty} \bar{U}_{\alpha,\mu}(t)\, \frac{2c^{\,\mu\theta+1}}{\mu\theta+1}\,, & \alpha \in \mathbb{Q}_{o,e}\,.
\end{cases}
\end{aligned}
$$

Calculating the bounded norms for the binomial solutions 2, 3 (4.1) and the series solutions 5, 6 (4.1) is performed exactly in the same way. □

Proposition 10.1 says that we cannot distinguish between the series solutions (4.1) using the norm of the functional space $H_a^1(-c,+c)$.

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Received 23.04.2020