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DECISION MAKING UNDER UNCERTAINTY -PROOF OF ELLSBERG`S EXPERIMENT

Decision making under uncertainty is a very important problem in real life problems. The paper is dedicated to the problem of decision making under uncertainty. In this paper the proof of Ellsberg's experiment is suggested.

***Keywords**: decision making under uncertainty, possibility measure, Choquet integral, Ellsberg`s Experiment*

1. Introduction

Ellsberg paradox [1] contradicts an axiom of expected utility called sure-thing principle [2]. This principle was introduced by Savage . According to this principle if two alternatives have shared result over the different states of nature, a choice of alternatives will not depend on this result. This paper explains Ellsberg paradox and has no disadvantages of classic methods.

There are a lot of theories and methods of decision making in the conditions of uncertainty [3 – 9]. Key concept of all these methods is the concept of utility. Utility represents some size which characterizes considered alternative in such a manner that among set of the alternatives accessible to a choice, decision maker chooses that alternative where utility is maximum,i.e. the best alternative. In other words, utility represents the mathematical description of preferences decision maker. Each model of utility used for the mathematical description of preferences DM, in general is based on those or other hypotheses of properties, i.e. the nature of these preferences. It is natural that the model, as a matter of fact, approximately describes the real phenomena and consequently different models of utility on a miscellaneous description choice of decision maker, i.e. in different degree are approached to real behavior of people in the conditions of uncertainty and risk. Utility theory is one of the main parts of decision analysis. The notion of utility function consists in construction of a function that represents an individual's preference relations defined over the set of possible alternatives. A utility function *u(.)* is such a real-valued function that for any two possible alternatives x and y an inequality $u(x) \geq u(y)$ holds and only if *x* is preferred or indifferent to *y*. In general, the existence of a utility function individual's is based on transitivity and completeness properties of individual's preferences. For decision making under uncertainty the first utility paradigm was the expected utility theory of Neumann and Morgenstern [10]. This model compares finite-outcome lotteries (alternatives) on the basis of their utility values under conditions of exactly known utilities and probabilities of possible outcomes. A utility value *u(x)* of a finite-outcome lottery

$$x = (x_1, p_1; ...; x_n, p_n)$$

is defined as $u(x) = \sum_{i=1}^{n} u(x_i)p_i$.

The given theory represents harmoniously enough, thus quite simple model of a choice in the condition of uncertainty. However, as the subsequent experimental researches of economists and psychologists have shown, the given model frequently is incapable to describe behavior of people.

Savage's SEU theory, a probability is considered as a decision maker's (DM's) degree of confidence concerning the occurrence of an event.

The most famous examples showing inconsistency of the expected utility model are Allais paradox and Ellsberg paradox[11].

The rest of the paper is organized as follows. In section 2 we give preliminaries. In Section 3 we formulate a statement of the problem. Section 4 is devoted to solution of the problem by using Choquet expected utility. Section 5 is conclusion.

2. Preliminaries

Definition 1.

Choquet expected utility model [12].

The Choquet expected utility (CEU) model has the form

$$CEU(f) = \sum_{i=1}^{n} u(f(s_i))w_i$$

Decision weights w_i are non negative. $u(f(s_i))$ - is the utility values.

Definition 2.

Possibility measure [8].

Assume that P(X) is a power fuzzy set of the universe X. Then the mapping $\Pi : P(X) \to [0,1]$ with the following properties:

$$\Pi(\varnothing) = 0, \Pi(X) = 1;$$
$$A \subseteq B \to \Pi(A) \leq \Pi(B);$$
$$\Pi(\bigcup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i).$$

is the possibility measure. Here I is index set.

Definition 3.

Linguistic lottery[13].

Linguistic lottery is a linguistic random variable with known linguistic probability distribution. Linguistic lottery is represented by a vector

$$L = \left(\tilde{P}_1, \tilde{x}_1; ...; \tilde{P}_1, \tilde{x}_1; ...; \tilde{P}_n, \tilde{x}_n\right)$$

Let $(\Omega, P(\Omega), Q)$ be probability space, where $\Omega = (\omega_1, \omega_2, ..., \omega_n)$ is finite, $P(\Omega)$ is a set of subsets of Ω ,Q is a probability measure. In reality very often Q is not exactly known. Only partial information is available. For example, instead of exact Q it is known a set P of possible probability distributions over Ω [14]. In this case it is possible to approximate $\overline{P}$ by getting upper and lower bounds called lower probability function and upper probability function respectively [14]. Given $B \in P(\Omega)$

$$F(B) = \wedge\left\{Q(B) : Q \in \overline{P}\right\},\ G(B) = \wedge\left\{Q(B) : Q \in \overline{P}\right\},$$

where functions F,G are called lower probability function and upper probability function respectively [14]. Here $F(B) \le P(B) \le G(B)$ holds for all B. F, G are not probability measures, not being additive. For them it is held:

1) $F(\varnothing) = G(\varnothing) = 0$,

2) $F(\Omega) = G(\Omega) = 1$

3) $B \subseteq C$, $F(B) \le F(C)$

and $G(B) \le G(C)$

3. Statement of the problem

Let`s consider an urn with 90 balls. It is known the urn contains 30 red and 60 black balls in unknown proportion. Then the decision maker is offered four lotteries (table 1). He receives \$100, if he draws a red ball in the first lottery and a black ball in the second lottery. The probability of a red ball is 1/3, but the probability of a black ball is unknown. The third and the forth lotteries are more complicated.

Table 1

The lotteries in Ellsberg`s experiment

Number of balls	30	60	
Lotteries	RED	BLACK	YELLOW
1	\$100	\$0	\$0
2	\$0	\$100	\$0
3	\$100	\$0	\$100
4	\$0	\$100	\$100

Decision maker receives \$100, if he draws red or yellow balls in the third lottery and black or yellow balls in the forth lottery. An experiment showed, that most people prefers lottery 1 to lottery 2 and lottery 4 to lottery 3. Thus the decision maker demonstrates an uncertainty aversion. He chooses the second lottery, as, according to expected utility theory, he thinks, that a number of red balls is more than a number of black ones. He also chooses the forth lottery as he thinks, that a number of black balls is more than a number of red ones. Thus the paradox appears when the decision maker thinks simultaneously that a number of red balls is more and less. The causes are in decision maker`s preferring of the lotteries where an uncertainty in probabilities is less.

Formally this problem can be formulated as following. There is a space of mutually exclusive and exhaustive states of nature $S = \{s_1, s_2, s_3\}$, consisting from three states of nature: red ball, black ball and yellow ball, X – a set of outcomes, $X \subset R$, R – set of real numbers. A is the set of actions that are functions h : S $\rightarrow$ X. Problem is in decision making on a base of Ellsberg paradox under condition that a number of black balls is more than a number of yellow ones.

4. Solving the problem using CEU

As it was mentioned above, there are a set of states of nature $S = \{s_{red}, s_{black}, s_{yellow}\}$, a set of alternatives $A = \{h_1, h_2, h_3, h_4\}$, consisting of alternatives of decision making over four lotteries, and a set of results

$$X = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}\}$$

We use a method of Choquet expected utility(CEU) and elements of fuzzy measure for decision making under uncertainty.

If choice is carried out on the basis of expected utility, then

U(L1)>U(L2)

and U(L4)>U(L3).

By using table 1, we obtain:

R*U(\$100)+(1-R)*U(\$0)>B*U(\$100)+(1-B)*U(\$0)

U(\$100) > U(\$0) .

R >B.

Analogously for L3 and L4

B*U(\$100)+Y*U(\$100)+R*U(\$0)>

R*U(\$100)+Y*U(\$100)+B*U(\$0).

B >R.

Contradiction is received from here.

We can use non-additive aggregator (Choquet integral) [2,15,16] for the liquidating this apparent contradiction. For four lotteries Choquet integral is in the following form:

U1=(g{R}-g{∅})*100+(g{R,B}-g{R})*0+
+(g{R,B,Y}-g{R,B})*0

U2=(g{B}-g{∅})*100+(g{B,R}-g{B})*0+
+(g{R,B,Y}-g{B,R})*0

U1=g{R}*100

U2=g{B}*100

Taking into account, U1>U2

Then g{R} >g{B}

Thus, comparing the utility functions for the first and the second lotteries we determine U1>U2.

Analogously

U3=(g{R}-g{∅})*100+(g{R,Y}-g{R})*100+
+(g{R,B,Y}-g{R,Y})*0

U4=(g{B}-g{∅})*100+(g{B,Y}-g{B})*100+
+(g{R,B,Y}-g{B,Y})*0

U3=g{R,Y}*100

U4=g{B,Y}*100

Taking into account, U4>U3

Then g{B,Y} >g{R,Y}

So, we obtain:

g(B)<g(R),

g(R,Y)<g(B,Y) .

Adding condition of monotonicity, we can write:

\g(r,y)>g(r),g(y);g(b,y)>g(b),g(y);g(r,b)>g(r),g(b)

for any measure which satisfy condition above mentioned inequalities Elsberg's experiment is true. For example, let us assume the following as the fuzzy measure :

$$g(\{R\}) := 1/3$$

$$g(\{B\}) := g(\{Y\}) := 2/9,$$

$$g(\{R,Y\}) := 5/9, g(\{B,Y\}) = g(\{R,B\}) := 2/3$$

$$g(\{R,B,Y\}) := 1$$

Then we obtain:

U1=(1/3-0)*100+(2/3-1/3)*0+(1-2/3)*0=100/3

U2=(2/9)*100+(2/3-2/9)*0+(1-2/3)*0=200/9

Thus, comparing the utility functions for the first and the second lotteries we determineU1<U2. Then it is necessary to take into consideration, that a crisp number is a private case of fuzzy set. It involves to state, that a number of the red balls is less than a number of black ones.

For the third lottery the value of utility function will be constant:

U3=(1/3-0)*100+(5/9-1/3)*100+(1-5/9)*0=

=100/3+200/9=500/9

For the forth lottery this situation will repeat:

U4=(2/9-0)*100+(2/3-2/9)*100+(1-2/3)*0=

=200/9+400/9=600/9

Thus, U4>U3

Continuing the experiment and supposing that a number of the black balls is less than a number of yellow ones, we can easily determine that preferences will change on opposite, i.e.,. U1>U2, U4>U3

Above mentioned results show that Ellsberg`s Experiment is true.

5. Conclusion

In this paper, the proof of Ellsberg's experiment is discussed. A method of decision making under uncertainty using the Choquet expected utility and possibility measure is suggested in this paper.

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ПРИНЯТИЕ РЕШЕНИЯ В УСЛОВИЯХ НЕОПРЕДЕЛЕННОСТИ – ДОКАЗАТЕЛЬСТВО ПАРАДОКСА ЭЛЛСБЕРГА

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В статье был предложен метод принятия решения в условиях неопределенности. При этом доказан известный парадокс Эллсберга, с использованием неаддитивной меры и меры возможности. Статья состоит из следующих частей: введение, постановка задачи, основные определения, доказательство парадокса Эллсберга и заключение.

Ключевые слова: *принятие решения в условиях неопределенности, мера возможности, интеграл Шоке, парадокс Эллсберга.*

ПРИЙНЯТТЯ РІШЕННЯ В УМОВАХ НЕВИЗНАЧЕНОСТІ – ДОКАЗ ПАРАДОКСУ ЕЛЛСБЕРГА

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У статті був запропонований метод прийняття рішення в умовах невизначеності. При цьому доведений відомий парадокс Еллсберга, з використанням неаддитивної міри і міри можливості. Стаття складається з наступних частин: введення, постановка завдання, основні визначення, доказ парадокса Еллсберга і висновок.

Ключові слова: *ухвалення рішення в умовах невизначеності, міра можливості, інтеграл Шоке, парадокс Еллсберга.*