ISSN № 9125 0912. ВІСНИК ДНУ. Серія "Моделювання". № 8. 2009. Вип. 1. С. 61–88

Проблеми математичного моделювання
та теорії диференціальних рівнянь

УДК 681.31

TOPOLOGICAL ASPECTS IN VECTOR OPTIMIZATION PROBLEMS

P. I. Kogut*, R. Manzo**, I. V. Nechay***

* *Department of Differential Equations Dnipropetrovsk National University Naukova str., 13, 49050 Dnipropetrovsk, Ukraine E-mail: p.kogut@i.ua*

** *Dipartimento di Ingegneria dell'Informazione e Matematica Applicata Università di Salerno Via Ponte don Melillo, 84084 Fisciano (SA), Italy. E-mail: manzo@diima.unisa.it*

*** *Department of Technical Cybernetics Dnipropetrovsk Technical University Acad. Lazarjan str., 2, 49010 Dnipropetrovsk, Ukraine. E-mail: i.nechay@i.ua*

In this paper we study vector optimization problems in partially ordered Banach spaces. We suppose that the objective mapping possesses a weakened property of lower semicontinuity and make no assumptions on the interior of the ordering cone. We derive sufficient conditions for existence of efficient solutions of the above problems and discuss the role of topological properties of the objective space. We discuss the scalarization of vector optimization problems when the objective functions are vector-valued mappings with a weakened property of lower semicontinuity. We also prove the existence of the so-called generalized efficient solutions via the scalarization process. All principal notions and assertions are illustrated by numerous examples.

Key words. Vector optimization problem, efficient solutions, objective mapping, property of lower semicontinuity, generalized efficient solutions.

1. Introduction

The main goal of this paper is to discuss one class of vector optimization problems in Banach spaces in the case when the objective vector-valued mapping possesses a weakened property of lower semicontinuity. The classical setting of vector optimization problems usually consists in the investigation of "optimal" elements of a non-empty subset of a partially ordered objective space, where by "optimal" elements one mainly means the minimal elements or several variants of this concept, for example, strongly minimal, properly minimal and weakly minimal elements. Therefore, an important aspect in the vector optimization is to find conditions which guarantee existence of the so-called efficient solutions, which are defined as inverse images of the minimal elements of the image set. The following result is well-known: *if the image of admissible solutions in an objective Banach space is compact then the set of efficient solutions is non-empty.* Since the compactness is a very restrictive assumption, at least in an infinite-dimensional setting, many authors have tried to weaken it. The typical way to do it is to endow the objective mapping with some lower semicontinuity properties. In the vector-valued case there are several possible ways to extend the "scalar" notion of lower semicontinuity (see, for example, [3, 4, 5, 7, 8, 13, 16, 20]). We could mention the lower semicontinuity, quasi lower semicontinuity, and order lower semicontinuity. However, the above properties for the objective functions may fail at an efficient solution, even for simple vector optimization problems with non-empty solution sets. This is an atypical situation for the scalar case

$$I(x^*) = \inf\{I(x) \ : \ x \in X\}, \tag{1.1}$$

where each solution x^* is always a point of lower semicontinuity of the cost functional $I : X \to \mathbb{R}$.

The next problem, which motivated our efforts in this field, concerns the following observation: if the scalar problem (1.1) has a non-empty set of solutions, then

$$\inf\{I(x) \ : \ x \in X\} = \min\{I(x) \ : \ x \in X\} = \min\left[\text{closure}\{I(x) \ : \ x \in X\}\right].$$

However, in the case of vector optimization, the typical situation is:

$$\mathrm{Min}(S) \neq \emptyset, \ \mathrm{Min}\left[\text{closure}(S)\right] \neq \emptyset, \quad \text{and} \quad \mathrm{Min}(S) \cap \mathrm{Min}\left[\text{closure}(S)\right] = \emptyset,$$

where by $\mathrm{Min}(S)$ we symbolically denote the family of all minimal elements of a subset S.

Thus our prime interest in this paper is to consider vector optimization problems in a new setting, which involves topological properties of the objective space, and discuss the problem of their scalarization. We deal with the case when the objective mappings take values in a real Banach space Y partially ordered by a pointed cone Λ with possibly empty interior. In contrast to the classical setting of the vector optimization problem

$$\text{Minimize} \quad f(x) \quad \text{with respect to the cone} \quad \Lambda \text{ subject to} \quad x \in X_\partial, \ f : X \to Y,$$

we study the problem in the following formulation:

$$\text{Realize} \quad \mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} f(x) \tag{1.2}$$

and associate this problem with the quaternary $\langle X_\partial, f, \Lambda, \tau\rangle$, where the essential counterpart is the choice of the topology τ on the objective space Y.

We also extend the concept of lower semicontinuity to vector-valued mappings, which is compatible with optimization problems in the form (1.2), and discuss the existence of the so-called (Λ, τ)-efficient solutions to the problem (1.2). In particular, we show that the extended concept of lower semicontinuity does not fail at (Λ, τ)-efficient solutions, however the topological properties of the spaces (X, σ) and (Y, τ), where this problem is considered, play an essential role. In view of this, our main intension deals with the scalarization of vector optimization problems (1.2) with the so-called $(\Lambda, \sigma \times \tau)$-lower semicontinuous mappings, using the "simplest" method of the "weighted sum". We show that in this case one of the fundamental requirements on the scalarizing vector optimization problems (according to Sawaragi et al. [18]): *solutions to the scalarized optimization problem must also be minimal solutions to the original vector optimization problem*, may not hold. Moreover, we show that for $(\Lambda, \sigma \times \tau)$-lower semicontinuous mappings $f : X_\partial \to Y$ a situation is possible, when none of the scalar functions, obtained by "weighted sum"approach, is sequentially lower semicontinuous. For this reason, we extend the notion of (Λ, τ)-efficient solutions to the so-called generalized solutions of the vector optimization problem. We study their main properties and derive sufficient conditions when the generalized solutions can be obtained via the scalarization process of (1.2).

2. Notation and Preliminaries

Let X and Y be two real Banach spaces. We assume that X is reflexive and Y is dual to some separable Banach space V (that is $Y = V^*$). We suppose that these spaces are endowed with some topologies $\sigma = \sigma(X)$ and $\tau = \tau(Y)$, respectively. By default σ is always associated with the weak topology of X, whereas τ is associated with the weak-$*$ topology of Y. For a subset $A \subset Y$ we denote by $\mathrm{int}_\tau\, A$ and $\mathrm{cl}_\tau\, A$ its interior and closure with respect to the τ-topology, respectively. We will omit this index if no confusion may occur. Let Λ be a τ-closed convex pointed cone in Y. No assumption is imposed on the topological interior of Λ. Throughout this paper, we suppose that Y is partially ordered with the ordering cone Λ. We denote with $\leq_\Lambda$ a partial ordering introduced by the cone Λ, that is, for any elements $y, z \in Y$, we will write $y \leq_\Lambda z$ whenever $z \in y + \Lambda$ and $y <_\Lambda z$ for $y, z \in Y$, if $z - y \in \Lambda \setminus \{0_Y\}$. We say that a sequence $\{y_k\}_{k=1}^\infty \subset Y$ is decreasing and we use the notation $y_k \searrow$ whenever, for all $k \in \mathbb{N}$, we have $y_{k+1} \leq_\Lambda y_k$. We also say that a sequence $\{y_k\}_{k=1}^\infty \subset Y$ is bounded below if there exists an element $y^* \in Y$ such that $y^* \leq_\Lambda y_k$ for all $k \in \mathbb{N}$.

For the investigation of "optimal" elements of a non-empty subset S of the partially ordered space Y one is mainly interested in minimal or maximal elements of S.

Definition 1. (see [11]) An element $y^* \in S \subset Y$ is said to be minimal of the set S, if there is no $y \in S$ such that $y \leq_\Lambda y^*,\ y \neq y^*$, that is

$$S \cap (y^* - \Lambda) = \{y^*\}.$$

Definition 2. (see [11]) An element $y^* \in S \subset Y$ is said to be weakly minimal of the set S, if

$$S \cap (y^* - cor(\Lambda)) = \emptyset,$$

where by $\mathrm{cor}\,(\Lambda)$ we denote the algebraic interior of Λ, that is,

$$\mathrm{cor}\,(\Lambda) := \{\widehat{z} \in V \mid \forall\, z \in V \text{ there is an } \ \widehat{\alpha} > 0 \text{ with} \\ \widehat{z} + \alpha z \in \Lambda \ \text{ for all } \ \alpha \in [0, \widehat{\alpha}]\}\,.$$

Let $\mathrm{Min}_\Lambda(S)$ denote the family of all minimal elements of S. We say that an element y^* is the *ideal minimal point* (or a strongly minimal element) of the set S, if $y^* \in S$ and $y^* \leq_\Lambda y$ for every $y \in S$.

Let us introduce two singular elements $-\infty_\Lambda$ and $+\infty_\Lambda$ in Y. We assume that these elements satisfy the following conditions:

$$1) -\infty_\Lambda \preceq y \preceq +\infty_\Lambda,\ \forall y \in Y; \quad 2) +\infty_\Lambda + (-\infty_\Lambda) = 0_Y.$$

Let $Y^\bullet$ denote a semi-extended Banach space: $Y^\bullet = Y \cup \{+\infty_\Lambda\}$ assuming that

$$\| + \infty_\Lambda\|_Y = +\infty \quad \text{and} \quad y + \lambda(+\infty_\Lambda) = +\infty \;\forall\, y \in Y \quad \text{and} \quad \forall\, \lambda > 0.$$

The following concept is a crucial point in this paper.

Definition 3. We say that a set E is the efficient infimum of a set $S \subset Y$ with respect to the τ topology of Y (or shortly (Λ, τ)-infimum) if E is the collection of all minimal elements of $\mathrm{cl}_\tau\, S$ in the case when this set is non-empty, and E is equal to $\{-\infty_\Lambda\}$ otherwise.

Hereinafter we denote the (Λ, τ)-infimum for S by $\mathrm{Inf}^{\Lambda,\tau}\, S$. Thus, in view of the definition given above, we have

$$\mathrm{Inf}^{\Lambda,\tau}\, S := \begin{cases} \mathrm{Min}_\Lambda(\mathrm{cl}_\tau\, S), & \mathrm{Min}_\Lambda(\mathrm{cl}_\tau\, S) \neq \emptyset \\ -\infty_\Lambda, & \mathrm{Min}_\Lambda(\mathrm{cl}_\tau\, S) = \emptyset. \end{cases}$$

The following example shows the significance of this definition and compares it with the notion of minimal elements.

Example 1. Let $Y = \mathbb{R}^2$ and let $\Lambda = \mathbb{R}^2_+$ be the natural ordering cone of positive elements in $\mathbb{R}^2$. Suppose that the set $S \subset Y$ is given as $S = \cup_{i=1}^3 X_i$, where

$$\begin{gathered} X_1 = \left\{z \in \mathbb{R}^2 \;:\; z_1 \geq 1,\; z_2 > 3,\; z_1 + z_2 \leq 5\right\}, \\ X_2 = \left\{z \in \mathbb{R}^2 \;:\; z_1 > 2,\; z_2 > 2,\; z_1 + z_2 \leq 5\right\}, \\ X_3 = \left\{z \in \mathbb{R}^2 \;:\; z_1 > 3,\; z_2 \geq 4,\; z_1 + z_2 \leq 5\right\}, \\ X_4 = \{(2;3), (3;2)\} \end{gathered}$$

(see Fig. 1). It is essential that the set S is not closed. Then the set $\mathrm{Min}_\Lambda(S)$ of

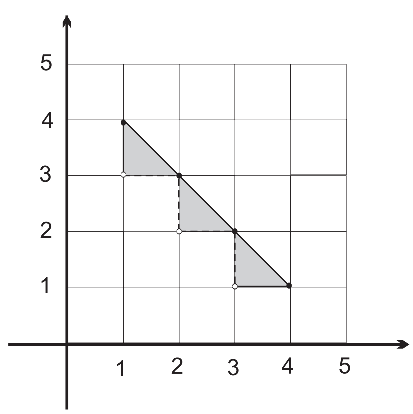

Fig. 1. The set S in Example 1

all minimal elements of S is given as

$$\mathrm{Min}_\Lambda(S) = \{(2;3), (3;2)\},$$

whereas the (Λ, τ)-infimum of the S reads as

$$\mathrm{Inf}^{\Lambda,\tau}(S) = \{(1;3), (2;2), (3;1)\},$$

where τ is the strong topology of $\mathbb{R}^2$. Consequently, in contrast to the scalar case where the inclusion $\mathrm{Min}_\Lambda(S) \subseteq \mathrm{Inf}^{\Lambda,\tau} S$ is always true, we have:

$$\mathrm{Inf}^{\Lambda,\tau}(S) \neq \emptyset, \quad \mathrm{Min}_\Lambda(S) \neq \emptyset, \quad \text{and} \quad \mathrm{Inf}^{\Lambda,\tau}(S) \cap \mathrm{Min}_\Lambda(S) = \emptyset.$$

Let X_∂ be a non-empty subset of the Banach space X, and $f : X_\partial \to Y$ be some mapping. Note that the mapping $f : X_\partial \to Y$ can be associated with its natural extension $\widehat{f} : X \to Y^\bullet$ to the whole space X, where

$$\widehat{f}(x) = \begin{cases} f(x), & x \in X_\partial, \\ +\infty_\Lambda, & x \notin X_\partial. \end{cases}$$

Following [1] a mapping $f : X_\partial \to Y^\bullet$ is said to be bounded below if there exists an element $z \in Y$ such that $z \leq_\Lambda f(x)$ for all $x \in X_\partial$.

Definition 4. A subset A of Y is said to be the efficient infimum of a mapping

$$f : \ X_\partial \to Y$$

with respect to the τ-topology of Y and is denoted by $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$, if A is the (Λ, τ)-infimum of the image $f(X_\partial)$ of X_∂ in Y, that is,

$$\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x) = \mathrm{Inf}^{\Lambda,\tau} \{f(x) \, : \, \forall\, x \in X_\partial\}.$$

Remark 1. It is clear now that if $a \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$ then

$$\mathrm{cl}_\tau \ \{f(x) \, : \, \forall\, x \in X_\partial\} \cap (a - \Lambda) = \{a\}$$

provided $\mathrm{Min}_\Lambda \left[\mathrm{cl}_\tau \ \{f(x) \ : \ \forall\, x \in X_\partial\}\right] \neq \emptyset$.

Let $\{y_k\}_{k=1}^\infty$ be a sequence in Y. Let $\mathrm{L}^\tau\{y_k\}$ denote the set of all its cluster points with respect to the τ-topology of Y, that is, $y \in \mathrm{L}^\tau\{y_k\}$ if there is a subsequence $\{y_{k_i}\}_{i=1}^\infty \subset \{y_k\}_{k=1}^\infty$ such that $y_{k_i} \xrightarrow{\tau} y$ in Y as $i \to \infty$. If this set is lower unbounded, i.e., $\mathrm{Inf}^{\Lambda,\tau}\,\mathrm{L}^\tau\{y_k\} = -\infty_\Lambda$, we assume that $\{-\infty_\Lambda\} \in \mathrm{L}^\tau\{y_k\}$. If $\mathrm{Sup}^{\Lambda,\tau}\,\mathrm{L}^\tau\{y_k\} = +\infty_\Lambda$, we assume that $\{+\infty_\Lambda\} \in \mathrm{L}^\tau\{y_k\}$. Let $x_0 \in X_\partial$ be a fixed element. In what follows for an arbitrary mapping $f : \ X_\partial \to Y$ we make use of the following sets:

$$\mathrm{L}^{\sigma\times\tau}(f, x_0) := \bigcup_{\{x_k\}_{k=1}^\infty \in \mathfrak{M}_\sigma(x_0)} \mathrm{L}^\tau\{\widehat{f}(x_k)\}, \tag{2.1}$$

$$\mathrm{L}^{\sigma\times\tau}_{\min}(f, x_0) := \mathrm{L}^{\sigma\times\tau}(f, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x), \tag{2.2}$$

where $\mathfrak{M}_\sigma(x_0)$ is the set of all sequences $\{x_k\}_{k=1}^\infty \subset X$ such that $x_k \to x_0$ with respect to the σ-topology of X. To illustrate the characteristic features of the set $\mathrm{L}^{\sigma\times\tau}_{\min}(f, x_0)$, we give the following example.

Example 2. Let $X_\partial = [1; 3]$, $Y = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}^2_+$ be the ordering cone of positive elements. We define a vector-valued mapping $f : X_\partial \to Y$ as follows:

$$f(x) = \begin{cases} \left[\begin{smallmatrix} x \\ 2 \end{smallmatrix}\right], & x \neq 1, \\ \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right], & x = 1. \end{cases} \tag{2.3}$$

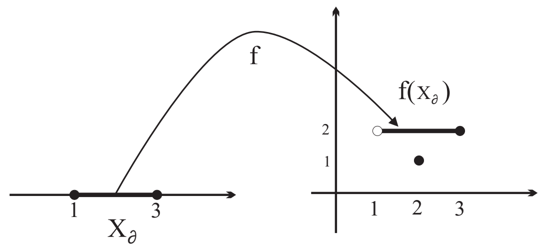

Fig. 2. Illustration of the set $\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)$

(see Fig. 2). Then

$$\mathrm{L}^{\sigma\times\tau}(f,x_0)=\{f(x_0)\}\quad \forall\, x_0\in(1;3],$$

$$\mathrm{L}^{\sigma\times\tau}(f,1)=\left\{\begin{bmatrix}1\\2\end{bmatrix};\begin{bmatrix}2\\1\end{bmatrix}\right\},\ \text{ and }\ \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)=\left\{\begin{bmatrix}1\\2\end{bmatrix};\begin{bmatrix}2\\1\end{bmatrix}\right\}$$

Therefore, $\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)=\emptyset$ in the case when $x_0\in(1;3]$, and

$$\mathrm{L}^{\sigma\times\tau}_{\min}(f,1)=\left\{\begin{bmatrix}1\\2\end{bmatrix};\begin{bmatrix}2\\1\end{bmatrix}\right\}.$$

Remark 2. It is easy to see that the set $\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)$ can be alternatively defined as

$$\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)=\Big\{y^*\in\mathrm{L}^{\sigma\times\tau}(f,x_0)\ \text{if}\ \ f(x_k)\xrightarrow{\tau}y^*,\\ f(x_k)\le_\Lambda y^*\ \forall k\in\mathbb{N},\ \forall\, x_k\xrightarrow{\sigma}x_0\Big\}. \quad (2.4)$$

Now we are able to introduce the notion of the lower limit for the vector-valued mappings.

Definition 5. We say that a subset $A\subset Y\cup\{\pm\infty_\Lambda\}$ is the Λ-lower sequential limit of the mapping $f:X_\partial\to Y$ at the point $x_0\in X_\partial$ with respect to the product topology $\sigma\times\tau$ of $X\times Y$, and we use the notation $A=\liminf^{\Lambda,\tau}_{x\xrightarrow{\sigma}x_0} f(x)$, if

$$\liminf^{\Lambda,\tau}_{x\xrightarrow{\sigma}x_0} f(x):=\begin{cases}\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0), & \mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)\ne\emptyset,\\ \mathrm{Inf}^{\Lambda,\tau}\,\mathrm{L}^{\sigma\times\tau}(f,x_0), & \mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)=\emptyset.\end{cases} \quad (2.5)$$

Remark 3. Note that in the scalar case ($f:X_\partial\to\mathbb{R}$) the sets

$$\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)\quad \text{and}\quad \mathrm{Inf}^{\Lambda,\tau}\,\mathrm{L}^{\sigma\times\tau}(f,x_0)$$

are singletons. Therefore, if $\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)\ne\emptyset$ then we have

$$\begin{aligned}\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)&=\mathrm{L}^{\sigma\times\tau}(f,x_0)\cap\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)\\ &=\mathrm{Inf}^{\Lambda,\tau}\,\mathrm{L}^{\sigma\times\tau}(f,x_0)\cap\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)\\ &=\mathrm{Inf}^{\Lambda,\tau}\,\mathrm{L}^{\sigma\times\tau}(f,x_0).\end{aligned}$$

Hence the choice rules in (2.5) coincide and we come to the classical definition of the lower limit.

To illustrate the crucial role of the conditions

$$\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)\neq\emptyset \quad\text{and}\quad \mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0)=\emptyset$$

in Definition (5), we give the following example.

Example 3. Under assumptions of Example 2 we consider the mapping $f: X_\partial \to Y$ defined as follows (see Fig. 3):

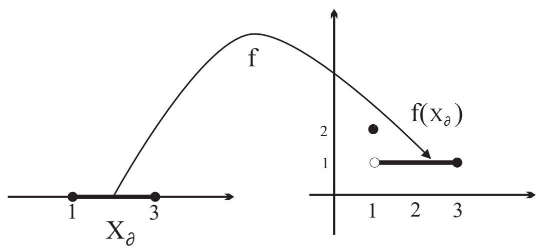

Fig. 3. Illustration of Definition 5 in Example 3

$$f(x)=\begin{cases}\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right], & x\neq 1,\\ \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right], & x=1.\end{cases}\tag{2.6}$$

Let us define the Λ-lower sequential limit of $f: X_\partial \to Y$ at two points: firstly at $x_0=1$, and after at $x_0\neq 1$. Then direct calculations show that

$$\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)=\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\},\ \mathrm{L}^{\sigma\times\tau}(f,1)=\left\{\begin{bmatrix}1\\2\end{bmatrix};\begin{bmatrix}1\\1\end{bmatrix}\right\},\ \text{and}$$
$$\mathrm{L}^{\sigma\times\tau}(f,x_0)=\left\{\begin{bmatrix}x_0\\1\end{bmatrix}\right\}\ \forall\, x_0\in(1;3].$$

Hence, since

$$\mathrm{L}^{\sigma\times\tau}_{\min}(f,x_0):=\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)\cap \mathrm{L}^{\sigma\times\tau}(f,x_0)=\emptyset\ \text{ for every }\ x_0\in(1;3],$$

it follows that

$$\liminf^{\Lambda,\tau}_{x\xrightarrow{\sigma}x_0} f(x)=\mathrm{Inf}^{\Lambda,\tau}\left\{\begin{bmatrix}x_0\\1\end{bmatrix}\right\}=\left\{\begin{bmatrix}x_0\\1\end{bmatrix}\right\}.$$

At the same time, in the case when $x_0=1$, we have

$$\mathrm{L}^{\sigma\times\tau}_{\min}(f,1):=\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)\cap \mathrm{L}^{\sigma\times\tau}(f,1)=\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

As a result, we conclude:

$$\liminf^{\Lambda,\tau}_{x\xrightarrow{\sigma}1} f(x)=\mathrm{L}^{\sigma\times\tau}_{\min}(f,1)=\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

3. The setting of vector optimization problems

Let X_∂ be a non-empty σ-closed subset of the reflexive Banach space X. Let Y be a partially ordered Banach space with a τ-closed pointed ordering cone $\Lambda \subset Y$. Let $f : X_\partial \to Y$ be a given mapping. Then the typical vector optimization problem can be stated in general manner as follows:

$$\left.\begin{array}{r} \text{Minimize} \quad f(x) \quad \text{with respect to the cone} \quad \Lambda \\ \text{subject to} \quad x \in X_\partial. \end{array}\right\} \tag{3.1}$$

Usually this problem is associated with the triplet $\langle X_\partial, f, \Lambda\rangle$, where the set X_∂ is called the set of admissible solutions to the problem (3.1). The problem consists in determining minimal (or weakly minimal) solutions $x^{min} \in X_\partial$ which are defined as the inverse image of the minimal (or weakly minimal) elements of the image set $f(X_\partial)$ in the sense of Definition 1 (or Definition 2, respectively). Let $\mathrm{Min}(X_\partial, f, \Lambda)$ and $\mathrm{WMin}(X_\partial, f, \Lambda)$ denote the sets of minimal and weakly minimal solutions to the problem (3.1), respectively. It is clear that the notions “minimal”and “weakly minimal”are closely related, moreover, the following inclusion is obvious

$$\mathrm{Min}(X_\partial, f, \Lambda) \subseteq \mathrm{WMin}(X_\partial, f, \Lambda).$$

However, the concept of weak minimality is rather of theoretical interest, and it is not an appropriate notion for applied problems.

In contrast to (3.1) we will consider the vector optimization problems in the following form:

$$\text{Realize} \quad \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x), \tag{3.2}$$

where the operator $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial}$ is defined in Definition 4. Note that in this case the optimization problem (3.2) can be associated with the quaternary

$$\langle X_\partial, f, \Lambda, \tau\rangle, \tag{3.3}$$

which indicates that the essential component of this setting is the choice of the τ-topology on the objective space Y.

Remark 4. It is clear that vector optimizations problems (3.1) and (3.2) are identical in the case when $Y = \mathbb{R}$ and $\Lambda = \mathbb{R}_+$, and they lead to the classical setting of a scalar constrained minimization problem. However, in general, there is a principal difference between the mentioned setting of vector optimizations problems. First, as follows from (3.2), it is natural to say that an element $x^* \in X_\partial$ is a solution to the problem (3.2) if

$$f(x^*) \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x). \tag{3.4}$$

Hence, $f(x^*) \in \mathrm{Min}_\Lambda\left(\mathrm{cl}_\tau f(X_\partial)\right)$. Since $f(x^*) \in f(X_\partial)$ it follows that

$$f(x^*) \in \mathrm{Min}_\Lambda f(X_\partial).$$

Therefore, x^* is a minimal solution to the problem (3.1), i.e. $x^* \in \mathrm{Min}(X_\partial, f, \Lambda)$. However, as follows from Example 4 given below, the converse statement is not true in general. Note that this situation is atypical for the scalar case when we always have the implication

$$\text{if} \;\; f(x^*) = \min_{x \in X_\partial} f(x), \;\text{then}\;\; x^* \in X_\partial \;\text{and}\;\; f(x^*) = \inf_{x \in X_\partial} f(x).$$

On the other hand, as follows from Definition 4, the problem (3.2), and hence the set of its solutions, essentially depend on the properties of the τ-topology of the objective space Y. Thereby, the problems (3.1) and (3.2) are essentially different.

We introduce now the following concept.

Definition 6. An element $x^{eff} \in X_\partial$ is said to be a (Λ, τ)-efficient solution to the problem (3.2) if x^{eff} realizes the (Λ, τ)-infimum of the mapping $f : X_\partial \to Y$, that is,

$$f(x^{eff}) \in \mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} f(x) = \mathrm{Inf}^{\Lambda,\tau} \{f(x) \,:\, \forall\, x \in X_\partial\}.$$

We denote by $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ the set of all (Λ, τ)-efficient solutions to the vectorial problem (3.2), i.e.

$$\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) = \left\{ x^{eff} \in X_\partial \;:\; f(x^{eff}) \in \mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} f(x) \right\}. \tag{3.5}$$

Taking into account the motivation of Remark 4, we come to the following obvious result:

Proposition 1. Let X and Y be two Banach spaces, let X_∂ be a non-empty subset of X, and let $f : X_\partial \to Y$ be an objective mapping. Assume that the space Y is partially ordered by a τ-closed pointed cone $\Lambda \subset Y$. Then the solution sets to the problems (3.1) and (3.2) satisfy the relation

$$\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) \subseteq \mathrm{Min}(X_\partial, f, \Lambda).$$

The sets $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ and $\mathrm{Min}(X_\partial, f, \Lambda)$ do not coincide in general. To illustrate this fact, we give the following example.

Example 4. (see [12]) Let $X = Y = \mathbb{R}^2$ and let $\Lambda = \mathbb{R}^2_+$ be the ordering cone of positive elements. We suppose that a vector-valued mapping $f : X \to Y$ and a set of admissible solutions X_∂ are such that $f(x) = x$ and $X_\partial = \cup_{i=1}^4 X_i$, where

$$\begin{aligned} X_1 &= \left\{ z \in \mathbb{R}^2 \;:\; z_1 \geq 1,\; z_2 > 3,\; z_1 + z_2 \leq 5 \right\}, \\ X_2 &= \left\{ z \in \mathbb{R}^2 \;:\; z_1 > 2,\; z_2 > 2,\; z_1 + z_2 \leq 5 \right\}, \\ X_3 &= \left\{ z \in \mathbb{R}^2 \;:\; z_1 > 3,\; z_2 \geq 4,\; z_1 + z_2 \leq 5 \right\}, \\ X_4 &= \{(2;3), (3;2), (3;1)\} \end{aligned}$$

(see Fig. 4). Then straightforward calculations show that

$$\mathrm{Min}_\Lambda(f(X_\partial)) = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, \quad \mathrm{Inf}^{\Lambda,\tau}(f(X_\partial)) = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$$

Hence

$$\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, \quad \mathrm{Min}(X_\partial, f, \Lambda) = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$$

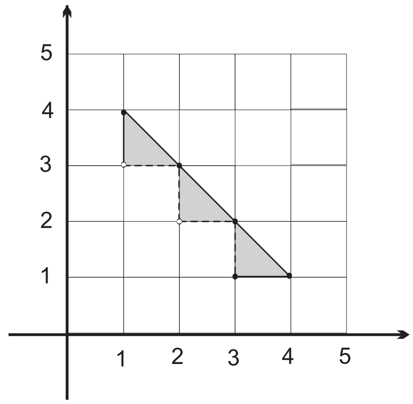

Fig. 4. The image of the set X_∂ in Example 4

The aim of this section is to obtain an existence theorem of the (Λ, τ)-efficient solutions for a vector optimization problem (3.2), that is, to find sufficient conditions which guarantee the relation $\mathrm{Eff}_\tau(X_\partial; f; \Lambda) \neq \emptyset$. Let $\widehat{f} : X \to Y^\bullet$ denote the natural extension of $f : X_\partial \to Y$ to the whole X. We begin with the following concept of lower semicontinuity for vector-valued mappings.

Definition 7. We say that a mapping $f : X_\partial \to Y$ is $(\Lambda, \sigma \times \tau)$-lower semicontinuous ($(\Lambda, \sigma \times \tau)$-lsc) at the point $x_0 \in X_\partial$ if

$$f(x_0) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x_0} \widehat{f}(x).$$

A mapping f is $(\Lambda, \sigma \times \tau)$-lsc if f is $(\Lambda, \sigma \times \tau)$-lsc at each point of X_∂.

The main motivation to introduce this concept is the following observation.

Proposition 2. Let X be a Banach space, and let Y be a partially ordered Banach space with an ordering τ-closed pointed cone Λ. Moreover, let X_∂ be a non-empty subset of X and let $f : X_\partial \to Y$ be a given mapping. If $x^0 \in X_\partial$ is any (Λ, τ)-efficient solution to the problem (3.2), then the mapping $f : X_\partial \to Y$ is $(\Lambda, \sigma \times \tau)$-lsc at this point for any Hausdorff topology σ on X.

Proof. Let $x^0 \in \mathrm{Eff}_\tau(X_\partial; f; \Lambda)$. Then $f(x^0) \in \mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} f(x)$. On the other hand

$$f(x^0) \in \mathrm{L}^{\sigma\times\tau}_{\min}(f, x^0)$$

for any Hausdorff topology σ on X. Hence

$$f(x^0) \in \mathrm{L}^{\sigma\times\tau}_{\min}(f, x^0).$$

As a result, by Definition 5, we have

$$f(x^0) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x^0} f(x).$$

This concludes the proof. $\square$

Before proceeding further, we note that the concept of $(\Lambda, \sigma\times\tau)$-lower semicontinuity for the vector-valued mappings, given above, is more general than well known extensions of the "scalar" notion of lower semicontinuity to the vector-valued case (see, for example, [3, 4, 5, 7, 8, 13, 16]). We recall now a few main definitions of lower semicontinuity of vector-valued mappings with respect to the product topology $\sigma \times \tau$ on $X \times Y$, introduced in [7, 8, 10, 19].

Definition 8. (see [8]) A mapping $f : X \to Y^\bullet$ is said to be sequentially lower semicontinuous (s-lsc) at $x^0 \in X$, if for any $y \in Y$ satisfying $y \leq_\Lambda f(x^0)$ and for any sequence $\{x_k\}_{k=1}^\infty$ of X σ-convergent to x^0, there exists a sequence $\{y_k\}_{k=1}^\infty \subset Y$ τ-converging to y in Y and satisfying condition $y_k \leq_\Lambda f(x_k)$, for any $k \in \mathbb{N}$.

Definition 9. (see [7]) A mapping $f : X \to Y^\bullet$ is said to be quasi lower semicontinuous (q-lsc) at $x^0 \in X$, if for each $b \in Y$ such that $b \ngeq_\Lambda f(x^0)$, there exists a neighborhood $\mathcal{O}$ of x^0 in the σ-topology of X such that $b \ngeq_\Lambda f(x)$ for each x in $\mathcal{O}$.

A mapping f is s-lsc (resp., q-lsc) if f is s-lsc (resp., q-lsc) at each point of X. It is clear that the s-lsc-property of f at x implies its q-lsc at this point. To characterize the properties of $(\Lambda, \sigma \times \tau)$-lower semicontinuity more precisely, we give the following result.

Proposition 3. (see [12]) If a mapping $f : X_\partial \to Y$ is q-lower semicontinuous at $x^0 \in X_\partial$ with respect to the $\sigma \times \tau$-topology on $X \times Y$, then f is $(\Lambda, \sigma \times \tau)$-lower semicontinuous at this point.

As a consequence of this result and the properties of quasi-lower semicontinuity, we have: if f is s-lsc then f is $(\Lambda, \sigma \times \tau)$-lsc. However, in general, $(\Lambda, \sigma \times \tau)$-ls continuity of the vector-valued mappings does not imply their q-lsc property. Indeed, let us consider the following example.

Example 5. Let $X_{ad} = [-3, -1]$, $Y = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}^2_+$ be the ordering cone of positive elements. We define a vector-valued mapping $f : X_{ad} \to Y$ as follows (see Fig. 5):

$$f(x) = \begin{cases} \begin{bmatrix} -x \\ 2 \end{bmatrix}, & x \neq -1, \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, & x = -1. \end{cases} \tag{3.6}$$

Let $x_0 = -1$. Then

$$f(x_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x_0} \widehat{f}(x) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \tag{3.7}$$

(see Fig. 5). Let us take $b = \begin{bmatrix} 1{,}5 \\ 3 \end{bmatrix}$. Obviously $b \ngeq_\Lambda f(x_0)$ and there is no neighborhood of the point x_0 such that $b \ngeq_\Lambda f(x)$ for all x from this neighborhood. Hence, this mapping is neither q-lsc nor lsc mapping at the point x_0. However, by (3.7), we have the inclusion

$$f(x_0) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x_0} \widehat{f}(x).$$

Hence, f is the $(\Lambda, \sigma \times \tau)$-lower semicontinuous mapping at $x_0 = -1$.

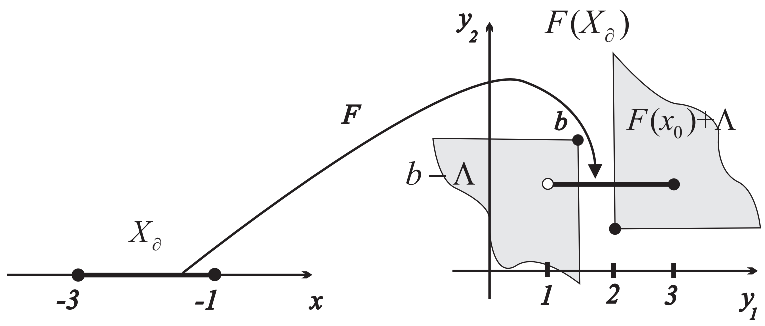

Fig. 5. The example of $(\Lambda, \sigma \times \tau)$-lsc mapping which is neither s-lsc nor q-lsc mapping

Before going on further, we prescribe some additional properties to the ordering cone Λ.

Definition 10. Let (Y, τ) be a real topological linear space with an ordering cone Λ. The convex cone Λ is called Daniell, if for every decreasing net (i.e. $i \le j \Longrightarrow y_j \le_\Lambda y_i$), which is lower bounded, τ-converges to its (Λ, τ)-infimum.

Condition ensuring the Daniall property are given by the next lemma.

Lemma 1. *Let (Y, τ) be a real topological linear space with an ordering cone Λ. If Y has compact intervals $[-z, z]$ and Λ is τ-closed and pointed, then Λ is Daniell.*

For this result see Borwein [6]. A typical example of Daniell cone with respect to the weak topology of $L^p(\Omega)$ $(1 < p < +\infty)$ is the so-called natural ordering cone in $L^p(\Omega)$ which is defined as

$$\Lambda_{L^p(\Omega)} = \{f \in L^p(\Omega) \ : \ f(x) \ge 0 \text{ almost everywhere on } \ \Omega\}.$$

Definition 11. We say that a non-empty subset Y_0 of a real topological space (Y, τ) with an ordering cone Λ is lower semibounded if every decreasing net $\{y_i\} \subset Y_0$ is bounded from below.

As a direct consequence of Definition 11, we have the following observation.

Remark 5. Let Y_0 be a lower semibounded subset of a partially ordered linear topological space Y with a τ-closed ordering cone Λ. Then, for any $z \in Y_0$ the section $Y_0^z = (\{z\} - \Lambda) \cap Y_0$ of Y_0 is bounded from below, that is, there exists an element $z^* \in Y$ such that $z^* \le_\Lambda y$ for all $y \in Y_0^z$. Hence, the lower semiboundedness of a subset Y_0 implies the lower semiboundedness of its τ-closure $\mathrm{cl}_\tau Y_0$.

Now we are ready to formulate the main result of this section.

Theorem 1. *Let (X, σ) and (Y, τ) be two real topological linear spaces, and let Y be partially ordered with the τ-closed pointed Daniell cone Λ. Moreover, let X_∂ be a non-empty sequentially σ-compact subset of X and let $f : X_\partial \to Y$ be a given $(\Lambda, \sigma \times \tau)$-lower semicontinuous mapping. Then the vector optimization problem (3.2) has a non-empty set of (Λ, τ)-efficient solutions.*

Remark 6. Before the proof, we note that in contrast to the scalar case for vector optimization problem (3.2) with a sequentially σ-compact subset of X_∂ and $(\Lambda, \sigma\times\tau)$-lower semicontinuous objective mapping $f : X_\partial \to Y$, the image set $f(X_\partial)$ can be unbounded from below. It means that, in general, there does not exist an element $y^* \in Y$ such that $f(X_\partial) \subset \{y^*\} + \Lambda$. Indeed, let us consider the following example: let $X = \mathbb{R}$, $X_\partial = [0; 1]$, $Y = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}^2_+$ be the ordering cone of positive elements. We suppose that a vector-valued mapping $f : X \to Y$ is defined as follows:

$$f(x) = \begin{bmatrix} -1/x \\ 1/x \end{bmatrix} \quad \text{if} \quad x \in [0; 1), \text{ and } \quad f(1) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Since

$$\mathrm{L}^{\sigma\times\tau}(f, 1) = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x) = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\},$$

it follows that

$$\liminf^{\Lambda,\tau}_{x \overset{\sigma}{\to} 1} \widehat{f}(x) = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}.$$

Hence this mapping is $(\Lambda, \sigma \times \tau)$-lower semicontinuous on X_∂. However the image set $f(X_\partial)$ is unbounded from below (see Fig. 6).

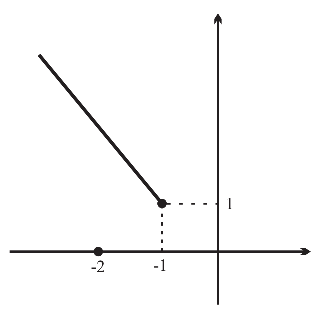

Fig. 6. The example of $(\Lambda, \sigma \times \tau)$-lsc mapping with lower unbounded image

Proof. Since the proof of this theorem is rather technical, we divide it into several steps.

Step 1. First we show that the image set $f(X_\partial)$ is lower semibounded in the sense of Definition 11. Indeed, let us assume the converse. Then, there exists a sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ such that the corresponding image sequence

$$\{y_k = f(x_k)\}_{k=1}^\infty \subset f(X_\partial)$$

is decreasing (i.e., $y_{k+1} \leq_\Lambda y_k \ \forall\, k \in \mathbb{N}$) and unbounded from below in Y. Hence $-\infty_\Lambda \in L^\tau\{y_k\}$, where $L^\tau\{y_k\}$ denotes the set of all its cluster points with respect to the τ-topology of Y. By the initial assumptions, the family $\{x_k\}_{k=1}^\infty \subset X_\partial$ is sequentially σ-compact, so we may suppose that $x_k \overset{\sigma}{\to} x^*$ in X, where x^* is some element of X_∂. Since the sequence $\{f(x_k)\}_{k=1}^{\infty}$ is unbounded from below, we have $\{-\infty_\Lambda\} \in \mathrm{L}^{\sigma\times\tau}_{min}(f, x^*)$. Hence, by Definition 5,

$$\liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x^*} f(x) = \{-\infty_\Lambda\}\,.$$

On the other hand, taking into account the $(\Lambda, \sigma\times\tau)$-lower semicontinuity property of f, we obtain

$$f(x^*) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x^*} f(x)$$

which contradicts the previous conclusion. This proves Step 1.

Step 2. Let us prove that the set $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$ is non-empty. We show that there exists at least one decreasing sequence $\{y_k\}_{k=1}^{\infty} \subset f(X_\partial)$ such that

$$y_k \xrightarrow{\tau} y^* \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x) = \mathrm{Inf}^{\Lambda,\tau}\{f(x) \,:\, \forall\, x \in X_\partial\}\,.$$

Let y be an arbitrary element of $\mathrm{cl}_\tau\, f(X_\partial)$. To begin with, we show that for any neighbourhood of zero $\mathcal{V}_\tau$ in (Y, τ) there exists an element $y^{\mathcal{V}} \in \mathrm{cl}_\tau\, f(X_\partial)$ such that

$$y^{\mathcal{V}} \leq_\Lambda y \quad\text{and}\quad \left(\{y^{\mathcal{V}}\} - \Lambda \setminus \{0_Y\}\right) \cap \left(\mathrm{cl}_\tau\, f(X_\partial) \setminus (\mathcal{V}_\tau + \{y^{\mathcal{V}}\})\right) = \emptyset. \tag{3.8}$$

Having assumed the converse, we suppose the existence of a sequence

$$\{y_k\}_{k=1}^{\infty} \subset \mathrm{cl}_\tau\, f(X_\partial)$$

such that

$$y_1 \in f(X_\partial), \quad y_{k+1} \in (\{y_k\} - \Lambda \setminus \{0_Y\}) \cap (\mathrm{cl}_\tau\, f(X_\partial) \setminus (\mathcal{V}_\tau + \{y_k\}))\ \ \forall\, k \in \mathbb{N}.$$

Since $y_{k+1} \in \{y_k\} - \Lambda \setminus \{0_Y\}$, this sequence is decreasing. Taking into account Remark 5, the set $\mathrm{cl}_\tau\, f(X_\partial)$ is lower semibounded. Therefore, there exists an element $y^* \in Y$ such that $y^* \leq_\Lambda y_k$ for all $k \in \mathbb{N}$. Hence, by Daniell property, this sequence τ-converges to its (Λ, τ)-infimum: $y_k \xrightarrow{\tau} \widetilde{y} \in Y$. However this contradicts the condition

$$y_{k+1} \in \mathrm{cl}_\tau\, f(X_\partial) \setminus (\mathcal{V}_\tau + \{y_k\}) \quad \forall\, k \in \mathbb{N}.$$

Thus the choice by the rule (3.8) is possible for any neighbourhood $\mathcal{V}_\tau$.

Let $\{\mathcal{V}_k\}_{k=1}^{\infty}$ be a neighbourhood system of zero in (Y, τ) such that $\mathcal{V}_{k+1} \subset \mathcal{V}_k$ for every $k \in \mathbb{N}$, and for any neighbourhood $\mathcal{V}(0_Y)$ in (Y, τ) there is an integer $k^* \in \mathbb{N}$ such that $\mathcal{V}_{k^*} \subseteq \mathcal{V}(0_Y)$. Then, using the choice rule (3.8), we can construct a sequence $\{u_k\}_{k=1}^{\infty} \subset \mathrm{cl}_\tau\, f(X_\partial)$, where u_1 is an arbitrary element of $f(X_\partial)$, as follows

$$\begin{aligned} u_{k+1} \leq_\Lambda u_k \quad\text{and}\quad & (\{u_k\} - \Lambda \setminus \{0_Y\}) \cap \\ & \cap\, (\mathrm{cl}_\tau\, f(X_\partial) \setminus (\mathcal{V}_k + \{u_k\})) = \emptyset \quad \forall\, k \geq 1. \end{aligned} \tag{3.9}$$

Since $u_{k+1} \in \{u_k\} - \Lambda$ it follows that

$$u_{k+1} \in \mathrm{cl}_\tau\, f(X_\partial) \quad\text{and}\quad u_{k+1} \notin \mathrm{cl}_\tau\, f(X_\partial) \setminus (\mathcal{V}_k + \{u_k\}).$$

Hence, in view of Daniell property, $\{u_k\}_{k=1}^{\infty}$ is the τ-converging decreasing sequence. As a result, there is an element

$$u^* \in \mathrm{Inf}^{\Lambda,\tau}\{u_k \in \mathrm{cl}_\tau f(X_\partial) \,:\, \forall\, k \in \mathbb{N}\}$$

such that $u_k \xrightarrow{\tau} u^*$. It is clear that $u^* \in \mathrm{cl}_\tau f(X_\partial)$. Our aim is to prove that $u^* \in \mathrm{Inf}^{\Lambda,\tau}\{f(x) \,:\, \forall\, x \in X_\partial\}$. To do this, we assume that there exists an element

$$q \in \mathrm{Inf}^{\Lambda,\tau}\{f(x) \,:\, \forall\, x \in X_\partial\}$$

such that $q \leq_\Lambda u^*$. Since $u^* \leq_\Lambda u_k$ for all $k \in N$, it follows that $q \leq_\Lambda u_k$ for all $k \in N$. Then (3.9) ensures that

$$(\{q\} - \Lambda \setminus \{0_Y\}) \cap (\mathrm{cl}_\tau f(X_\partial) \setminus (\mathcal{V}_k + \{u_k\})) = \emptyset \quad \forall\, k \in \mathbb{N}. \tag{3.10}$$

Hence (3.10) and the fact that $q \in \mathrm{cl}_\tau f(X_\partial)$ imply $q \in \mathcal{V}_k + \{u_k\}$ for every $k \in \mathbb{N}$, that is, $u_k \xrightarrow{\tau} q$ in Y. Thus $u^* = q$ and this concludes the Step 2.

Step 3: We show that the set $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ is non-empty. Let ξ be any element of $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$. Then, by Definition 4, there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that $y_k \xrightarrow{\tau} \xi$ in Y. We define a sequence $\{x_k\}_{k=1}^{\infty} \subset X_\partial$ as follows $x_k = f^{-1}(y_k)$ for all $k \in \mathbb{N}$. Since the set X_∂ is sequentially σ-compact, we may suppose that there exists $x_0 \in X_\partial$ such that $x_k \xrightarrow{\sigma} x_0$ in X. Hence $\xi \in \mathrm{L}^{\sigma\times\tau}(f, x_0)$, and we get

$$\mathrm{L}^{\sigma\times\tau}(f, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x) \neq \emptyset.$$

Then, due to the $(\Lambda, \sigma \times \tau)$-lower semicontinuity of the mapping f on X_∂ and Definition 5, we obtain

$$f(x_0) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x_0} f(x) = \mathrm{L}^{\sigma\times\tau}(f, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x).$$

Thus, on the one hand,

$$f(x_0) \in \mathrm{L}^{\sigma\times\tau}(f, x_0),$$

which implies the equality

$$f(x_0) = \xi = \tau - \lim_{k\to\infty} y_k.$$

On the other hand, $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$. Hence, $x_0 \in \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ and this concludes the proof. $\square$

4. Vector optimization problems for $(\Lambda, \sigma \times \tau)$-lower semicontinuous objective mappings and their scalarization

Typically, scalarization means the replacement of a vector optimization problem by a suitable scalar optimization problem that is an optimization problem with a real-valued objective functional. It is a fundamental principle in vector optimization that optimal (minimal) elements of a subset of a partially ordered linear space can be characterized as optimal solutions of certain scalar optimization problems. For the problem (3.1), a wide family of scalar problems is known, which fully describe the set of all minimal elements $\mathrm{Min}(X_\partial, f, \Lambda)$ under suitable assumptions (see, for instance, [9, 11, 14, 15] and the references therein). However, our prime interest is to describe the set $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ of (Λ,τ)-efficient solutions to the vector problem (3.2) (see (3.5)), which involves some topological properties of the objective mapping f and the space Y. In order to do it, we will consider the problem of scalar representation of vector optimization problem (3.2) with a $(\Lambda, \sigma\times\tau)$-lower semicontinuous mapping $f : X_\partial \to Y$, using the "simplest" method of the "weighted sum".

To begin with, we introduce some additional suppositions. As was mentioned above, the objective space Y is dual to some separable Banach space V (that is $Y = V^*$). Suppose that the space V is partially ordered with a nontrivial pointed ordering cone $K \subset V$ for which Λ is the dual cone, that is,

$$\Lambda = K^* := \Big\{ y \in Y \ : \ \langle y, \lambda\rangle_{Y;V} \geq 0 \ \text{ for all } \ \lambda \in K \Big\}. \tag{4.1}$$

Definition 12. We say that $\lambda \in V$ is a quasi-interior point of the cone K if $\lambda \in K$ and $\langle b, \lambda\rangle_{Y;V} > 0$ for all $b \in \Lambda \setminus \{0\}$.

We denote by $K^\sharp$ the set of all quasi-interior points of K. Note that, in general, we have the inclusion $\mathrm{cor}\,(K) \subseteq K^\sharp$, where $\mathrm{cor}\,K$ is an algebraical interior of the cone K (for more details we refer to [11]).

In what follows, we associate with the vector optimization problem (3.2) the following scalar minimization problem

$$f_\lambda(x) = \langle f(x), \lambda\rangle_{Y;V} \to \inf \qquad \text{subject to } \ x \in X_\partial \subset X \tag{4.2}$$

where λ is an element of the cone K.

The main property of this problem can be characterized as follows.

Theorem 2. *Let X and $Y = V^*$ be two real Banach spaces, let Y be endowed with the weak-$*$ topology τ, and let Y be partially ordered with the cone $\Lambda = K^*$, where K is an ordering cone in V with a non-empty quasi-interior $K^\sharp$. Let also X_∂ be a non-empty subset of X, and let $f : X_\partial \to Y$ be a given mapping. Assume that there are elements $x^0 \in X_\partial$ and $\lambda \in K^\sharp$ such that $x^0 \in \operatorname*{Argmin}_{x\in X_\partial} \langle f(x), \lambda\rangle_{Y;V}$. Then x^0 is a (Λ,τ)-efficient solution to the problem (3.2).*

Proof. By the initial assumptions, we have

$$f_\lambda(x^0) - f_\lambda(x) = \big\langle f(x^0) - f(x), \lambda\big\rangle_{Y;V} \leq 0, \quad \forall\, x \in X_\partial. \tag{4.3}$$

Let z be any element of the image set $\mathrm{cl}_\tau\, f(X_\partial)$. Then there exists a sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ such that $f(x_k) \xrightarrow{\tau} z$ in Y as $k \to \infty$. Hence, in view of (4.3), we get

$$\big\langle f(x^0) - f(x_k), \lambda\big\rangle_{Y;V} \leq 0, \quad \forall\, k \in \mathbb{N}. \tag{4.4}$$

Passing to the limit in (4.4) as $k \to \infty$, we obtain

$$\big\langle f(x^0) - z, \lambda\big\rangle_{Y;V} \leq 0, \quad \forall\, z \in \mathrm{cl}_\tau\, f(X_\partial). \tag{4.5}$$

Let us assume that $x^0 \notin \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$. Then there exists an element $h \in \mathrm{cl}_\tau\, f(X_\partial)$ such that $h <_\Lambda f(x^0)$. So, $f(x^0) - h \in \Lambda \setminus \{0_Y\}$. Hence, by Definition 12,

$$\big\langle f(x^0) - h, \lambda \big\rangle_{Y;V} > 0,$$

and we come to a contradiction with (4.5). So, $x^0 \in \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ and this concludes the proof. □

As an evident consequence of this result, we have the following conclusion.

Corollary 1. *Under suppositions of Theorem 2, we have*

$$\bigcup_{\lambda \in K^\sharp} \operatorname*{Argmin}_{x \in X_\partial} \langle f(x), \lambda \rangle_{Y;V} \subseteq \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda). \tag{4.6}$$

Remark 7. Note that Theorem 2 generally fails when $\lambda \in K \setminus K^\sharp$. Indeed, let $V = Y = \mathbb{R}^2$, $X_\partial = [1,2]$, and let $\Lambda = \mathbb{R}^2_+$ be the ordering cone of positive elements (then $K = \Lambda$). We define the objective mapping $f : X_\partial \to Y$ as follows:

$$f(x) = \begin{bmatrix} x \\ 1 \end{bmatrix} \text{ if } x \in (1,2], \quad \text{and} \quad f(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ at the point } x = 1$$

(see Fig. 7). Straightforward calculations show that

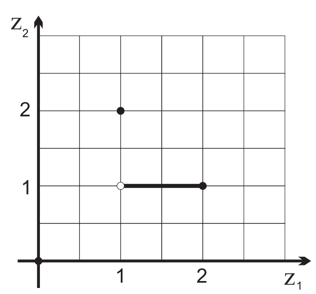

Fig. 7. The example of the problem for which $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) = \emptyset$

$$\liminf{}^{\Lambda,\tau}_{x \to 1} f(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and hence $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) = \emptyset$. However, if we take $\lambda = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \in K \setminus K^\sharp$, then

$$\langle f(x), \lambda \rangle_{V^*;V} = x$$

and hence

$$\operatorname*{Argmin}_{x \in [1,2]} \langle f(x), \lambda \rangle_{V^*;V} = \{1\} \notin \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda).$$

Before proceeding further, we note that the objective mapping in Theorem 2 does not possess the $(\Lambda, \sigma \times \tau)$-lower semicontinuity property, in general. So the question is about the solvability of the associated scalar minimization problems (4.2) with $\lambda \in K^\sharp$. Following the direct method in the Calculus of Variations, the constrained minimization problem (4.2) has a non-empty set of solutions, provided X_∂ is a σ-compact subset and

$$f_\lambda(\cdot) = \langle f(\cdot), \lambda \rangle_{Y;V} : X_\partial \to \overline{\mathbb{R}}$$

is a proper lower σ-semicontinuous function. However, the characteristic feature of vector optimization problems (3.2) is the fact that with any $(\Lambda, \sigma \times \tau)$-lower semicontinuous mapping $f : X_\partial \to Y$, which is neither lower semicontinuous nor quasi-lower semicontinuous on X_∂, there can be always associated a scalar minimization problem (4.2) for which the corresponding cost functional $f_\lambda : X_\partial \to \mathbb{R}$ is not lower σ-semicontinuous on X_∂. Indeed, let τ be the weak-$*$ topology on Y, and let x^0 be a point of X_∂ where the quasi-lower semicontinuity of f is failed. Then there exists at least one element $a^* \in \mathrm{cl}_\tau\,(f(X_\partial))$ such that

$$a^* \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x^0} f(x), \;\; f(x^0) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x^0} f(x), \quad \text{and} \quad a^* \neq f(x^0). \tag{4.7}$$

Let $\{x_k\}_{k=1}^\infty \subset X_\partial$ be a sequence such that $x_k \xrightarrow{\sigma} x^0$ in X and $f(x_k) \xrightarrow{\tau} a^*$ in Y. Since $a^* \not\geq_\Lambda f(x^0)$, it follows that $a^* - f(x^0) \notin \Lambda$ and hence there exists a vector $\lambda^* \in K$ such that

$$\left\langle a^* - f(x^0), \lambda^* \right\rangle_{Y;V} < 0.$$

As a result, we have

$$\begin{aligned} &\liminf_{k\to\infty} f_{\lambda^*}(x_k) = \lim_{k\to\infty} \langle f(x_k), \lambda^* \rangle_{Y;V} \\ &= \langle a^*, \lambda^* \rangle_{Y;V} < \left\langle f(x^0), \lambda \right\rangle_{Y;V} = f_{\lambda^*}(x^0). \end{aligned}$$

Thus, the lower σ-semicontinuity property for f_{λ^*} fails at x^0. Moreover, as the following example shows, for $(\Lambda, \sigma \times \tau)$-lower semicontinuous mappings $f : X_\partial \to Y$ a situation is possible when none of the scalar functions $f_\lambda(x) = \langle f(x), \lambda \rangle_{Y;V}$ is lower σ-semicontinuous for any $\lambda \in K^\sharp$.

Example 6. Let $X_\partial = [1, 2] \subset \mathbb{R}$, and let $\Lambda = \mathbb{R}^2_+$ be the ordering cone of positive elements in $Y = \mathbb{R}^2$. It is clear that in this case $V = Y$ and $K = \Lambda$. Let us consider the mapping $f : X_\partial \to \mathbb{R}^2$ defined by (see Fig. 8)

$$f(x) = \begin{cases} \begin{bmatrix} x \\ 1 \end{bmatrix}, & \text{if } x \in [1,2] \setminus \{1 + 1/k,\ k \in \mathbb{N}\}, \\ \begin{bmatrix} 0 \\ 1+k \end{bmatrix}, & \text{if } x = 1 + 1/k,\ k \in \mathbb{N}. \end{cases}$$

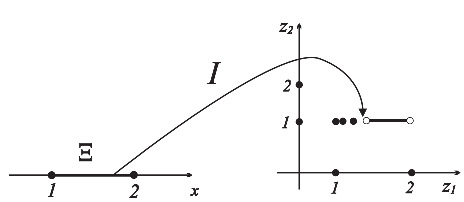

Fig. 8. The vector-valued mapping in Example 6

Straightforward calculations show that

$$\liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} 1} f(x) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

$$\liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} (1+1/k)} f(x) = \left\{ \begin{bmatrix} 0 \\ 1+k \end{bmatrix}, \begin{bmatrix} 1+1/k \\ 1 \end{bmatrix} \right\}.$$

Since

$$f(1) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} 1} f(x) \quad \text{and} \quad f(1+1/k) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} (1+1/k)} f(x),$$

it means that the mapping $f : X_\partial \to \mathbb{R}^2$ is $(\Lambda, \sigma \times \tau)$-lower semicontinuous at these points and in fact on the whole domain X_∂. Let $\lambda = \left[\begin{smallmatrix} \lambda_1 \\ \lambda_2 \end{smallmatrix}\right]$ be any vector with non-negative components, i.e. $\lambda \in K$. Then the scalar function f_λ, associated with the vector-valued mapping f by the scheme of the “weighted sum”, can be represented in the form

$$f_\lambda(x) := \langle f(x), \lambda \rangle_{Y;V} = \begin{cases} \lambda_1 x + \lambda_2, & \text{if } x \neq 1+1/k, \\ \lambda_2(1+k), & \text{if } x = 1+1/k, \end{cases} \qquad \forall\, k \in \mathbb{N},\ \forall\, x \in X_\partial. \tag{4.8}$$

To be sure that the lower σ-semicontinuity property for this function at the points $x_k = 1+1/k$ is valid, we have to choose the parameters λ_1 and λ_2 so that the inequality

$$\lambda_2(1+k) \leq \lambda_1(1+1/k) + \lambda_2 \tag{4.9}$$

holds true for every $k \in \mathbb{N}$.

However, taking into account the non-negativeness of λ_i and passing in (4.9) to the limit as $k \to \infty$, we obtain $\lambda_2 = 0$. As a result, we have

$$f_\lambda(x) = \begin{cases} \lambda_1 x, & \text{if } x \neq 1+1/k, \\ 0, & \text{if } x = 1+1/k, \end{cases} \qquad \forall\, k \in \mathbb{N},\ \forall\, x \in X_\partial. \tag{4.10}$$

Nevertheless, as follows from (4.10), the inequality

$$f_\lambda(1) \leq \liminf_{k\to\infty} f_\lambda(x_k)$$

does not hold for any $\lambda_1 > 0$ with the exception of $\lambda_1 = 0$. Thus, there is a unique scalar function in the collection (4.8) satisfying the lower semicontinuity property in the domain $X_\partial = [1,2]$. This function is $f_\lambda(x) \equiv 0$.

This example motivates the introduction of the following notion.

Definition 13. Let $f : X_\partial \to Y$ be a given mapping. The cone

$$K_f^\sigma := \{\lambda \in K \ : \ f_\lambda \ \text{ is lower } \ \sigma\text{-semicontinuous on } \ X_\partial\} \tag{4.11}$$

is called the cone of σ-semicontinuity for the mapping f.

As a result, Theorem 2 can be sharped as follows.

Theorem 3. *Let X be a reflexive Banach space, let V be a separable Banach space, and let $Y = V^*$ be endowed with the weak-$*$ topology τ and partially ordered with a pointed Daniell cone $\Lambda = K^*$, where K is a weakly closed ordering cone in V. Let also X_∂ be a non-empty bounded weakly closed subset of X, and let $f : X_\partial \to Y$ be a $(\Lambda, \sigma \times \tau)$-lower semicontinuous mapping, where σ is the weak topology of X. Assume that $K_f^\sigma \setminus 0_V \neq \emptyset$. Then*

$$\operatorname*{Argmin}_{x \in X_\partial} \langle f(x), \lambda \rangle_{Y;V} \cap \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) \neq \emptyset \quad \forall\, \lambda \in K_f^\sigma \setminus 0_V. \tag{4.12}$$

Proof. As follows from Theorem 1, under the above assumptions, we have

$$\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) \neq \emptyset.$$

Let λ be any element of $K_f^\sigma \setminus 0_V$. Then, by the direct method in the Calculus of Variations, we obtain

$$\operatorname*{Argmin}_{x \in X_\partial} \langle f(x), \lambda \rangle_{Y;V} \neq \emptyset.$$

If $\lambda \in K^\sharp$ then relation (4.12) is obvious by Theorem 2. So, we suppose that $\lambda \in K_f^\sigma \setminus \big(K^\sharp \cup 0_V\big)$. Assume that

$$\operatorname*{Argmin}_{x \in X_\partial} \langle f(x), \lambda \rangle_{Y;V} \nsubseteq \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda).$$

Then, there exists an element $x^* \in X_\partial$ such that

$$x^* \in \operatorname*{Argmin}_{x \in X_\partial} \langle f(x), \lambda \rangle_{Y;V}, \tag{4.13}$$

$$x^* \notin \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda). \tag{4.14}$$

Hence, by (4.14), there exists an element

$$y^* \in \mathrm{Min}_\Lambda\,(\mathrm{cl}_\tau f(X_\partial)) \subseteq \mathrm{cl}_\tau f(X_\partial) \ \text{ such that } \ y^* <_\Lambda f(x^*).$$

However, in view of (4.13) and (4.1), this leads us to the equality

$$f_\lambda(x^*) = \langle f(x^*), \lambda \rangle_{Y;V} = \langle y^*, \lambda \rangle_{Y;V}. \tag{4.15}$$

Let $\{x_k\}_{k=1}^\infty$ be a sequence in X_∂ such that

$$f(x_k) \xrightarrow{\tau} y^* \ \text{ as } \ k \to \infty. \tag{4.16}$$

Since the set X_∂ is sequentially weakly compact, we may suppose that there exists $x_0 \in X_\partial$ such that $x_k \xrightarrow{\sigma} x_0$ in X. On the other hand, $y^* \in \mathrm{Min}_\Lambda\,(\mathrm{cl}_\tau f(X_\partial))$. Hence, $y^* \in \mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} f(x)$ by Definition 6. As a result, we have $x_0 \in \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$. Taking into account the lower σ-semicontinuity of the functional $f_\lambda : X_\partial \to \mathbb{R}$, we get

$$\langle f(x_0), \lambda \rangle_{Y;V} \le \liminf_{k \to \infty} \langle f(x_k), \lambda \rangle_{Y;V} \overset{\text{by (4.16)}}{=} \langle y^*, \lambda \rangle_{Y;V}.$$

Then, combining this with (4.15), we obtain

$$\langle f(x_0), \lambda \rangle_{Y;V} \le \langle f(x^*), \lambda \rangle_{Y;V},$$

i. e.

$$x_0 \in \operatorname*{Argmin}_{x \in X_\partial} \langle f(x), \lambda \rangle_{Y;V}.$$

Thus, we have shown that there exists at least one element $x_0 \in X_\partial$ which is a joint point of the sets $\operatorname*{Argmin}_{x \in X_\partial} \langle f(x), \lambda \rangle_{Y;V}$ and $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$, respectively. This completes the proof. □

As an evident consequence of this theorem, we have the following conclusion:

Corollary 2. *Assume that in addition to the conditions of Theorem 3 there exists an element $\lambda \in K_f^\sigma \setminus 0_V$ such that the infimum in the scalar problem*

$$\textit{Minimize } f_\lambda(x) = \langle f(x), \lambda \rangle_{Y;V} \ \textit{ subject to } \ x \in X_\partial \tag{4.17}$$

is attained at a unique point $x^* \in X_\partial$. Then $x^* \in \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$.

Note that, we do not give the conditions which would guarantee the fulfilment of the relation $K_f^\sigma \setminus 0_V \neq \emptyset$. However, as a hypothesis, we can make the following conjecture:

If the image set $f(X_\partial)$ is bounded in $\langle Y, \|\cdot\| \rangle$ and K has a non-empty quasi-interior ($K^\sharp \neq \emptyset$), then under conditions of Theorem 3, the cone K_f^σ contains at least one nontrivial element.

To motivate this hypothesis, we note that if a uniformly bounded mapping $f : X_\partial \to Y$ is quasi-lower semicontinuous on X_∂ then f is lower semicontinuous (see [2]). In this case the functions $f_\lambda(x) = \langle f(x), \lambda \rangle_{Y;V}$ are lower σ-semicontinuous on X_∂ for every $\lambda \in K$. Hence $K_f^\sigma \setminus 0_V \neq \emptyset$. Let x^0 be a point of X_∂ where the quasi-lower semicontinuity of f fails. Then there exists at least one element $a^* \in \mathrm{cl}_\tau\left(f(X_\partial)\right)$ with properties (4.7). Let λ^* be an element of K such that

$$\left\langle f(x^0), \lambda^* \right\rangle_{Y;V} \le \langle a^*, \lambda^* \rangle_{Y;V} \quad \forall\, a^* \in \liminf\nolimits^{\Lambda,\tau}_{x \xrightarrow{\sigma} x^0} f(x). \tag{4.18}$$

The existence of λ^* immediately follows from the fact that

$$f(x^0) \not>_\Lambda a^* \ \text{ for all } \ a^* \in \liminf\nolimits^{\Lambda,\tau}_{x \xrightarrow{\sigma} x^0} f(x).$$

Let $\{x_k\}_{k=1}^\infty \subset X_\partial$ be a sequence such that $x_k \xrightarrow{\sigma} x^0$ in X. Since each of elements a^* belongs to the set

$$\mathrm{L}^{\sigma\times\tau}(f, x^0) := \bigcup_{\{x_k\}_{k=1}^\infty \in \mathfrak{M}_\sigma(x^0)} \mathrm{L}^\tau\{\widehat{f}(x_k)\}$$

of τ-cluster points of the sequences $\left\{\widehat{f}(x_k)\right\}_{k=1}^\infty$, it follows from (4.18) that

$$\left\langle f(x^0), \lambda^* \right\rangle_{Y;V} \le \liminf_{k\to\infty} \langle f(x_k), \lambda^* \rangle_{Y;V}.$$

Thus, the function f_{λ^*} is sequentially lower σ-semicontinuous at the point x^0.

5. The ill-posed vector optimization problems and their generalized solutions

Let λ be an arbitrary element of the cone K. Denote by

$$\mathrm{Sol}(X_\partial;\, f_\lambda) := \operatorname*{Argmin}_{x\in X_\partial} f_\lambda(x)$$

the solution set to the scalar problem (4.17). We recall that the problem (4.17) is said to be well-posed in the generalized sense when every minimizing sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ (i.e. such that $f_\lambda(x_k) \to \inf_{x\in X_\partial} f_\lambda(x)$) has a subsequence σ-converging to some element of $\mathrm{Sol}(X_\partial;\, f_\lambda)$. We recall also a generalization of the above mentioned notion. The problem (4.17) is said to be well-set when every minimizing sequence contained in $X_\partial \setminus \mathrm{Sol}(X_\partial;\, f_\lambda)$ has a σ-cluster point in $\mathrm{Sol}(X_\partial;\, f_\lambda)$. However, as follows from the arguments of this section (see also Example 7 given below), the problem (4.17) can be neither well-posed nor well-set, in general. The main reason is the $(\Lambda, \sigma\times\tau)$-lower semicontinuity property of the objective mapping f which is the weakened property of lower semicontinuity for vector-valued mappings in Banach spaces.

Example 7. Let $X_\partial = \{x \in X \;:\; \|x\| \le 1\}$ be a unit closed ball in a reflexive Banach space X. Let $Y = \mathbb{R}^2$ be the objective space partially ordered with the cone $\Lambda = \mathbb{R}^2_+$ of positive elements in $\mathbb{R}^2$. We suppose that X and Y are endowed with the strong topologies σ and τ, respectively. Let the objective mapping $f : X_{ad} \to \mathbb{R}^2$ be defined as

$$f(x) = \begin{bmatrix} 2-\|x\| \\ 1+\|x\| \end{bmatrix} \text{ if } x \in X_{ad} \setminus \{0_X \cup S\}\,,$$

$$f(x) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ if } x \in S,\;\; f(0_X) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $S = \{x \in \mathbb{X} \;:\; \|x\| = 1\}$ is the unit sphere in X. Since

$$\mathrm{Min}_\Lambda\left(\mathrm{cl}_\tau f(X_\partial)\right) = \mathrm{Min}_\Lambda\left(f(X_\partial)\right) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

it follows that

$$\liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} 0_X} f(x) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

and hence f is $(\Lambda, \sigma\times\tau)$-lower semicontinuous on X_∂. Then, by Theorem 1, the corresponding vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$ is solvable and, moreover, $x^{eff} = 0_X$ is its unique (Λ, τ)-efficient solution.

Let us consider the following scalar problem

$$\text{Minimize } f_\lambda(x) = (f(x), \lambda)_{\mathbb{R}^2} \text{ subject to } \; x \in X_\partial, \tag{5.1}$$

associated with the vector problem $\langle X_\partial, f, \Lambda, \tau\rangle$, where

$$\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_\lambda(x) := (f(x), \lambda)_{\mathbb{R}^2} = \begin{cases} 2-\|x\|, & \text{if } \; \|x\| < 1 \text{ and } x \neq 0_X, \\ 2, & \text{if } \; x \in S, \\ 1, & \text{if } \; x = 0_X \end{cases}$$

Through direct verification we can show that $\mathrm{Sol}(X_\partial;\, f_\lambda) = \{0_X\}$. However, this scalar problem is neither well-posed nor well-set with respect to the strong topology of X, because all minimizing sequences for (5.1) containing in $X_\partial \setminus \mathrm{Sol}(X_\partial;\, f_\lambda)$ have σ-cluster points on the unit sphere $S = \{x \in X \,:\, \|x\| = 1\}$.

In many applications it has a sense to weaken the requirement on efficient solutions to the vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$. In particular, we may let the objective mapping to attain its efficient infimum on the set X_∂ with some error. On the other hand, the set of (Λ, τ)-efficient solutions to such problem can possibly be empty, i.e., the efficient infimum of the objective mapping is often unattainable on the given set X_∂. Nevertheless, the absence of its infimum does not mean that the vector optimization problem makes no sense, since its efficient infimum exists and hence can be approached with some accuracy.

Definition 14. We say that a sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ is minimizing to the vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$, if $f(x_k) \overset{\tau}{\to} \xi$ in Y, where ξ is an element of $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$.

Definition 15. We say that the vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$ is well-posed in the Tikhonov sense with respect to the σ-topology of X, if it is solvable and every minimizing sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ has a subsequence σ-converging to some element of $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$. In this case a minimizing sequence is called a Tikhonov minimizing sequence. We also say that the vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$ is well-set in the Tikhonov sense with respect to the σ-topology of X, if it is solvable and every minimizing sequence contained in $X_\partial \setminus \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ has a σ-cluster point in $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$.

Note that having a Tikhonov minimizing sequence, we can guarantee both the proximity of the corresponding values of the objective mapping to its efficient infimum and the proximity of the approximation itself to one of the (Λ, τ)-efficient solutions of the problem. Nevertheless it should be stressed that even in simple applied problems the construction of Tikhonov minimizing sequences and corresponding Tikhonov approximate solutions usually turns out to be a very complicated and sometimes unsolvable problem. In view of this, it is reasonable to weaken the requirements on approximate solutions to the vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$.

Definition 16. We say that an element $x^* \in X_\partial$ is the (σ, τ)-generalized solution to vector optimization problem (3.2), if there exist a sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ and an element $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$ such that $x_k \overset{\sigma}{\rightharpoonup} x^*$ in X and $f(x_k) \overset{\tau}{\to} \xi$ in Y.

Thus, a vector optimization problem may have an approximate solution even in the absence of its solvability. It is clear that any Tikhonov approximate solution to the problem $\langle X_\partial, f, \Lambda, \tau\rangle$ is also a (σ, τ)-generalized solution. However, even if a (Λ, τ)-efficient solution is available ($x^{eff} \in \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$), we cannot guarantee the proximity of an (σ, τ)-generalized solution x^* to $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$ in the σ-topology of X.

We denote by $\mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda)$ the set of all (σ, τ)-generalized solutions to the problem $\langle X_\partial, f, \Lambda, \tau\rangle$. It is clear that

$$\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) \subseteq \mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda).$$

Moreover, as evident consequence of Theorem 1, we have the following obvious result:

Proposition 4. Under suppositions of Theorem 1, the vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$ is well-set in the Tikhonov sense with respect to the topology of X, and in addition $\mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda) = \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$.

However, as the next example indicates, the inverse inclusion

$$\mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda) \subset \mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda)$$

does not generally hold.

Example 8. Let $X_\partial = \{x \in \mathbb{X} \ :\ \|x\| \le 1\}$ be a unit ball in a Banach space X, let $Y = \mathbb{R}^2$ be partially ordered with the cone $\Lambda = \mathbb{R}^2_+$ of positive elements in $\mathbb{R}^2$. Let the mapping $f : X_\partial \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{bmatrix} 1+\|x\| \\ 1+\|x\| \end{bmatrix} \ \text{ if } x \in X_\partial \setminus \{0_X \cup S\},$$

$$f(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ \text{ if } x \in S,\ \ f(0_X) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

where $S = \{x \in \mathbb{X} \ :\ \|x\| = 1\}$ is the unite sphere in X. We endow the spaces X

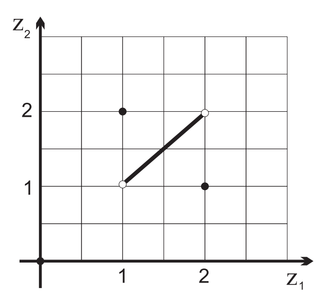

Fig. 9. The set $f(X_\partial)$ to Example 8

and Y with the weak (σ) and the strong (τ) topologies, respectively. Since

$$\mathrm{Min}_\Lambda\left(f(X_\partial)\right) = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \ \text{ and } \ \mathrm{Min}_\Lambda\left(\mathrm{cl}_\tau f(X_\partial)\right) = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\},$$

it follows that $\mathrm{Min}(X_\partial, f, \Lambda) = \{0_X\} \cup S$ whereas $\mathrm{Eff}_\tau(X_\partial;\, f;\, \Lambda) = \emptyset$. However, the set of (σ,τ)-generalized solutions to the problem $\langle X_\partial, f, \Lambda, \tau\rangle$ is non-empty. Indeed, let us fix a sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ such that

$$x_k \rightharpoonup 0_X \ \text{ in } \ X \ \text{ and } \ f(x_k) \to \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

Then, following Definition 16, we have

$$0_X \in \mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda)$$

and, in fact,

$$\mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda) = \{0_X\}\,.$$

Having taken $\lambda^* = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, we consider the following scalar problem associated with the vector problem $\langle X_\partial, f, \Lambda, \tau\rangle$:

$$f_\lambda(x) := (f(x), \lambda)_{\mathbb{R}^2} = \begin{cases} 1 + \|x\|, & \text{if } \|x\| < 1 \text{ and } x \neq 0_X, \\ \quad 1, & \text{if } \|x\| = 1, \\ \quad 2, & \text{if } x = 0_X \end{cases} \tag{5.2}$$

Straightforward calculations show that

$$\operatorname*{Argmin}_{x\in X_\partial} f_\lambda(x) = \{x \in X_\partial \ :\ \|x\| = 1\}\,.$$

As a result, we have

$$\mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda) \cap \operatorname*{Argmin}_{x\in X_\partial} f_\lambda(x) = \emptyset.$$

Thus, any solution of the scalar problem (5.2) is neither a (Λ, τ)-efficient solution nor a generalized one to the vector problem $\langle X_\partial, f, \Lambda, \tau\rangle$. Thus, in view of Definition 15, $\langle X_\partial, f, \Lambda, \tau\rangle$ can be characterized as the ill-posed vector optimization problem.

To obtain the sufficient conditions which would guarantee that the set of (σ, τ)-generalized solutions to the problem $\langle \Xi, I, \Lambda, \tau\rangle$ is non-empty, we use the scalarization of this problem in the form (4.2).

Let $\mathrm{sc}^-_\sigma f_\lambda : X_\partial \to \mathbb{R}$ denote the lower σ-semicontinuous envelope of the functional $f_\lambda(x) = \langle f(x), \lambda\rangle_{Y;V}$ with some $\lambda \in K$, that is, $\mathrm{sc}^-_\sigma f_\lambda$ is the greatest lower σ-semicontinuous functional majorized by f_λ on X_∂. Then, following the direct method in the Calculus of Variations, we get:

Proposition 5. Let X_∂ be a sequentially closed subset of a linear topological space (X, σ). Assume that for a fixed $\lambda \in K$ the functional $\mathrm{sc}^-_\sigma f_\lambda : X_\partial \to \mathbb{R}$ is countably σ-coercive, i.e. the σ-closure of the set $\{x \in X_\partial \ :\ \mathrm{sc}^-_\sigma f_\lambda(x) \leq t\}$ is countably σ-compact for every $t \in \mathbb{R}$. Then every minimizing sequence for $\inf_{x\in X_\partial} \mathrm{sc}^-_\sigma f_\lambda(x)$ has a σ-cluster point which is a minimum point of $\mathrm{sc}^-_\sigma f_\lambda$ on X_∂, i.e., $\mathrm{Sol}(X_\partial;\, \mathrm{sc}^-_\sigma f_\lambda) \neq \emptyset$.

Remark 8. It is clear that this theorem remains valid if instead of the countable σ-coerciveness of $\mathrm{sc}^-_\sigma f_\lambda$ on X_∂ we assume the sequential σ-compactness of the set X_∂.

Now we are able to prove the main result of this paper.

Theorem 4. *Let X be a reflexive Banach space, σ be the weak topology on X, V be a separable Banach space, and the Banach space $Y = V^*$ be endowed with the weak-$*$ topology τ and partially ordered with a pointed cone $\Lambda = K^*$, where K is a convex pointed cone in V with non-empty algebraic interior* $\mathrm{cor}\,(K)$*. Let also X_∂ be a non-empty sequential σ-compact subset of X, and let $f : X_\partial \to Y$ be a* *given mapping (not necessary $(\Lambda, \sigma \times \tau)$-lower semicontinuous on X_∂). Then the following inclusion is valid:*

$$\bigcup_{\lambda \in K^\sharp} \operatorname*{Argmin}_{x \in X_\partial} \mathrm{sc}_\sigma^- f_\lambda(x) \subseteq \mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda). \tag{5.3}$$

Proof. To begin with, we note that the convexity of the pointed cone K and condition $\mathrm{cor}\,(K) \neq \emptyset$ imply the inclusion $\mathrm{cor}\,(K) \subset K^\sharp$ (see [11]). Hence the quasi interior $K^\sharp$ of K is non-empty. Let λ be any element of $K^\sharp$. Then, by Proposition 5, there exists at least one element $x^* \in X_\partial$ such that

$$x^* \in \operatorname*{Argmin}_{x \in X_\partial} \mathrm{sc}_\sigma^- f_\lambda(x). \tag{5.4}$$

Since $\mathrm{sc}_\sigma^- f_\lambda(x)$ is the lower σ-semicontinuous envelope of the

$$f_\lambda(x) = \langle f(x), \lambda \rangle_{Y;V}\,,$$

it follows that there exists a sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ such that $x_k \xrightarrow{\sigma} x^*$ and

$$\lim_{k \to \infty} \langle f(x_k), \lambda \rangle_{Y;V} = \mathrm{sc}_\sigma^- f_\lambda(x^*) \leq$$

$$\text{by condition (5.4)}$$

$$\leq \mathrm{sc}_\sigma^- f_\lambda(x) \leq \langle f(x), \lambda \rangle_{Y;V} \tag{5.5}$$

$\forall x \in X_\partial$. Since $K^\sharp \cup 0_V$ is a nontrivial convex cone in V with non-empty algebraical interior, it follows that it is a reproducing cone in V, that is,

$$\left[K^\sharp \cup 0_V\right] - \left[K^\sharp \cup 0_V\right] = V$$

(see [11]). Then, following Peressini [17] and Borwein [6], we have that in the dual space $Y = V^*$ the ordering cone $\Lambda = K^*$ is normal with respect to the norm topology of Y, that is,

$$y <_\Lambda z \quad \Longrightarrow \quad \|y\| < \|z\|. \tag{5.6}$$

Now, turning back to the formula (5.5), we get: there exist an integer $\widehat{k} \in \mathbb{N}$ and an element $\widehat{y} \in Y$ such that

$$\langle f(x_k), \lambda \rangle_{Y;V} < \langle \widehat{y}, \lambda \rangle_{Y;V} \quad \forall\, k > \widehat{k}.$$

Since $\lambda \in K^\sharp$, this implies $f(x_k) <_\Lambda \widehat{y}$ for all $k > \widehat{k}$. Using the normality property (5.6) of the cone Λ for the norm topology of Y, we come to the conclusion: there exists a constant $c > 0$ such that

$$\|f(x_k)\|_Y \leq C \quad \text{for all} \quad k > \widehat{k}.$$

Hence, without loss of generality, we may suppose that the sequence $\{f(x_k)\}_{k=1}^\infty$ is bounded in Y. So, by Banach-Alaoglu Theorem, there exist an element $\eta \in Y$ and a subsequence of $\{f(x_k)\}_{k=1}^\infty$ (still denoted by suffix k) such that $f(x_k) \xrightarrow{\tau} \eta$ in Y as $k \to \infty$.

For now we assume that

$$x^* \notin \mathrm{GenEff}_{\sigma,\tau}(X_\partial;\, f;\, \Lambda). \tag{5.7}$$

Then, as follows from Definition 16, $\eta \notin \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$. Hence, there can be found an element $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$ such that $\xi <_\Lambda \eta$. Therefore, $\eta - \xi \in \Lambda \setminus \{0_Y\}$, and using the fact that $\lambda \in K^\sharp$, we just come to the inequality

$$\langle \eta, \lambda\rangle_{Y;V} > \langle \xi, \lambda\rangle_{Y;V}$$

which is equivalent to

$$\lim_{k\to\infty} \langle f(x_k), \lambda\rangle_{Y;V} > \langle \xi, \lambda\rangle_{Y;V}\,. \tag{5.8}$$

On the other hand, for the element $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$ there exists a sequence $\{v_k\}_{k=1}^\infty \subset X_\partial$ such that $f(v_k) \xrightarrow{\tau} \xi$ in Y. Since the set X_∂ is sequentially σ-compact, we may suppose that $v_k \xrightarrow{\sigma} v^* \in X_\partial$. Then, by inequality (5.5), we deduce

$$\lim_{k\to\infty} \langle f(x_k), \lambda\rangle_{Y;V} \le \langle f(v_i), \lambda\rangle_{Y;V}\,, \quad \forall\, i \in \mathbb{N}. \tag{5.9}$$

Passing to the limit in (5.9) as $i \to \infty$, we get

$$\lim_{k\to\infty} \langle f(x_k), \lambda\rangle_{Y;V} \le \langle \xi, \lambda\rangle_{Y;V}\,.$$

However, this contradicts (5.8) and hence (5.7). Thus, x^* is the (σ,τ)-generalized solution to vector optimization problem $\langle X_\partial, f, \Lambda, \tau\rangle$. □

References

1. *Вулих Б. З.* Введение в функциональный анализ / Б. З. Вулих. — М. : Наука, 1967. — 415 с.
2. *Довженко А. В.* Квазі-неперервна знизу регуляризація відображень в банахових просторах // А. В. Довженко, П. І. Когут // Вісник ДНУ, 2009. — Т. 17, № 6/1. — С. 91–109.
3. *Ait Mansour M., Metrane A., Théra M.* Lower Semicontinuous Regularization for Vector-Valued Mappings, J. Global Optimization, 2006. — Vol. 35, N. 2. — P. 283–309.
4. *Akian M., Singer I.* Topologies on lattice ordered groups, separations from closed downwards and conjugations of type Lau // Optimization,2003. — Vol. 52, № 4. — P. 629–673.
5. *Aubin J. P., Frankowska H.* Set-Valued Analysis. — Cambridge : Birkhäuser, 1990.
6. *Borwein J.M.* Continuity and differentiability properties of convex operators // Optimization, 1982. — Vol. 44, N 3. — P. 420–444.
7. *Borwein J. M., Penot J. P., Théra M.* Conjugate Convex Operators // Math. Anal. and Appl., 1984. — Vol. 102. — P. 399–414.
8. *Combari C., Laghdir M., Thibault L.* Sous-différentiel de fonctions convexes composées // Ann. Sci. Math. Québec, 1994. — Vol. 18, N 2. — P. 119–148.
9. *Ehrgott M.* Multicriteria Optimization. — Berlin : Springer, 2005.
10. *Finet C., Quarta L., Troestler C.* Vector-valued variational principles // Institute de Mathématique et d'Informatique, Université de Mons-Hainaut, 2001. — (*Preprint* ♯4, January 19).

11. *Jahn J.* Vector Optimization. Theory, Applications, and Extensions. — Berlin : Springer-Verlag, 2004.
12. *Kogut P. I., Manzo R., Nechay I. V.* On existence of efficient solutions to vector optimization problems in Banach spaces // Note de Matematica, 2009 (to appear).
13. *Luc D. T.* Theory of Vector Optimization. — N. Y. : Springer-Verlag, 1989.
14. *Luc D. T.* Scalarization of vector optimization problems// Journal of Optim. Theory and Appl. (JOTA), 2005. — Vol. 55, N 1. — P. 85–102.
15. *Miglierina E., Molho E.* Scalarization and its stability in vector optimization// Journal of Optim. Theory and Appl. (JOTA), 2002. — Vol. 111, N 1. — P. 657–670.
16. *Penot J. P., Théra M.* Semi-continuous mappings in general topology// Arch. Math., 1982. — Vol. 38. — P. 158–166.
17. *Peressini A. L.* Ordered Topological Vector Spaces. — N. Y. : Harpet and Row, 1967.
18. *Sawaragi Y., Nakayama H., Tanino T.* Theory of Multiobjective Optimization. — Orlando : Academic Press, 1985.
19. *Théra M.* Études des fonctions convexes vectorielles semi-continues // Thèse de 3^e cycle., Université de Pau, 1978.
20. *Yu P. L.* Cone convexity, cone extreme points and non-dominated solutions in decision problems with multiobjectivity // Journal of Optim. Theorey and Appl. (JOTA), 1974. — Vol. 14. — P. 319–377.

Надійшла до редакції 01.08.2009