DOI 10.15673/2072-9812.3/2014.40574
Proc. Intern. Geom. Center 2014 **7**(3) 48–57 *dω*

Hyperspaces and spaces of probability measures on $\mathbb{R}$-trees

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Abstract We prove that the "sliced" hyperspaces and spaces of probability measures of the rooted $\mathbb{R}$-trees are also rooted $\mathbb{R}$-trees.

Keywords $\mathbb{R}$-tree, hyperspace, probability measure

Mathematics Subject Classification (2010)37E25, 54B20, 60B05

1 Introduction

The real trees ($\mathbb{R}$-trees) were introduced by Tits [12]. Since then, they found numerous applications in different parts of mathematics. In particular, Kirk [9] established connections between $\mathbb{R}$-trees and the hyperconvex metric spaces introduced by Aronszajn and Panitchpakdi [1].

Some applications of $\mathbb{R}$-trees are also described in [2]. In particular, it is mentioned that $\mathbb{R}$-trees arise also in the coarse setting of word-hyperbolic groups.

Outside of mathematics, $\mathbb{R}$-trees are used in biology, medicine and computer science. In particular, applications in biology and medicine are related to the notion of phylogenetic tree [11].

In the paper [7], connections between geodesically complete rooted $\mathbb{R}$-trees and ultrametric spaces are established. The results of [7] are formulated in terms of categorical equivalence. This makes reasonable studying functorial constructions in appropriate categories of $\mathbb{R}$-trees. In the present note, we consider the hyperspaces and the spaces of probabilily measures of rooted $\mathbb{R}$-trees that are also rooted $\mathbb{R}$-trees.

A geodesic segment with endpoint $x, y \in X$ is the image of an isometric embedding $\alpha\colon [0, d(x, y)] \to X$. By $[x, y]$ we denote a geodesic segment with endpoints x and y.

Definition 1 We say that a metric space (X, d) is a geodesic space if for every $x, y \in X$ there exists a geodesic joining x and y.

Definition 2 A metric space (X, d) is called an $\mathbb{R}$-tree if

1. (X, d) is a geodesic space;
2. if $[x, y] \cap [x, z] = \{x\}$, then $[y, z] = [x, y] \cup [x, z]$;
3. for every $x, y, z \in X$ there exists $w \in X$ such that $[x, y] \cap [x, z] = [x, w]$.

It is known that a geodesic metric space X is an $\mathbb{R}$-tree if and only if X is 0-hyperbolic. It is also known that a geodesic space is an $\mathbb{R}$-tree if and only if for every two distinct points x, y of this space there exists a unique arc with endpoints x, y.

Definition 3 A rooted $\mathbb{R}$-tree consists of an $\mathbb{R}$-tree (X, d) and a point $x_0 \in X$ called the root.

Definition 4 A rooted $\mathbb{R}$-tree (X, d, x_0) is geodesically complete if every isometric embedding $f\colon [0, t] \to X$, where $t > 0$, with $f(0) = x_0$, extends to an isometric embedding $\bar{f}\colon [0, \infty) \to X$.

In this case the map $\bar{f}$ is said to be a geodesic ray.

Given a rooted $\mathbb{R}$-tree (X, d, x_0), we let $|x| = d(x, x_0)$, for every $x \in X$. For every $t > 0$, let $X_t = \{y \in X \mid |y| = t\}$ and $X_{\le t} = \cup\{X_s \mid s \le t\}$. If $0 \le s \le t$, we define a map $\pi_{ts}\colon X_t \to X_s$ by the condition $\pi_{ts}(x) = y$ if $\{y\} = [x, x_0] \cap X_s$. Remark that π_{ts} is uniquely determined.

Also, we define a retraction $\pi_t\colon X \to X_{\le t}$ by the condition $\pi_t(x) = \pi_{st}(x)$, for every $x \in X_s$, where $s \ge t$.

Recall that a metric ϱ on a set Z is said to be an ultrametric if it satisfies the following *strong triangle inequality*:

$$\varrho(x, y) \le \max\{\varrho(x, z), \varrho(z, y)\},\ x, y, z \in Z.$$

Lemma 1 *The restriction of the metric d onto X_t is an ultrametric.*

Proof Let $x, y, z \in X_t$. There exist $a, b \in X$ such that $[x, x_0] \cap [y, x_0] = [a, x_0]$, $[y, x_0] \cap [z, x_0] = [b, x_0]$. Without loss of generality, one may suppose that $[b, x_0] \subset [a, x_0]$. Then $[x, a] \cup [a, b] \cup [b, z]$ is a geodesic segment containing x and z.

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Since $d(x,y) = 2d(x,a)$, $d(y,z) = 2d(y,b)$, and

$$d(x,z) = d(x,a) + d(a,b) + d(b,z) = d(x,b) + d(b,z),$$

we conclude that $d(x,z) \leq d(y,z) = \max\{d(x,y), d(y,z)\}$.

Denote by $\mathbb{R}$**-TREE** the category whose objects are rooted $\mathbb{R}$-trees and whose morphisms are $|\cdot|$-preserving continuous maps.

2 Hyperspaces

Given a metric space (X,d), by $\exp X$ we denote the hyperspace of X, i.e. the set of all nonempty compact subsets of X. We endow $\exp X$ with the Hausdorff metric d_H,

$$d_H(A,B) = \inf\{r > 0 \mid A \subset O_r(B),\ B \subset O_r(A)\},$$

where $O_r(C)$ denotes the r-neighborhood of $C \in \exp X$. For every $n \in \mathbb{N}$, denote by $\exp_n X$ the subspace

$$\{A \in \exp X \mid \text{ the cardinality of } A \text{ is at most } n\}$$

of $\exp X$.

In the sequel, we suppose that (X, d, x_0) is a rooted $\mathbb{R}$-tree. Let

$$\tilde{\exp} X = \{A \in \exp X \mid A \subset X_t \text{ for some } t > 0\}.$$

Given $A \in \tilde{\exp} X$, we write $|A| = t$ whenever $A \subset X_t$. By $\tilde{d_H}$ we denote the restriction of the Hausdorff metric onto the subspace $\tilde{\exp} X$.

Let us consider the function $\tilde{d}$: $\tilde{\exp} X \times \tilde{\exp} X \to \mathbb{R}$ defined as follows:

$$\tilde{d}(A,B) = \inf\{|A| + |B| - 2u \mid \pi_{|A|u}(A) = \pi_{|B|u}(B)\}.$$

Lemma 1 *The metric* $\tilde{\exp} X$ *on* $\tilde{\exp} X$ *coincides with the function* $\tilde{d}$.

Proof Let $A, B \in \tilde{\exp} X$, $|A| = t$, $|B| = s$. Suppose that $\tilde{d}(A,B) = r$, then there exists a unique $u \in \mathbb{R}_+$ such that $(t-u) + (s-u) = r$ C– $\pi_{tu}(A) = \pi_{su}(B)$.

Let $a \in A$, then there exists $b \in B$ such that $\pi_{tu}(a) = \pi_{su}(b)$. We conclude that $d(a,b) = t + s - 2u = r$.

Similarly, for every $b \in B$ there exists $a \in A$ such that $d(a,b) = r$.

Summing up, $\tilde{d}_H(A,B) \leq r$.

Conversely, if $\tilde{d}_H(A,B) \leq r$, then for every $a \in A$ there exists $b \in B$ such that $d(a,b) \leq r$. Let $[a,b]$ be a geodesic segment connecting a and b. Let c be a point of this segment with the minimal norm. Then $t - |c| + s - |c| \leq r$ and therefore $|c| \geq u = \frac{1}{2}(t + s - r)$.

It is easy to see that then $\pi_{tu}(A) = \pi_{su}(B)$, and therefore $\tilde{d}(A,B) \leq r$.

Corollary 1 *The space* $\tilde{\exp}X_t$ *is zero-dimensional for every* $t > 0$.

Proposition 1 *The map* $|\cdot|\colon \tilde{\exp}X \to \mathbb{R}_+$ *is nonexpanding.*

Proof Let $A, B \in \tilde{\exp}X$, $|A| = t$, $|B| = s$. Then there exists $r \le \min\{t, s\}$ such that

$$\tilde{d_H}(A, B) = |t-r| + |s-r| = |t-r| + |r-s| \ge |t-r+r-s| = |t-s|$$

and we are done.

Proposition 2 *For every* $\mathbb{R}$*-tree* X *the space* $\tilde{\exp}X$ *is geodesic.*

Proof Let $A, B \in \tilde{\exp}X$, $|A| = t$, $|B| = s$, and $\tilde{d_H}(A, B) = c$. Then there exists $r \le \min\{t, s\}$ such that $\pi_{tr}(A) = \pi_{sr}(B)$ and $|t-r| + |s-r| = c$.

Consider a map $\gamma\colon [0, c] \to \tilde{\exp}X$ defined by the formula:

$$\gamma(x) = \begin{cases} \pi_{t,t-x}(A), \text{ if } x \in [0, t-r], \\ \pi_{s,s-c+x}(B), \text{ if } x \in [t-r, c]. \end{cases}$$

Then $\gamma(0) = \pi_{t,t}(A) = A$, $\gamma(c) = \pi_{s,s}(B) = B$.

It is easy to see that γ is a geodesic segment that connects A and B.

Proposition 3 *Let* $\gamma\colon [0, 1] \to \tilde{\exp}X$ *be an embedding. Then the function* $t \mapsto |\gamma(t)|$ *satisfies one of the three conditions:*

1. *it is increasing;*
2. *it is decreasing;*
3. *it is decreasing on* $[0, t_0]$ *and is increasing on* $[t_0, 1]$, *for some* $t_0 \in [0, 1]$.

Proof If none of the condition holds, then there exist $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, for which $|\gamma(t_1)| = |\gamma(t_2)|$ and $|\gamma(t)| \ge |\gamma(t_1)|$, for all $t \in [t_1, t_2]$.

Let $|\gamma(t_1)| = c$. Then the map $t \mapsto \pi_c\gamma(t)$, $t \in [t_1, t_2]$, is a map into a zero-dimensional space, and therefore is a constant map. Thus, $\gamma(t_1) = \gamma(t_2)$. This contradicts to the fact that γ is an embedding.

Proposition 4 *The space* $\tilde{\exp}X$ *does not contain an embedded* S^1.

Proof Otherwise, there exist $A, B \in \tilde{\exp}X$ and a geodesic $\gamma\colon [0, d_H(A, B)]$ such that $|\gamma(t)| \ge |A| = |B|$, for every $t \in [0, d_H(A, B)]$. However, this contradicts to Proposition 3.

Corollary 2 *The space* $\tilde{\exp}X$ *is an* $\mathbb{R}$*-tree.*

Proof This follows from the known characterization of $\mathbb{R}$-trees; see, e.g., [10].

DOI 10.15673/2072-9812.3/2014.40574

Proposition 5 *The set* $\tilde{\exp} X$ *is a closed subset in the hyperspace* $\exp X$.

Proof Since the map $f\colon X \to \mathbb{R}_+$, $f(x) = |x|$, is continuous, so is the map $\exp f\colon \exp X \to \exp \mathbb{R}_+$. Then

$$\tilde{\exp} X = (\exp f)^{-1}(\{\{t\} \mid t \in \mathbb{R}_+\}) = (\exp f)^{-1} \exp_1(\mathbb{R}_+)$$

and therefore is closed.

Corollary 3 *For every complete rooted* $\mathbb{R}$*-tree* X*, the* $\mathbb{R}$*-tree* $\tilde{\exp} X$ *is complete.*

The following example demonstrates that the $\mathbb{R}$-tree $\tilde{\exp} X$ is not geodesically complete even for a geodesically complete $\mathbb{R}$-tree X. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1],\ y = 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid x \in \{1\} \cup \{(n-1)/n \mid n \in \mathbb{N}\},\ y \in \mathbb{R}_+\}$, we endow X with the geodesic metric inherited from $\mathbb{R}^2$.

We suppose that $(0, 0)$ is the root. Then X_1 is homeomorphic to a convergent sequence. It is easy to see that, for every $r > 0$, the space X_{1+r} is a countable discrete space.

Define $\gamma\colon [0, 1] \to \tilde{\exp} X$ by the formula $\gamma(t) = X_t$. Then this geodesic segment cannot be extended onto the set $\mathbb{R}_+$.

The hyperspace construction determines an endofunctor in the category $\mathbb{R}$-**TREE**.

3 Probability measures on $\mathbb{R}$-trees

Let $P(X)$ denote the set of probability measures of compact support on a space X. It is known that the construction of probability measures of compact support determines a functor on the category of Tychonov spaces and continuous maps [3]. If (X, d) is a metric space, then the set $P(X)$ can be endowed with the Kantorovich metric [8]; if d is an ultrametric, then the set $P(X)$ can be endowed with an ultrametric d_{HV},

$$d_{HV}(\mu, \nu) = \inf\{r > 0 \mid \mu(B_r(x)) = \nu(B_r(x)),\ \text{for every } x \in X\}$$

(see [5, 13]; here $B_r(x)$ denotes the r-ball centered at x). Some categorical properties of this metric were investigated in [6].

If X is a rooted $\mathbb{R}$-tree, we let

$$\tilde{P}(X) = \{\mu \in P(X) \mid \operatorname{supp}(\mu) \in \tilde{\exp} X\}.$$

Given $\mu \in \tilde{P}(X)$, let $|\mu| = |\operatorname{supp}(\mu)|$.

We endow $\tilde{P}(X)$ with a metric $\hat{d}$:

$$\hat{d}(\mu,\nu)=\inf\{|\mu|+|\nu|-2s \mid s\in[0,\min\{|\mu|,|\nu|\}],\ P(\pi_s)(\mu)=P(\pi_s)(\nu)\}.$$

Note first that the function $\hat{d}$ is well-defined. Indeed, $P(\pi_0)(\mu)=P(\pi_0)(\nu)=\delta_{x_0}$, for every $\mu,\nu\in\tilde{P}(X)$.

If $\hat{d}(\mu,\nu)=0$, then there exists a sequence (s_i) in $\mathbb{R}_+$ such that $|\mu|+|\nu|-2s_i\to 0$ and $s_i\le\min\{|\mu|,|\nu|\}$. This implies, in particular, that $\lim_{i\to\infty}s_i=|\mu|=|\nu|$.

Since the sequence of maps (π_{s_i}) uniformly converges to $\pi_{|\mu|}$, we obtain

$$\begin{aligned}\mu=&P(\pi_{|\mu|})(\mu)=P(\lim_{i\to\infty}\pi_{s_i})(\mu)=\lim_{i\to\infty}P(\pi_{s_i})(\mu)\\ &=\lim_{i\to\infty}P(\pi_{s_i})(\nu)=P(\lim_{i\to\infty}\pi_{s_i})(\nu)=\nu.\end{aligned}$$

Symmetry of the function $\hat{d}$ is obvious.

We are going to verify the triangle inequality. Let $\mu,\nu,\tau\in\tilde{P}(X)$, then there exist sequences

$$s_i\in[0,\min\{|\mu|,|\nu|\}],\ t_i\in[0,\min\{|\nu|,|\tau|\}]$$

such that

$$P(\pi_{s_i})(\mu)=P(\pi_{s_i})(\nu),\ P(\pi_{t_i})(\nu)P(\pi_{t_i})(\tau)$$

and

$$\hat{d}(\mu,\nu)=\lim_{i\to\infty}(|\mu|+|\nu|-2s_i),\hat{d}(\nu,\tau)=\lim_{i\to\infty}(|\nu|+|\tau|-2t_i).$$

Without loss of generality, one may assume that $s_i\le t_i$, for all $i\in\mathbb{N}$. Then $P(\pi_{s_i})(\mu)=P(\pi_{s_i})(\nu)=P(\pi_{s_i})(\tau)$ and we obtain

$$\hat{d}(\mu,\tau)\le\lim_{i\to\infty}(|\mu|+|\tau|-2s_i)\le\lim_{i\to\infty}(|\mu|+|\nu|-2s_i+|\nu|+|\tau|-2t_i)$$

(since $2|\nu|-2t_i\ge 0$)

$$\le\hat{d}(\mu,\nu)+\hat{d}(\nu,\tau).$$

Proposition 1 *The restriction of the metric $\hat{d}$ on the set $\tilde{P}(X)_t$ is an ultrametric for every $t\in\mathbb{R}_+$.*

Proof If $\mu,\nu,\tau\in\tilde{P}(X)_t$, then there exist $s_i,t_i\in\mathbb{R}_+$ such that

$$P(\pi_{s_i})(\mu)=P(\pi_{s_i})(\nu),\ P(\pi_{t_i})(\nu)P(\pi_{t_i})(\tau)$$

and

$$\hat{d}(\mu,\nu)=\lim_{i\to\infty}(|\mu|+|\nu|-2s_i),\hat{d}(\nu,\tau)=\lim_{i\to\infty}(|\nu|+|\tau|-2t_i).$$

Without loss of generality, one may assume that $s_i\le t_i$, for all $i\in\mathbb{N}$. Then $P(\pi_{s_i})(\mu)=P(\pi_{s_i})(\nu)=P(\pi_{s_i})(\tau)$ and we obtain

$$\hat{d}(\mu,\tau)\le\max\{\hat{d}(\mu,\nu),\hat{d}(\nu,\tau)\}.$$

We can prove even more, namely

Proposition 2 *The restriction of the metric $\hat{d}$ on the set $\tilde{P}(X)_t$ coincides with the metric d_{HV}, for every $t \in \mathbb{R}_+$.*

Proof Suppose that $d_{HV}(\mu,\nu) < r$. Then, for every $x \in X$, $\mu(B_r(x)) = \nu(B_r(x))$. We are going to show that $P(\pi_{t,t-(r/2)})(\mu) = P(\pi_{t,t-(r/2)})(\nu)$.

Indeed,

$$P(\pi_{t,t-(r/2)})(\mu) = \sum_{i=1}^{k} \mu(B_r(x_i))\delta_{\pi_{t,t-(r/2)}(x_i)}, \tag{1}$$

where $x_1, \dots, x_k \in \operatorname{supp}(\mu)$ are such that $\{B_r(x_i) \mid i = 1, \dots, k\}$ is a disjoint cover of $\operatorname{supp}(\mu)$. It is easy to see that the right-hand side is well-defined, i.e. does not depend on the choice of $x_1, \dots, x_k$. Applying the same arguments to the measure ν one easily concludes that the right-hand side of (1) is equal to $P(\pi_{t,t-(r/2)})(\nu)$.

On the other hand, suppose that $\hat{d}(\mu,\nu) < r$. Then $P(\pi_{t,t-(r/2)})(\mu) = P(\pi_{t,t-(r/2)})(\nu)$ and therefore, for every $x \in X_{t-(r/2)}$ and every $\varepsilon > 0$, we have

$$P(\pi_{t,t-(r/2)})(\mu)(B_\varepsilon(x)) = P(\pi_{t,t-(r/2)})(\nu)(B_\varepsilon(x)).$$

Then $\mu(B_{r+\varepsilon}(x)) = \nu(B_{r+\varepsilon}(x))$, for every $x \in X$ and therefore $d_{HV}(\mu,\nu) \le r+\varepsilon$, for every $\varepsilon > 0$. Thus, $d_{HV}(\mu,\nu) \le r$.

Denote by $\tilde{d}$ the restriction of the metric $\hat{d}$ onto $\tilde{P}(X)$.

Theorem 1 *The metric space $(\tilde{P}(X), \tilde{d})$ is an $\mathbb{R}$-tree.*

Proof The proof of this fact can be performed analogously to that of Corollary 3. We use the fact that the metric d_{HV} is an ultrametric.

Proposition 3 *The map* $\operatorname{supp}\colon \tilde{P}(X) \to \tilde{\exp} X$ *is nonexpanding.*

Proof Suppose that $\hat{d}(\mu,\nu) < r$, for some $r > 0$. Then there exists $c \in [0, \min\{|\mu|, |\nu|\}]$ such that $P(\pi_c)(\mu) = P(\pi_c(\nu)$ and $|\mu| + |\nu| - 2c < r$.

Then $|\operatorname{supp}(\mu)| = |\mu|$, $|\operatorname{supp}(\nu)| = |\nu|$ and $\pi_c(\operatorname{supp}(\mu)) = \pi_c(\operatorname{supp}(\nu))$, whence

$$\tilde{d}_H(\operatorname{supp}(\mu), \operatorname{supp}(\nu)) \le |\operatorname{supp}(\mu)| + |\operatorname{supp}(\nu)| < r.$$

Theorem 2 *Let X be a complete $\mathbb{R}$-tree. Then $\tilde{P}(X)$ is also a complete $\mathbb{R}$-tree.*

Proof Let (μ_i) be a Cauchy sequence in $\tilde{P}(X)$. Since, by Corollary 3, the space $\tilde{\exp} X$ is complete and the map supp is nonexpanding, the Cauchy sequence $(\mathrm{supp}(\mu_i))$ is convergent.

We follow the idea of the proof of [5, Theorem 3.5]. Define $\mu \in P(X)$ as follows. Let $x \in A$ and $r > 0$. We put $\mu(B_r(x)) = \lim_{i\to\infty} \mu_i(B_r(x))$.

Since (μ_i) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $\mu_m(B_r(x)) = \mu_n(B_r(x))$, for every $m, n > n_0$. This means that the sequence $\mu_i(B_r(x))$ is eventually constant and, therefore, is convergent. Clearly, the function μ, which is defined on the balls, uniquely extends to a probability measure; we keep the notation μ for the latter.

By the definition, $\mu = \lim_{i\to\infty} \mu_i$.

Similarly as above one can demonstrate that the $\mathbb{R}$-tree $\tilde{P}(X)$ is not necessarily geodesically complete even if so is X. Actually, the example at the end of the previous section works.

The construction of space of probability measures determines an endofunctor in the category $\mathbb{R}$-**TREE**. The class of maps supp comprises a natural transformation from $\tilde{\exp}$ to $\tilde{P}$.

4 Open problems

In [7], the category of geodesically complete, rooted $\mathbb{R}$-trees and equivalence classes of isometries at infinity is introduced. This leads to the following question.

Question 1 Are there counterparts of the hyperspace functor and the probability measure functor in the mentioned category?

The notion of ultrametric has its counterpart in the theory of fuzzy metric spaces (see [4]). A continuous operation $(a, b) \mapsto a * b\colon [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm, if $*$ is associative, commutative, monotonic and 1 is its neutral element.

A function $M\colon X \times X \times (0, \infty) \to [0, 1]$ is said to be a fuzzy metric on a set X, if it satisfies the following conditions: (i) $M(x, y, t) > 0$; (ii) $M(x, y, t) = 1$ if and only if $x = y$; (iii) $M(x, y, t) = M(y, x, t)$; (iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$; (v) the function $M(x, y, -)\colon (0, \infty) \to [0, 1]$ is continuous.

The triple $(X, M, *)$ is called a fuzzy metric space ([3, 4]). If condition (iv) in the definition of a fuzzy metric $M\colon X \times X \times (0, \infty) \to [0, 1]$ is replaced with the stronger condition (iv') $M(x, y, t) * M(y, z, t) \le M(x, z, t)$, then this function is called a fuzzy ultrametric.

A metric space (X, d) is an $\mathbb{R}$-tree if and only if it is complete, path-connected, and satisfies the so-called four point condition, that is,

$$d(x_1, x_2) + d(x_3, x_4) \leq \max\{d(x_1, x_3) + d(x_2, x_4), d(x_1, x_4) + d(x_2, x_3)\}$$

for all $x_1, \ldots, x_4 \in X$.

This leads to the following question.

Question 2 Is there a fuzzy counterpart of the four point condition? of the notion of $\mathbb{R}$-tree?

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