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Pursuit differential game of many pursuers and one evader in a convex hyperspace

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A pursuit differential game of many pursuers and one evader in a nonempty closed convex compact hyperspace is studied. Pursuit is completed when at least one pursuer coincides with the evader. Control functions of players are constrained by geometric constraints. A pursuit game in a set containing a closed convex compact set is solved, and pursuit is shown to be completed in a pursuit game within a finite-dimensional cube. Parallel strategy and fictitious pursuers are used to solve the game, and a guaranteed pursuit time is obtained.

Keywords: *differential game, parallel strategy, pursuit, cube, convex set.*

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1. Introduction

Differential game is basically a study about the mathematical strategy of a party to achieve its objective against an opposing party in the game. It is either pursuit or evasion differential game and was started by Rufus Isaacs in 1965 [1] and was further elaborated by Petrosjan [2] and Friedman [3]. Pursuit differential game is where a strategy is constructed for pursuer to capture evader (see for example, [1, 2, 4–11]). In many pursuit games, parallel strategy is used as one of the methods for pursuer to capture evader. It was first introduced by Petrosjan [2] as an admissible strategy for pursuit to be completed by any pursuer.

On the other hand, a strategy for evader is constructed in evasion differential game, to avoid being captured by pursuer (see for example, [12–18]).

Movement of players are controlled by their respective control function which is constrained either by geometric or integral constraint. Games with geometric constraint, for example, were discussed in [6, 12–14], and the games with integral constraint in [15–18].

Differential game was studied in various spaces such as Hilbert space (see for example, [5, 7, 19, 20]) and in $\mathbb{R}^n$ for some integer n (see for example, [1, 3, 5, 7, 19, 20]). The game is said to have a state constraint if trajectories of all players are only within a specific subset of $\mathbb{R}^n$, for some integer n. The following works [4, 8–11] have stated the constraint in the form of convex space in $\mathbb{R}^n$.

The game of pursuit of many pursuers versus one evader in a compact convex set was studied in [4]. In this work, the motions of all players were described by the following differential equation:

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad |u_i| \leqslant 1, \quad i = 0, 1, 2, \dots, m,$$

where $x_i, u_i, x_{i0} \in \mathbb{R}^n$, u_i is the control parameter of the ith pursuer and u_0 is the control parameter of the evader. During the game, all players must not leave a given compact convex subset A of $\mathbb{R}^n$ and their control functions are constrained by geometric constraint. It was proven that if the number m of pursuers is strictly less than the dimension n of the space, then evasion is possible. Otherwise, that is if

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$m \geqslant n$, then pursuit can be completed and the guaranteed pursuit time obtained is $d(n^3 - 2n^2 + n + 1)$, where d is the length of the side of the cube.

A two-person differential game in a convex space was then studied in [9] but with integral constraint on the control function of each player. The motion of players was described in a linear system as follows:

$$\dot{z}(t) = A(t)z + B(t)(v - u), \quad z(0) = z_0,$$

where $z, z_0, u, v \in \mathbb{R}^n$. The terminal time of the game was prefixed to the time when the pursuer and evader coincide by reaching a closed convex subset M and the game is completed. This work was then followed by a study on pursuit differential game of many pursuers and many evaders on a closed convex set of $\mathbb{R}^n$ for $n \geqslant 2$. The study was done by [8] with control function of each player is constrained by integral constraint, and motion of players were described as follows:

$$\begin{aligned} \dot{x}_i &= \varphi(t)u_i, \qquad x_i(0) = x_{i0}, \qquad i = 1, 2, \ldots, m, \\ \dot{y}_j &= \varphi(t)v_j, \qquad y_j(0) = y_{j0}, \qquad j = 1, 2, \ldots, k, \end{aligned}$$

where $x_i, y_j, u_i, v_j \in \mathbb{R}^n$, u_i and v_j are control parameters for the pursuers and evaders respectively. This work used a scalar function $\varphi(t)$ that satisfies the following expression:

$$a(\tau) \triangleq \left(\int_0^\tau \varphi^2(t)dt \right)^{1/2} < \infty, \quad \tau > 0, \quad \lim_{\tau \to \infty} a(\tau) = \infty.$$

It was shown that if the total resource of the pursuers is greater than that of the evaders, then pursuit can be completed. In this study, the concept of fictitious pursuers were used.

On the other hand, the work of [10] studied a pursuit differential game of many pursuers and one evader with geometric constraint. The work was motivated by [4] with the purpose of improving the guaranteed pursuit time obtained in [4]. The game was first studied in an n-dimensional cube in which the method of fictitious pursuers was used and motion of players were described as follows:

$$\begin{aligned} \dot{x}_i &= u_i, \qquad x_i(0) = x_{i0}, \qquad |u_i| \leqslant 1, \quad i = 1, 2, \ldots, m, \\ \dot{y} &= v, \qquad y(0) = y_0, \qquad |v| \leqslant 1. \end{aligned}$$

where $x_i, y, x_{i0}, y_0 \in \mathbb{R}^n$. The game was then studied in a nonempty closed bounded convex subset of $\mathbb{R}^n$ contained in the cube. Parallel strategy was used for each pursuer but with a different method of trajectory from the one in [4]. The guaranteed pursuit time was found to be $\frac{d}{2}(n^2 - n + \sqrt{n} + 1)$, where d is the length of the side of the cube. The time was an improved result from the result in [4].

Pursuit differential game in a convex space was studied by [11] with a different condition on the control function of each player, which was subjected to coordinate-wise integral constraint. It was a game in a convex set of many players against one evader. The motion of players were described by:

$$\begin{aligned} \dot{x}_i &= u_i, \quad x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, m, \\ \dot{y} &= v, \quad y(0) = y_0, \end{aligned}$$

where $x_i, y, u_i, v \in \mathbb{R}^2$. The game is said to be completed when at least one pursuer can catch the evader. Parallel strategy was applied by each pursuer to capture the evader. The sufficient condition for the completion of the game was obtained and the study also provided a guaranteed pursuit time.

The purpose of the current project is to solve a pursuit game with geometric constraint between finitely many pursuers against one evader, in a closed convex compact set contained in $\mathbb{R}^n$. The game is completed when at least one pursuer captures the evader. The first study is on a pursuit game where the pursuers can move everywhere in a set M that contain a closed convex compact set S, but the evader can only move within S. In this part, parallel strategy and the concept of fictitious pursuers are used.

The second part is on a pursuit game where all players are confined within an n-dimensional cube. A new method of trajectories for pursuers using parallel strategy and concept of fictitious pursuers, is constructed to obtain a sufficient condition for the pursuit to be completed with a guaranteed pursuit time.

The final part of this study use the results in the first two parts to obtain solution of a pursuit differential game of finitely many pursuers against one evader in a closed convex compact hyperspace, with an improved guaranteed pursuit time from the one in [10]. We note that in this article, real pursuer is meant to be pursuer, when it is necessary to distinguish from fictitious pursuer.

2. Notation and preliminary results

In this section we state some necessary basic definitions for this study.

Definition 1. *$G(A, B)$ is a game where all pursuers are in space A and evader in space B.*

Definition 2. *Control function for pursuer in the space A of the game is a measurable function $u_i(\cdot) = u_i(t)$, $t \geqslant 0$, if $| u_i(t) | \leqslant 1$, $t \geqslant 0$, and the solution $x_i(\cdot) = x_i(t)$, $t \geqslant 0$ of the initial value problem $\dot{x}_i = u_i(t)$, $x_i(0) = x_{i0}$, $i \in \{1, 2, \dots n\}$ satisfies the inclusion*

$$x_i(t) \in A.$$

Definition 3. *Control function for evader in the space A of the game is a measurable function $v(\cdot) = v(t)$, $t \geqslant 0$, if $| v(t) | \leqslant 1$, $t \geqslant 0$, and the solution $y(\cdot) = y(t)$, $t \geqslant 0$, of the initial value problem $\dot{y} = v(t)$, $y(0) = y_0$, satisfies the inclusion*

$$y(t) \in A.$$

Definition 4. *Strategy of the pursuer in the space A of the game is a function $U_i \colon \mathbb{R}^n \times \mathbb{R}^n \times H(0, 1) \to H(0, 1)$, such that $U_i = U_i(x_i, y, v)$ for $i \in \{1, 2, \dots, n\}$, for any control of the evader $v(t)$, if the initial value problem*

$$\dot{x}_i = U_i(x_i, y, v(t)), \quad x_i(0) = x_{i0}, \tag{1}$$

$$\dot{y} = v(t), \qquad y(0) = y_0, \tag{2}$$

has a unique absolutely continuous solution $(x_i(t), y(t))$, $t \geqslant 0$, with $x_i(t), y(t) \in A$, $t \geqslant 0$.

Definition 5. *Pursuit can be completed at some time $\tau \geqslant 0$ by the pursuer x_i, if $x_i(\tau) = y(\tau)$ for some $i \in \{1, 2, \dots, n\}$.*

Definition 6. *T is called guaranteed pursuit time if there exist strategies of pursuers $U_1, U_2, \dots, U_n$ such that for any control of the evader $v(\cdot)$, an equality $x_i(\tau) = y(\tau)$ holds for some $i \in \{1, 2, \dots, n\}$ and $0 \leqslant \tau \leqslant T$, where $(x_i(t), y(t))$ is the solution of (1), (2) at $v = v(t)$, $t \geqslant 0$.*

3. Main result

3.1. Pursuit game in a set containing a closed convex compact set

In this subsection, we study a pursuit game where fictitious pursuers can move within a set which is beyond the area of movement allowed for real pursuers and evader, which is a closed convex compact set. It is proved that pursuit can be completed in this game. It is also shown that, the guaranteed pursuit time of this game is the same for the guaranteed pursuit time of the pursuit game confined in the area of trajectories of the real pursuers and the evader.

Theorem 1. *Let M be a set containing a closed convex compact set S and $y(t) \in S$ for $t \geqslant 0$. Then pursuit can be completed in $G(M, S)$.*

Proof. Suppose $S \subset M$ and $y(t) \in S$ for $t \geqslant 0$. Now consider game $G(S, S)$ which means $x_i(t) \in S$ for $i = 1, 2, \dots, n$ and $y(t) \in S$ for $t \geqslant 0$. Note that S is contained in M and pursuers could only move within S for game $G(S, S)$ using parallel strategy. Therefore, in order to maintain the parallel movement of pursuers against evader at all time, we introduce fictitious pursuers $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$ that can move inside or outside of S, but still within M, such that

$$x_i(t) = F(\hat{x}_i(t)), \quad i = 1, 2, \ldots, n, \tag{3}$$

for time $t \geqslant 0$, and the function $F\colon M \to S$ is a projection function defined by the equation:

$$|F(\hat{x}_i) - \hat{x}_i| = \min_{a \in S} |a - \hat{x}_i|, \quad \hat{x}_i \in M. \tag{4}$$

By (4), it is clear that the line $x_i\hat{x}_i$ is a normal line at point x_i, on the border of S. Indeed, if $\hat{x}_i \in S$, then $|F(\hat{x}_i) - \hat{x}_i| = \min_{a \in S} |a - \hat{x}_i| = 0$ which implies $\hat{x}_i = F(\hat{x}_i) = x_i$. In other words, when movement of real pursuers are in S then fictitious pursuers coincide with the real pursuers, that is, if $\hat{x}_i(t) \in S$ then $\hat{x}_i(t) = x_i(t) = F(\hat{x}_i(t))$. On the other hand, when fictitious pursuers are outside S but still within M, the real pursuers will move according to the function F defined in equation (4).

As a note, the dynamics of the fictitious pursuer are defined by the following equations:

$$\dot{\hat{x}}_i = \hat{u}_i, \quad \hat{x}_{i0} = x_{i0}, \quad |\hat{u}_i| \leqslant 1, \quad i = 1, 2, \ldots, n, \tag{5}$$

where $\hat{x}_i(t), \hat{u}_i \in M$.

We note that the strategy of $\hat{x}_i$ in M is the same parallel strategy as that of x_i in S.

By the property of function F, $x_i(t)$ is an absolutely continuous function and hence differentiable almost everywhere in S. Thus, the control $u_i(t)$ of $x_i(t)$ is as follows:

$$\dot{x}_i(t) = \frac{d}{dt} x_i(t) = \frac{d}{dt} F(\hat{x}_i(t)) = u_i(t), \quad i = 1, 2, \ldots, n. \tag{6}$$

Indeed, the control $u_i(t)$ in S is admissible, that is $|u_i(t)| \leqslant 1$ as shown below:

$$\begin{aligned} |u_i(t)| &= \lim_{h \to 0^+} \left| \frac{x_i(t+h) - x_i(t)}{h} \right| \\ &= \lim_{h \to 0^+} \left| \frac{F(\hat{x}_i(t+h)) - F(\hat{x}_i(t))}{h} \right| \\ &\leqslant \lim_{h \to 0^+} \left| \frac{\hat{x}_i(t+h) - \hat{x}_i(t)}{h} \right| \\ &= |\hat{u}_i(t)| \\ &\leqslant 1. \end{aligned} \tag{7}$$

Finally, by the parallel strategy, pursuit can be completed in game $G(M, S)$. ■

The second theorem justifies the use of fictitious pursuers in finding the guaranteed pursuit time for the pursuit game in our study.

Theorem 2. *The guaranteed pursuit time for $G(S, S)$ and $G(M, S)$ are the same.*

Proof. Suppose pursuit can be completed in $G(M, S)$ with guaranteed time T. Hence, there exist strategies $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ in M by fictitous pursuers $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ such that $\hat{x}_i(\tau) = y(\tau)$ for some $i \in \{1, 2, \ldots, n\}$ and $0 \leqslant \tau \leqslant T$, where $\hat{x}_i(t) \in M$ for $t \in [0, T]$, $i = 1, 2, \ldots, n$. However, $y(\tau) \in S$, and so we obtain $y(\tau) = \hat{x}_i(\tau) = x_i(\tau) \in S$ and thus pursuit is achieved in $G(S, S)$ with the same guaranteed pursuit time T. ■

3.2. Many pursuers and one evader game in an n-dimensional cube

In this subsection, we begin by solving a pursuit game within a geometrical structure in the form of an n-dimensional cube with the center at the origin. The guaranteed pursuit time is obtained thru a new method in applying parallel strategy for the pursuers.

The pursuit differential game of geometric constraint with many pursuers against one evader, where movement of all players are confined within a cube N of side a, is defined as follows:

For $n \geqslant 3$,

$$N = \left\{ (q_1, q_2, \ldots, q_n) \middle| \frac{-a}{2} \leqslant q_i \leqslant \frac{a}{2},\ i = 1, 2, \ldots, n,\ a > 0 \right\}. \tag{8}$$

The center of the cube defined in our project is the origin, unlike the cube in [10]. Motion of players are described by the following differential equation;

$$\begin{aligned} x_i\colon \dot{x}_i = u_i, \quad & x_i(0) = x_{i0}, \quad |u_i| \leqslant 1, \quad i = 1, 2, \ldots, n, \\ y\colon \dot{y} = v, \quad & y(0) = y_0, \quad |v| \leqslant 1, \end{aligned} \tag{9}$$

where $x_i, y, u_i, v, x_{i0}, y_0 \in N$. Note that from (9), the maximum speed of each player is assumed to be 1. The following theorem is to be proven.

Theorem 3. *For $n \geqslant 3$, the guaranteed pursuit time in the pursuit differential game of n pursuers against one evader in the cube N is the time*

$$T = \frac{a}{2}\left(\sqrt{n} + n\sqrt{n-1} + n\right).$$

Proof.

I. **Constructed Strategy.**

We divide the motion of each pursuer into two stages. The strategy of movement of each pursuer is different from the one in [10], especially in Stage 2. The difference here is largely based on movement of players to catch the projection of the evader in a space, rather than on the side of cube.

Stage 1.

To consider the guaranteed pursuit time, we first assume that $x_{i0} \neq (0, 0, \ldots, 0)$ for $i = 1, 2, \ldots, n$ and each pursuer x_i starts from time zero. Also, all pursuers move on the time interval $[0, |x_{i0}|]$ along their respective distinct straight line towards the center $O = (0, 0, \ldots, 0)$ of the cube N. The control function of each x_i in this stage is then defined as follows:

$$u_i(t) = -\frac{x_{i0}}{|x_{i0}|}, \quad 0 \leqslant t \leqslant |x_{i0}|, \quad i = 1, 2, \ldots, n. \tag{10}$$

Note that $|x_{i0}| \neq 0$ since $x_{i0} \neq (0, 0, \ldots, 0)$ and also at time $t_i = |x_{i0}|$, the pursuer x_i reaches the point O. The control function $u_i(t)$ is a unit vector which means, each pursuer moves with maximum speed 1.

Stage 2.

Stage 2 describes a new method of constructing the strategy for pursuer x_i when $t > t_i$. In [10], movement of pursuers after reaching the center of its cube, are on the side of the cube but here, each pursuer moves from the origin O of N and captures the projection of the evader on the $(n-1)$-dimensional cube Q_i, $i \in \{1, 2, \ldots, n\}$ defined as follows:

$$Q_i = \left\{(s_1, s_2, \ldots, s_n) \,\middle|\, s_i = 0, -\frac{a}{2} \leqslant s_j \leqslant \frac{a}{2},\ s_j \in \mathbb{R},\ j \neq i,\ j \in \{1, 2, \ldots, n\}\right\}. \tag{11}$$

From the definition above, note that the length of the diagonal of Q_i is $a\sqrt{n-1}$ and thus, at any time of the game, the distance of each pursuer from the center O of N is at most $\frac{a\sqrt{n-1}}{2}$.
Also observe that the center of each Q_i is the same as the center O of N, and each Q_i is perpendicular to q_i-axis, $i \in \{1, 2, \ldots, n\}$.
Now for $t > t_i$, pursuer x_i moves from stage 1 to stage 2. Hence, time $t_i = |x_{i0}|$ is called switch time and x_i moves from the center of Q_i and capture the projection of the evader y on Q_i. We consider the projection $z_i(t)$ of $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ on Q_i as $z_i(t) = (y_1(t), y_2(t), \ldots, y_{i-1}(t), 0, y_{i+1}(t), \ldots, y_n(t))$, with $y_i(t_i) = y_{i0}$, and $z_{i0} = z_i(t_i) = (y_{10}, y_{20}, \ldots, y_{i-1,0}, 0, y_{i+1,0}, \ldots, y_{n0})$.
Also for $t > t_i$, the velocity of the evader on Q_i is $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$, and the velocity $w_i(t)$ of projection $z_i(t)$ is $w_i(t) = (v_1(t), v_2(t), \ldots, v_{i-1}(t), 0, v_{i+1}(t), \ldots, v_n(t))$.
The pursuit differential game in Q_i is then described by the following equations:

$$
\begin{aligned}
&x_i\colon \dot{x}_i = u_i, \quad x_i(t_i) = x_{i0}, \quad |u_i| \leqslant 1, \quad i = 1, 2, \ldots, n,\\
&z_i\colon \dot{z}_i = w_i, \quad z_i(t_i) = z_{i0}, \quad |w_i| \leqslant 1,
\end{aligned}
\tag{12}
$$

where $x_i, z_i, u_i, w_i, x_{i0}, z_{i0} \in Q_i$.

Observe that $|w_i|^2 = |v|^2 - v_i^2$ for $i = 1, 2, \ldots, n$. Let τ_i be the first time when $x_i(\tau_i) = z_i(\tau_i)$, that is, the time when x_i captures the projection z_i of the evader in Q_i. Next, we specify the strategy of x_i in Q_i when $x_i(t) \neq z_i(t)$ as

$$
u_i = w_i - (w_i, e_i)e_i + e_i\sqrt{1 - |w_i|^2 + (w_i, e_i)^2}, \quad t_i < t \leqslant \tau_i, \tag{13}
$$

where $e_i = \frac{z_{i0}}{|z_{i0}|}$.

Now let $\overline{Q_i} = \{(s_1, s_2, \ldots, s_n) \,|\, s_i = 0,\ s_j \in \mathbb{R},\ j \neq i,\ j \in \{1, 2, \ldots, n\}\}$ be an unbounded $(n-1)$-dimensional hyperplane that contain Q_i. By Theorem 1, the pursuit can be completed in $G(\overline{Q_i}, Q_i)$, that is, $z_i(t') = x_i(t')$ for some $t' > t_i$.

Now, since Q_i is an $(n-1)$-dimensional cube and the length of its diagonal is $a\sqrt{n-1}$, then for $t > t_i$, we have the dot product

$$
(z_i(t) - x_i(t), e_i) \leqslant \frac{a\sqrt{n-1}}{2}.
$$

This implies that if x_i travel the distance of $a\frac{\sqrt{n-1}}{2}$, then the projection z_i of the evader is already captured at time t', meaning that pursuit is completed and thus,

$$
|x_i(t')| \leqslant \frac{a\sqrt{n-1}}{2}. \tag{14}
$$

Therefore,

$$
(x_i(t'), e_i) \leqslant \frac{a\sqrt{n-1}}{2}.
$$

For $t > \tau_i$, pursuer x_i will pursue the real evader y within the hypercube N and use the following strategy,

$$
u_i = \xi_i\sqrt{1 - |w_i|^2}, \tag{15}
$$

where

$$
\xi_i = \begin{cases} 1, & y_i(\tau_i) > 0,\\ -1, & y_i(\tau_i) < 0. \end{cases}
$$

II. **Guaranteed Pursuit Time.**

Next, we calculate the guaranteed pursuit time for the game within the cube. In stage 1, the distance D_1 between all pursuers and the center O of N is estimated from above as

$$
D_1 \leqslant \frac{1}{2}a\sqrt{n}. \tag{16}
$$

All pursuers move along distinct respective straight lines at the same time from their respective initial positions to the center O of N. Since $0 < |x_{i0}| \leqslant \frac{1}{2}a\sqrt{n}$, each pursuer is at the center O of N by the time $T_1 = \frac{1}{2}a\sqrt{n}$, where each pursuer moves with maximum speed 1.

In stage 2, strategies (13) and (15) are parallel strategies of which pursuit can be completed. Recall that τ_i is the time such that $z_i(\tau_i) = x_i(\tau_i)$ for $i = 1, 2, \ldots, n$. To obtain the guaranteed pursuit time for the whole game, we assume $z_{i0} \neq 0$ for $i = 1, 2, \ldots, n$.

Now, for $t_i < t \leqslant \tau_i$, the total distance D_2 between the pursuers and the projections of the evader is estimated from above as follows:

$$
D_2 \leqslant \sum_{i=1}^{n} \frac{a\sqrt{n-1}}{2} = \frac{1}{2}an\sqrt{n-1}, \tag{17}
$$

where the diagonal of Q_i for $i = 1, 2, \ldots, n$ is $a\sqrt{n-1}$.

For $t > \tau_i$, the total distance D_3 between the pursuers and the evader is estimated from above as follows:

$$D_3 \leqslant \sum_{i=1}^{n} \frac{a}{2} = \frac{an}{2}. \tag{18}$$

Note that the length of the side of N is a.

By strategies (13) and (15) and noting that $0 \leqslant \sqrt{1-|w_i|^2} \leqslant 1$, these distance decreases at the rate of $\alpha(t)$ which is calculated as follows:

$$\begin{aligned}
\alpha(t) &= \sum_{i\colon t_i<t\leqslant\tau_i}^{n} \left(\sqrt{1-|w_i|^2+(w_i,e_i)^2}\right) + \sum_{i\colon t>\tau_i}^{n} \left(\sqrt{1-|w_i|^2}\right) \\
&\geqslant \sum_{i\colon t>\tau_i}^{n} (1-|w_i|^2) \\
&= \sum_{i\colon t>\tau_i}^{n} \left[1-\left(|v|^2-v_i^2\right)\right] \\
&\geqslant 1.
\end{aligned} \tag{19}$$

Hence by stage 1 and 2, the total time T' taken by n pursuers to catch the evader is estimated from above as:

$$\begin{aligned}
T' &\leqslant \frac{D_1+D_2+D_3}{1} \\
&\leqslant \frac{a}{2}\left(\sqrt{n}+n\sqrt{n-1}+n\right)
\end{aligned} \tag{20}$$

and hence, the guaranteed pursuit time is $T = \frac{a}{2}\left(\sqrt{n}+n\sqrt{n-1}+n\right)$. ■

3.3. Guaranteed pursuit time of the game in a closed convex compact set

Any closed convex compact set S can be contained in the hypercube N defined by (8). Motion of the real pursuers and the evader only occur within S but the fictitious pursuers can move either inside or outside of S but still within N at all time. If a fictitious pursuer $\hat{x}_i$ for $i \in \{1,2,\ldots,n\}$ is inside S then it coincides with the real pursuer x_i and they move as one player. On the other hand, if $\hat{x}_i$ is outside S but still inside N, the corresponding real pursuer x_i will move according to the projection of $\hat{x}_i$ in S as described by (3). Therefore, we have game $G(S,S)$ and $G(N,S)$. We note that $\hat{x}_{i0} = x_{i0}$ for $i = 1,2,\ldots,n$ and the strategy of $\hat{x}_i$ and x_i is the same as described by (13). Now, we state our main result which is the guaranteed pursuit time in $G(S,S)$.

Theorem 4. *Let S be a closed convex compact set of diameter a which is contained in a hypercube N of side a in $\mathbb{R}^n$. Then, a guaranteed pursuit time in the pursuit differential game $G(S,S)$ of n pursuers against one evader in S for $n \geqslant 3$, is the time*

$$T = \frac{a}{2}\left(\sqrt{n}+n\sqrt{n-1}+n\right).$$

Proof. Since $S \subset N$, then by Theorem 2 and Theorem 3, the result is obtained. ■

4. Conclusion

In this project, it is proven that pursuit can be completed for a pursuit game that occur in a set that contain a closed convex compact set, with its guaranteed pursuit time, is the same as the guaranteed pursuit time for the pursuit game confined within the closed convex compact set. This result leads to our finding of the guaranteed pursuit time T for a pursuit differential game in any closed convex compact hyperspace by solving a pursuit problem within a hypercube with side a and of n-dimension that contain the convex hyperspace. The guaranteed pursuit time obtained is $T = \frac{a}{2}\left(\sqrt{n}+n\sqrt{n-1}+n\right)$.

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Диференціальна гра переслідування одного утікача багатьма переслідувачами в опуклому гіперпросторі

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Досліджено диференціальну гру переслідування одного утікача багатьма переслідувачами в непорожньому замкненому опуклому компактному гіперпросторі. Переслідування вважається завершеним, якщо хоча б один переслідувач співпадає з утікачем. Функції керування гравцями обмежені геометрично. Розв'язується гра переслідування у множині, що містить замкнений опуклий компактний набір, і показано, що переслідування завершується у межах скінченновимірного куба. Для розв'язання гри використовуються паралельна стратегія та фіктивні переслідувачі, що забезпечує гарантований час переслідування

Ключові слова: *диференціальна гра, паралельна стратегія, переслідування, куб, опукла множина.*