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NUMERICAL INVESTIGATION OF A PLAIN STRAIN STATE FOR A BODY WITH THIN COVER USING DOMAIN DECOMPOSITION

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Резюме. Розглядається модель, яка описує напружено деформований стан двовимірного гетерогенного тіла з тонким покриттям. Спочатку доведено збіжність ітеративного алгоритму, побудованого на основі поєднання методу скінченних елементів (МСЕ) та методу граничних елементів (МГЕ) з використанням декомпозиції областей. Після цього алгоритм проілюстровано на прикладі двовимірної задачі для тіла з покриттям.

Abstract. We consider a model, that describes the plain stress state of the $2D$ heterogeneous elastic body with the thin cover. First we prove the convergence of the iterative algorithm based on finite element method/boundary element method (FEM/BEM) coupling using domain decomposition. Further we illustrate this algorithm with an example of $2D$ problem for the body with a cover.

1. Introduction

A lot of structures, both natural and artificial, contain thin covers or thin inclusions. Therefore, the problem of analyzing the stress-strain state of such bodies is of great importance. Typically they consist of two or more homogeneous parts that have a big differences in physical dimensions and properties between them. A lot of aspects of the problems, related to this subject, were analyzed (see for example [2, 4, 5, 7, 8]). In this paper we use the combined model, where the parts of the body with comparable physical dimensions are described by the linear elasticity equations, whereas the sress state of the thin cover is described by Tymoshenko shell theory equations [5]. These parts are connected using the appropriate coupling conditions on the common boundaries.

In order to perform numerical analysis of our model we solve the corresponding problems in thin shells by finite element method (FEM) with bubble basis functions, and the other parts of the body are solved numerically using boundary element method (BEM) with linear basis functions; the iterative domain decomposition algorithm is then used to connect the solutions in both domains.

In this paper we also prove the properties of our model and prove the convergence of the algorithm.

[†] *Key words.* Elasticity theory, boundary element method, finite element method, domain decomposition.

2. Problem statement

Let us consider a problem of plane strain of cylindrical body Ω_1 with the cover Ω_2.

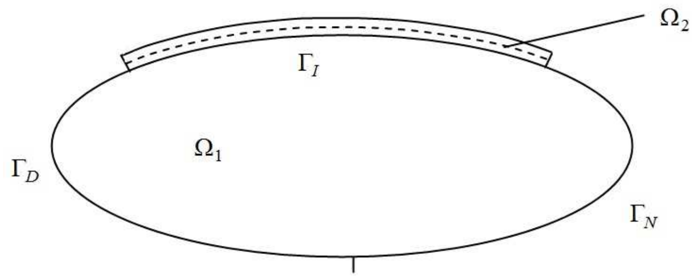

Fig. 1. Body with cover

The plane strain stress of the body in Ω_1 can be described by [1]

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = f_1, \qquad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = f_2 \tag{1}$$

that holds for $x \in \Omega_1$, $x = x_1, x_2$. Here $f = f_1, f_2$ denotes the volume forces that act on the body in Ω_1. From the Hook's law it follows that the components of the stress tensor can be written as

$$\sigma_{ij} = \frac{1}{2} E_1 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad i, j = 1, 2,$$

where $u(x) = u_1(x), u_2(x)$ is the displacement vector with u_i being the displacements in the directions x_i for $i = 1, 2$; E_1 is the Young's modulus of the body in Ω_1. In the following we assume that no volume forces act on the body in Ω_1.

Let us denote by n the outer normal vector to Ω_1, and by τ – the tangent vector. Equations (1) are considered together with the boundary conditions

$$u_v = 0, \quad u_\tau = 0, \quad x \in \Gamma_D$$

and

$$\sigma_{vv} = 0, \quad \sigma_{v\tau} = 0, \quad x \in \Gamma_N,$$

where u_v and u_τ are the components of the stress tensor in the coordinate system n, τ. Similarly, σ_{vv} and $\sigma_{v\tau}$ are the components of the stress tensor in the n, τ coordinate system.

For the description of the cover in Ω_2 we use the equations of Timoshenko shell theory for the cylindrical shell of the form [5]

$$
\begin{gathered}
-\frac{1}{A_1}\frac{dT_{11}}{d\xi_1} - k_1 T_{13} = p_1, \\
-\frac{1}{A_1}\frac{dT_{13}}{d\xi_1} + k_1 T_{11} = p_3, \qquad (2) \\
-\frac{1}{A_1}\frac{dM_{11}}{d\xi_1} + T_{13} = m_1, \quad -1 \le \xi_1 \le 0,
\end{gathered}
$$

where v_1, w, γ_1 are the displacements and angle of revolution in the shell; T_{11}, T_{13}, M_{11} are the forces and moments in the shell; $A_1 = A_1(\xi_1)$, $k_1 = k_1(\xi_1)$ correspond to Lame parameter and median surface curvature parameter; p_1, p_3, m_1 are given functions; it holds

$$
T_{11} = \frac{E_2 h}{1 - v_2^2}\varepsilon_{11}, \quad T_{13} = k' G' h \varepsilon_{13}, \quad M_{11} = \frac{E_2 h^3}{12\left(1 - v_2^2\right)}\chi_{11}, \qquad (3)
$$

$$
\varepsilon_{11} = \frac{1}{A_1}\frac{d\mathrm{v}_1}{d\xi_1} + k_1 w, \quad \varepsilon_{13} = \frac{1}{A_1}\frac{dw}{d\xi_1} + \gamma_1 - k_1 \mathrm{v}_1, \quad \chi_{11} = \frac{1}{A_1}\frac{d\gamma_1}{d\xi_1}, \qquad (4)
$$

$$
\begin{gathered}
p_1 = \left(1 + k_1\frac{h}{2}\right)\sigma_{13}^{+} - \left(1 - k_1\frac{h}{2}\right)\sigma_{13}^{-}, \\
p_3 = \left(1 + k_1\frac{h}{2}\right)\sigma_{33}^{+} - \left(1 - k_1\frac{h}{2}\right)\sigma_{33}^{-}, \qquad (5) \\
m_1 = \frac{h}{2}\left(\left(1 + k_1\frac{h}{2}\right)\sigma_{13}^{+} - \left(1 - k_1\frac{h}{2}\right)\sigma_{13}^{-}\right).
\end{gathered}
$$

Here E_2 is the Young's modulus for the shell, v_2 is the Poisson's ratio; g_1, g_3 are the components of the volume forces vector, that act on the shell; $\sigma_{ij}^{+}, \sigma_{ij}^{-}$, $i, j = 1, 3$ are the components of the stress tensor on the outer ($\xi_3 = \frac{h}{2}$) and inner ($\xi_3 = -\frac{h}{2}$) surfaces of the shell. It is known, that in the case of isotropic bodies we have $k' = \frac{5}{6}$, $G' = \frac{E_2}{2(1+v_2)}$.

At each end of the thin cover we impose boundary conditions either on the displacements v_1, w and γ_1 or on the forces T_{11}, T_{13} and moment M_{11} in the shell (if the end is subjected to load or free). At the outer surface of the shell we prescribe to σ_{13}^{+} and σ_{33}^{+} some given stresses.

Remark 1. *The choice of 2D curvilinear coordinate system for the shell as ξ_1, ξ_3 (instead of ξ_1, ξ_2) is based on the fact, that 2D problem is obtained from the 3D case by assuming the cylinder being infinite in the direction of ξ_2.*

On the boundary Γ_I, common to both Ω_1 and Ω_2 we prescribe the following coupling conditions [5]*:*

$$u_v = w, \quad u_\tau = \mathrm{v}_1 - \frac{h}{2}\gamma_1,$$
$$\sigma_{vv} = \sigma_{33}^{-}, \quad \sigma_{v\tau} = \sigma_{13}^{-}. \tag{6}$$

Let us rewrite the coupling conditions (6) on Γ_I as follows:

$$u_v = w, \quad u_\tau = \mathrm{v}_1 - \frac{h}{2}\gamma_1,$$
$$A_1\left(1-k_1\frac{h}{2}\right)\sigma_{vv} - A_1\left(1-k_1\frac{h}{2}\right)\sigma_{33}^{-} = 0, \tag{7}$$
$$A_1\left(1-k_1\frac{h}{2}\right)\sigma_{v\tau} - A_1\left(1-k_1\frac{h}{2}\right)\sigma_{13}^{-} = 0.$$

3. The properties of the Steklov-Poincare operators and convergence of the domain decomposition iterative algorithm

Let us suppose that on the inferface Γ_I the displacement is equal to $\varphi = \varphi_1, \varphi_2$, $\varphi_i \in H^1(\Gamma_I)$, $i = 1, 2$. In the following we consider the Steklov-Poincare operator S for our problem as well as local Steklov-Poincare operators S_i, that correspond to Ω_i, $i = 1, 2$. Therefore, we have from (7)

$$\langle S\varphi, \psi\rangle_{\Gamma_I} = \langle S_1\varphi, \psi\rangle_{\Gamma_I} + \langle S_2\varphi, \psi\rangle_{\Gamma_I}, \qquad \forall \varphi, \psi \in H^1(\Gamma_I) \times H^1(\Gamma_I)$$

$$\langle S_1\varphi, \psi\rangle_{\Gamma_I} = \left\langle A_1\left(1-k_1\frac{h}{2}\right)G_I\sigma_{vv}(\varphi), \psi_1\right\rangle_{\Gamma_I} +$$
$$+\left\langle A_1\left(1-k_1\frac{h}{2}\right)G_I\sigma_{v\tau}(\varphi), \psi_2\right\rangle_{\Gamma_I}, \tag{8}$$
$$\langle S_2\varphi, \psi\rangle_{\Gamma_I} = \left\langle -A_1\left(1-k_1\frac{h}{2}\right)\sigma_{33}^{-}(\varphi), \psi_1\right\rangle_{\Gamma_I} +$$
$$+\left\langle A_1\left(1-k_1\frac{h}{2}\right)\sigma_{13}^{-}(\varphi), \psi_2\right\rangle_{\Gamma_I},$$

where $G_I\sigma$ is the trace of σ on Γ_I; $\langle u, v\rangle_{\Gamma_I}$ denotes the bilinear form which formally can be written as

$$\langle u, v\rangle_{\Gamma_I} = \int_{\Gamma_I} uv d\Gamma_I.$$

First we prove that there exists a unique solution to the problem for Steklov-Poincare operators. For this purpose we will use the Lax-Milgram lemma.

Let Ω_2^* be a midline of Ω_2. Without loss of generality we assume that $g_1 = g_3 = \sigma_{13}^{+} = \sigma_{33}^{+} = 0$. Moreover, one notices that all the displacements defined in Ω_2 are continuous with respect to ξ_3, since both equations and boundary conditions are independent of ξ_3. Using the coupling conditions (7), one can rewrite (8) as

$$\begin{aligned}
\langle S_2\varphi,\psi\rangle_{\Gamma_I} &= \left\langle -A_1\left(1-k_1\frac{h}{2}\right)\sigma_{33}^{-}(\varphi),\tilde{w}\right\rangle_{\Gamma_I} + \\
&+\left\langle -A_1\left(1-k_1\frac{h}{2}\right)\sigma_{13}^{-}(\varphi),\left(\tilde{\mathrm{v}}_1-\frac{h}{2}\tilde{\gamma}_1\right)\right\rangle_{\Gamma_I} = \\
&= \left(-A_1\left(1-k_1\frac{h}{2}\right)\sigma_{33}^{-},\tilde{w}\right)_{\Omega_2^*} + \left(-A_1\left(1-k_1\frac{h}{2}\right)\sigma_{13}^{-},\tilde{\mathrm{v}}_1\right)_{\Omega_2^*} + \\
&+\left(A_1\frac{h}{2}\left(1-k_1\frac{h}{2}\right)\sigma_{13}^{-},\tilde{\gamma}_1\right)_{\Omega_2^*},
\end{aligned} \tag{9}$$

where

$$(u,v)_{\Omega_2^*} = \int\limits_{\Omega_2^*} uv\, d\Omega_2^*.$$

Let us substitute into (9) the corresponding left sides of the system of equations (2)-(5):

$$\begin{aligned}
\langle S_2\varphi,\psi\rangle_{\Gamma_I} &= \left(-\frac{dT_{13}}{d\xi_1}+k_1A_1T_{11},\tilde{w}\right)_{\Omega_2^*} + \\
&+\left(-\frac{dT_{11}}{d\xi_1}-k_1A_1T_{13},\tilde{\mathrm{v}}_1\right)_{\Omega_2^*} + \left(-\frac{dM_{11}}{d\xi_1}+A_1T_{13},\tilde{\gamma}_1\right)_{\Omega_2^*}.
\end{aligned}$$

After integrating by parts one can easily notice that the coerciveness and symmetry of the Steklov-Poincare operator S_2 follows from the properties of the corresponding operator defined on the midline Ω_2^* which has been proven in [2]. Therefore, one obtains

$$\begin{aligned}
\langle S_2\varphi,\varphi\rangle_{\Gamma_I} &\geq c^2\int\limits_{-1}^{0}\left(\left(\frac{d\mathrm{v}_1}{d\xi_1}\right)^2+\left(\frac{dw}{d\xi_1}\right)^2+\left(\frac{d\gamma_1}{d\xi_1}\right)^2\right)d\Omega_2^* + \\
&+c^2\int\limits_{-1}^{0}\left(\mathrm{v}_1^2+w^2+\gamma_1^2\right)d\Omega_2^*, \quad c\neq 0.
\end{aligned}$$

Further,

$$\langle S_2\varphi,\varphi\rangle_{\Gamma_I} \geq c_1^2\int\limits_{-1}^{0}\left(\left(\frac{dw}{d\xi_1}\right)^2+\left(\frac{d\mathrm{v}_1}{d\xi_1}-\frac{h}{2}\frac{d\gamma_1}{d\xi_1}\right)^2\right)d\Omega_2^* +$$

$$+c_1^2 \int_{-1}^{0} \left(w^2 + \left(\mathrm{v}_1 - \frac{h}{2}\gamma_1 \right)^2 \right) d\Omega_2^*, \quad c_1 \neq 0.$$

Thus, S_2 is coercive. The linearity of S_2 follows directly from the linearity of the corresponding operator in Ω_2^*.

Let us now prove the continuity of S_2. For this purpose, firstly one proves the continuity of the following operator in Ω_2^*

$$(Ay, \tilde{y})_{\Omega_2^*} = \left(-\frac{dT_{13}}{d\xi_1} + k_1 A_1 T_{11}, \tilde{w} \right)_{\Omega_2^*} +$$
$$+ \left(-\frac{dT_{11}}{d\xi_1} - k_1 A_1 T_{13}, \tilde{\mathrm{v}}_1 \right)_{\Omega_2^*} + \left(-\frac{dM_{11}}{d\xi_1} + A_1 T_{13}, \tilde{\gamma}_1 \right)_{\Omega_2^*},$$

where $y = \mathrm{v}_1, w, \gamma_1$, $\quad \tilde{y} = \tilde{\mathrm{v}}_1, \tilde{w}, \tilde{\gamma}_1$. Using Cauchy-Schwarz inequality, one obtains for $y, \tilde{y} \in H^1(\Gamma_I) \times H^1(\Gamma_I) \times H^1(\Gamma_I)$

$$(Ay, \tilde{y})_{\Omega_2^*} = \int_{-1}^{0} \left(T_{13} \frac{d\tilde{w}}{d\xi_1} + k_1 A_1 T_{11} \tilde{w} \right) d\xi_1 +$$

$$+ \int_{-1}^{0} \left(T_{11} \frac{d\tilde{\mathrm{v}}_1}{d\xi_1} - k_1 A_1 T_{13} \tilde{\mathrm{v}}_1 \right) d\xi_1 + \int_{-1}^{0} \left(M_{11} \frac{d\tilde{\gamma}_1}{d\xi_1} + A_1 T_{13} \tilde{\gamma}_1 \right) d\xi_1 =$$

$$= \int_{-1}^{0} \left(k' G' h \left(\frac{1}{A_1} \frac{dw}{d\xi_1} + \gamma_1 - k_1 \mathrm{v}_1 \right) \frac{d\tilde{w}}{d\xi_1} + \right.$$

$$\left. + k_1 A_1 \frac{E_2 h}{1 - v_2^2} \left(\frac{1}{A_1} \frac{d\mathrm{v}_1}{d\xi_1} + k_1 w \right) \tilde{w} \right) d\xi_1 +$$

$$+ \int_{-1}^{0} \left(\frac{E_2 h}{1 - v_2^2} \left(\frac{1}{A_1} \frac{d\mathrm{v}_1}{d\xi_1} + k_1 w \right) \frac{d\tilde{\mathrm{v}}_1}{d\xi_1} - \right.$$

$$\left. - k_1 A_1 k' G' h \left(\frac{1}{A_1} \frac{dw}{d\xi_1} + \gamma_1 - k_1 \mathrm{v}_1 \right) \tilde{\mathrm{v}}_1 \right) d\xi_1 +$$

$$+ \int_{-1}^{0} \left(\frac{E_2 h^3}{12 (1 - v_2^2)} \frac{1}{A_1} \frac{d\gamma_1}{d\xi_1} \frac{d\tilde{\gamma}_1}{d\xi_1} + A_1 k' G' h \left(\frac{1}{A_1} \frac{dw}{d\xi_1} + \gamma_1 - k_1 \mathrm{v}_1 \right) \tilde{\gamma}_1 \right) d\xi_1 \leq$$

$$\leq k' G' h \frac{1}{A_1^m} \left[\int_{-1}^{0} \left(\frac{dw}{d\xi_1} \right)^2 d\xi_1 \right]^{\frac{1}{2}} \left[\int_{-1}^{0} \left(\frac{d\tilde{w}}{d\xi_1} \right)^2 d\xi_1 \right]^{\frac{1}{2}} +$$

$$+k'G'h\left[\int_{-1}^{0}(\gamma_1)^2\,d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}\left(\frac{d\tilde{w}}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}+$$

$$+k'G'h\left|k_1^M\right|\left[\int_{-1}^{0}(\mathrm{v}_1)^2\,d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}\left(\frac{d\tilde{w}}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}+$$

$$+\frac{E_2h}{1-v_2^2}\left|k_1^M\right|\left[\int_{-1}^{0}\left(\frac{d\mathrm{v}_1}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}(\tilde{w})^2\,d\xi_1\right]^{\frac{1}{2}}+$$

$$+\left(A_1\left|k_1\right|\right)^M\frac{E_2h}{1-v_2^2}\left[\int_{-1}^{0}(w)^2\,d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}(\tilde{w})^2\,d\xi_1\right]^{\frac{1}{2}}+$$

$$+\frac{E_2h}{1-v_2^2}\frac{1}{A_1^m}\left[\int_{-1}^{0}\left(\frac{d\mathrm{v}_1}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}\left(\frac{d\tilde{\mathrm{v}}_1}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}+$$

$$+\frac{E_2h}{1-v_2^2}\left|k_1^M\right|\left[\int_{-1}^{0}(w)^2\,d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}\left(\frac{d\tilde{\mathrm{v}}_1}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}+$$

$$+\frac{E_2h}{1-v_2^2}\left|k_1^M\right|\left[\int_{-1}^{0}\left(\frac{dw}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}(\tilde{\mathrm{v}}_1)^2\,d\xi_1\right]^{\frac{1}{2}}+$$

$$+k'G'h\left(A_1\left|k_1\right|\right)^M\left[\int_{-1}^{0}(\gamma_1)^2\,d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}(\tilde{\mathrm{v}}_1)^2\,d\xi_1\right]^{\frac{1}{2}}+$$

$$+k'G'h\left(A_1k_1^2\right)^M\left[\int_{-1}^{0}(\mathrm{v}_1)^2\,d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}(\tilde{\mathrm{v}}_1)^2\,d\xi_1\right]^{\frac{1}{2}}+$$

$$+\frac{E_2h^3}{12\left(1-v_2^2\right)}\frac{1}{A_1^m}\left[\int_{-1}^{0}\left(\frac{d\gamma_1}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}\left[\int_{-1}^{0}\left(\frac{d\tilde{\gamma}_1}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}+$$

$$+k'G'h\left[\int\limits_{-1}^{0}\left(\frac{dw}{d\xi_1}\right)^2 d\xi_1\right]^{\frac{1}{2}}\left[\int\limits_{-1}^{0}(\tilde{\gamma}_1)^2 d\xi_1\right]^{\frac{1}{2}}+$$

$$+k'G'hA_1^M\left[\int\limits_{-1}^{0}(\gamma_1)^2 d\xi_1\right]^{\frac{1}{2}}\left[\int\limits_{-1}^{0}(\tilde{\gamma}_1)^2 d\xi_1\right]^{\frac{1}{2}}+$$

$$+k'G'h\left(A_1\left|k_1\right|\right)^M\left[\int\limits_{-1}^{0}(\mathrm{v}_1)^2 d\xi_1\right]^{\frac{1}{2}}\left[\int\limits_{-1}^{0}(\tilde{\gamma}_1)^2 d\xi_1\right]^{\frac{1}{2}}\leq$$

$$\leq C^2\left\|y\right\|_{H^1(\Omega_2^*)}\left\|\tilde{y}\right\|_{H^1(\Omega_2^*)}, \quad C\neq 0.$$

In the above $f^M = \sup\limits_{\Omega_2^*} f$, $f^m = \inf\limits_{\Omega_2^*} f$. As a result, the continuity of the operator A is proven. Taking into account the continuity of the operator A, we can conclude

$$\langle S_2\varphi,\psi\rangle_{\Gamma_I}\leq$$

$$\leq C^2\left[\int\limits_{-1}^{0}\left(\left(\frac{d\mathrm{v}_1}{d\xi_1}\right)^2+\left(\frac{dw}{d\xi_1}\right)^2+\left(\frac{d\gamma_1}{d\xi_1}\right)^2+\mathrm{v}_1^2+w^2+\gamma_1^2\right)d\Omega_2^*\right]^{\frac{1}{2}}\times$$

$$\times\left[\int\limits_{-1}^{0}\left(\left(\frac{d\tilde{\mathrm{v}}_1}{d\xi_1}\right)^2+\left(\frac{d\tilde{w}}{d\xi_1}\right)^2+\left(\frac{d\tilde{\gamma}_1}{d\xi_1}\right)^2+\tilde{\mathrm{v}}_1^2+\tilde{w}^2+\tilde{\gamma}_1^2\right)d\Omega_2^*\right]^{\frac{1}{2}}, \quad C\neq 0.$$

Thus, one obtains

$$\langle S_2\varphi,\psi\rangle_{\Gamma_I}\leq$$

$$\leq C_1^2\left[\int\limits_{-1}^{0}\left(\left(\frac{dw}{d\xi_1}\right)^2+\left(\frac{d\mathrm{v}_1}{d\xi_1}-\frac{h}{2}\frac{d\gamma_1}{d\xi_1}\right)^2+w^2+\left(\mathrm{v}_1-\frac{h}{2}\gamma_1\right)^2\right)d\Omega_2^*\right]^{\frac{1}{2}}\times$$

$$\times\left[\int\limits_{-1}^{0}\left(\left(\frac{d\tilde{w}}{d\xi_1}\right)^2\left(\frac{d\tilde{\mathrm{v}}_1}{d\xi_1}-\frac{h}{2}\frac{d\tilde{\gamma}_1}{d\xi_1}\right)^2+\tilde{w}^2+\left(\tilde{\mathrm{v}}_1-\frac{h}{2}\tilde{\gamma}_1\right)^2\right)d\Omega_2^*\right]^{\frac{1}{2}}, \quad C_1\neq 0.$$

Let us consider now the local Steklov-Poincare operator S_1.

$$\langle S_1\varphi,\psi\rangle_{\Gamma_I}=\left\langle A_1\left(1-k_1\frac{h}{2}\right)G_I\sigma_{vv}\left(\varphi\right),\psi_1\right\rangle_{\Gamma_I}+$$

$$+\left\langle A_1\left(1-k_1\frac{h}{2}\right)G_I\sigma_{v\tau}\left(\varphi\right),\psi_2\right\rangle_{\Gamma_I}.$$

It can be shown similarly to the case of linear elasticity that the operator S_1 is coercive, symmetric, linear and continuous on $H^{1/2}(\Gamma_I)$ [3, 6]. From the equivalence of the $H^{1/2}(\Gamma_I)$ and $L_2(\Gamma_I)$ norms with the use of Friedrichs' inequality, we obtain, that the operator S_1 is linear, continuous, symmetric and coercive on $H^1(\Gamma_I)$.

To conclude, the Steklov-Poincare operator S is linear, continuous, symmetric and coercive on $H^1(\Gamma_I)$ as the sum of the operators having such properties. By the Lax-Milgram lemma, our problem for the Steklov-Poincare operator has a unique solution on $H^1(\Gamma_I)$.

We remark that for the case of nonzero volume forces as well as nonzero boundary conditions, the proof can be carried out in a similar way.

Let Q, Q_1 and Q_2 be the corresponding preconditioners in the domain decomposition algorithm [6]. It is known, that in the case of Dirichlet-Neumann iterations these preconditioners can be expressed through S_1 and S_2 as [6]

$$\begin{gathered}Q=Q_1+Q_2,\\ \langle Q_1\varphi,\psi\rangle_{\Gamma_I}=\langle S_1\varphi,\psi\rangle_{\Gamma_I},\\ \langle Q_2\varphi,\psi\rangle_{\Gamma_I}=\langle S_2\varphi,\psi\rangle_{\Gamma_I}\end{gathered}\tag{10}$$

Since the Steklov-Poincare operators S_1 and S_2 are linear, continuous, symmetric and coercive on $H^1(\Gamma_I)$, we conclude that the operators Q, Q_1 and Q_2 also possess these properties.

Therefore, by the convergence of the Dirichlet-Neumann iterations, the following method is convergent for $0<\theta<\theta_{max}$:

$$\varphi^{k+1}=\varphi^k+\theta Q_2^{-1}\left(G-Q\varphi^k\right),\quad k=0,1,2,...$$

where G is the right-hand side of the equation $Q\varphi=G$.

It is worth mentioning that all the properties of the continuous operators can be transferred to the corresponding discrete operators, and in the case of quasi-uniform mesh, these properties also hold for the discrete operators [6].

4. Numerical example

In this section we consider a rectangular object lying in Ω that consists of a concrete main part in Ω_1 with a thin steel cover Ω_2 attached to its top. The physical dimensions are as follows: $x_1^b=0.05$, $x_2^b=0.05$, $x_1^e=1.05$, $x_2^e=0.55$, $h=0.02$. The physical parameters for the main part are $\nu=0.33$, $E=25000MPa$, for the shell – $\nu=0.33$, $E=200000MPa$. The body is kept fixed on both sides and subjected to the load on the bottom of $p = 1MPa/m^2$ (see Fig. 2) with zero load on top.

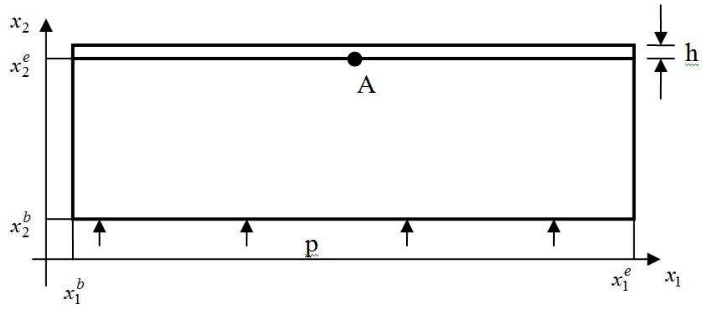

FIG. 2. Numerical Example

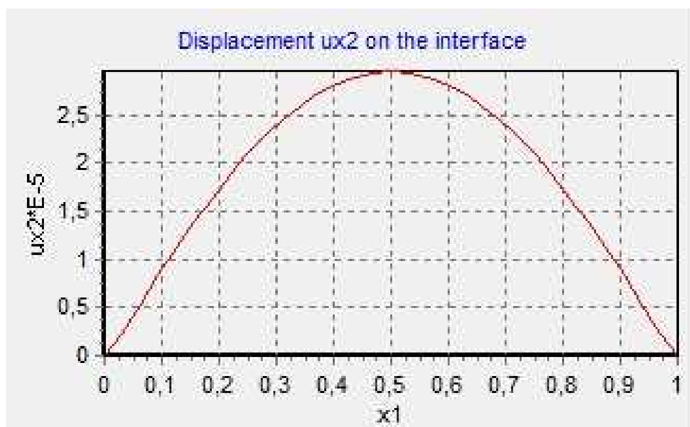

FIG. 3. Displacements in x_2 direction on the interface

The solution on each iteration in the main part is done by BEM with linear basis functions with the Galerkin method applied to integral representation formula [1]

$$\frac{1}{2}u_i = \int_\Gamma (F_{ij}(x,y)\,t_j(y))\,d\Gamma + \int_\Gamma (G_{ij}(x,y)\,u_j(y))\,d\Gamma, \quad i = 1,2,$$

where F_{ij} and G_{ij} are the Green's function and the co-normal derivative of Green's function respectively; $t_i = \sigma_{ij}n_j$ are the tractions.

The solution in Ω_2 is seeked as the linear combination of bubble basis functions which are defined on each element by

$$\Phi_0(\xi) = \frac{1-\xi}{2}, \quad \Phi_1(\xi) = \frac{1+\xi}{2}$$

$$\Phi_j(\xi) = \sqrt{\frac{2j-1}{2}}\int_{-1}^{\xi} P_{j-1}(t)\,dt, \quad j = 2,3...,$$

where $P_j(\xi)$ are the Legendre polynomials. The solution in both domains is then combined using the iterative algorithm (10).

For our example we choose 96 equally spaced boundary elements. The relaxation parameter θ is taken to be equal 0.00225

In Fig. 3 the displacement in x_2 direction along the interface is shown. The displacement achieves its maximum in the middle point A of the interface.

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