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Проблеми математичного моделювання
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ON OPTIMAL CONTROL PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE $P(X)$-LAPLACIAN

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An optimal control problem for the Dirichlet boundary value problem for the nonlinear elliptic equation with $p(x)$-Laplacian is considered. It is shown that this problem has at least one solution with respect to certain set of admissible pairs. The corresponding theoretic framework regarding Sobolev – Orlicz spaces is given.

Keywords: nonlinear elliptic equation, optimal control problem, $p(x)$-Laplacian, Lavrientiev phenomenon, existence theorem.

1. Setting of Optimal Control Problem

Let Ω be a bounded open connected subset of $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary $\partial\Omega$. Let $p : \Omega \to \mathbb{R}$ be a measured real-valued scalar function such that $1 < \alpha \leq p(x) \leq \beta < +\infty$ for almost all $x \in \Omega$. Let $f \in L^\infty(\Omega)^N$, $p_d \in L^2(\Omega)$, and $y_d \in L^\alpha(\Omega)$ be given distributions. We consider the following optimal control problem (OCP):

$$\text{Minimize} \quad \left\{ J(p,y) = \int_\Omega |y(x) - y_d(x)|^\alpha \, dx + \gamma \int_\Omega |\nabla y(x)|^{p(x)} \, dx \right.$$
$$\left. + \int_\Omega |p(x) - p_d(x)|^2 \, dx \right\} \tag{1.1}$$

subject to the constraints

$$-\operatorname{div}\left(|\nabla y|^{p(x)-2}\nabla y\right) = -\operatorname{div} f, \quad x \in \Omega, \tag{1.2}$$
$$y = 0 \ \text{ on } \ \partial\Omega, \tag{1.3}$$
$$p \in \mathbb{P}_{ad} = \left\{ p \in L^2(\Omega) \ : \ 1 < \alpha \leq p(x) \leq \beta, \ \text{ a.e. in } \ \Omega \right\}, \tag{1.4}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$.

To the best of the authors' knowledge, the existence of solutions for the optimal control problem (1.1)–(1.4) remains an open question. Only very few articles deal with distributed or boundary optimal control problems for the systems of similar type (see, for instance, [6,7] and the references therein). There are several reasons for this:

- it is unknown whether the set of admissible solutions to the problem (1.1)–(1.4) is weakly closed in the corresponding functional space;
- we have no a priori estimates for the weak solutions (in the sense of Minty) to the boundary value problem (1.2)–(1.3);
- the asymptotic behaviour of a minimizing sequence to the cost functional (1.1) is unclear in general.

To see these and other characteristic features of the optimization problem (1.1)–(1.4) more clearly, we introduce the well-known notions of solutions for nonlinear elliptic problems with variable exponent and discuss how the equation (1.2) can be interpreted.

To begin with, we note that if the exponent p is constant, then the Dirichlet boundary value problem (1.2)–(1.3) is well-posed in the classical Sobolev space $W_0^{1,p}(\Omega)$. For the variable measurable exponent, we look for the solution of this problem in the Sobolev-Orlicz space (see [4,15])

$$W_0^{1,p(\cdot)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) \ : \ \int_\Omega |\nabla u|^{p(x)}\,dx < +\infty \right\} \tag{1.5}$$

equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)^N}.$$

Here, $L^{p(\cdot)}(\Omega)^N$ stands for the set of all measurable vector-valued functions $f : \Omega \to \mathbb{R}^N$ such that

$$\rho_p(f) := \int_\Omega |f(x)|^{p(x)}\,dx < +\infty, \tag{1.6}$$

and $L^{p(\cdot)}(\Omega)^N$ is endowed with the so-called Luxemburg norm (see [3,4] for the details)

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \inf\left\{\lambda > 0 \ : \ \rho_p(\lambda^{-1} f) \le 1\right\}.$$

It is well-known that, unlike classical Sobolev spaces, smooth functions are not necessarily dense in $W = W_0^{1,p(\cdot)}(\Omega)$. Hence, with variable exponent $p = p(x)$ $(1 < \alpha \le p \le \beta)$ it can be associated another Sobolev space,

$$H = H_0^{1,p(\cdot)}(\Omega) \ \text{ as the closure of the set } \ C_0^\infty(\Omega) \ \text{ in } W_0^{1,p(\cdot)}(\Omega)\text{-norm.}$$

Since the identity $W = H$ is not always valid, it makes sense to say that an exponent $p(x)$ is regular if $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.

Definition 1.1. A function $y \in W_0^{1,p(\cdot)}(\Omega)$ is said to be a weak solution to the boundary value problem (1.2)–(1.3), if the integral identity

$$\int_\Omega |\nabla y|^{p(x)-2} (\nabla y, \nabla \varphi)\, dx = \int_\Omega (f, \nabla \varphi)\, dx \tag{1.7}$$

holds true for all $\varphi \in C_0^\infty(\Omega)$.

Here, $(\cdot,\cdot)$ stands for the scalar product in $\mathbb{R}^N$.

Since we can lose the density of the set $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ for some (irregular) variable exponents $p(x)$, it follows that a weak solution to the problem (1.2)–(1.3) is not unique, in general. Moreover, as it was shown in [16], the set of all weak solutions in not necessary convex in spite of the fact that the operator $\mathcal{A} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$, given by the equality

$$(\mathcal{A}u, v) = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u, \nabla\varphi\right)\, dx, \quad \forall\, v \in W_0^{1,p(\cdot)}(\Omega), \tag{1.8}$$

is strictly monotone. The question as to whether this set is weakly closed in $W_0^{1,p(\cdot)}(\Omega)$ remains open. At the same time, the following result is well-known.

Theorem 1.1 ([15],p.472). *If the domain $\Omega \subset \mathbb{R}^N$ is sufficiently smooth and the constant β in* (1.4) *is such that*

$$\beta < \frac{\alpha(N-1)}{N-1-\alpha} \quad \textit{for} \quad \alpha < N-1, \quad \textit{and} \quad \beta < +\infty, \quad \textit{for} \quad \alpha \geq N-1,$$

then the Dirichlet problem (1.2)–(1.3) *has a weak solution $y \in W_0^{1,p(\cdot)}(\Omega)$ satisfying the energy inequality*

$$\int_\Omega |\nabla y|^{p(x)}\, dx \leq \int_\Omega (f, \nabla y)_{\mathbb{R}^N}\, dx \tag{1.9}$$

Remark 1.1. The main idea of the proof of Theorem 1.1 is based on the fact that some weak solutions to the Dirichlet problem (1.2)–(1.3) can be attained through C^1-regularization of the exponent $p = p(x)$ or through some approximation of operator $\mathcal{A}$ using its perturbation by $\varepsilon\Delta_\beta$-Laplacian. Here, by attainability of a weak solution $y \in W_0^{1,p(\cdot)}(\Omega)$, we mean the existence of a sequence $\{y_\varepsilon\}_{\varepsilon>0}$, where y_ε are the solutions of 'more regular' boundary value problems, such that $y_\varepsilon \to y$ in some appropriate topology as ε tends to zero. However, because of the fact that the energy inequality (1.9) can be strict for some irregular variable exponents $p(x)$, it is unknown whether each weak solution to the Dirichlet problem (1.2)–(1.3) can be attained in such way.

Let $p(x)$ be an irregular exponent and let V be an arbitrary intermediate space between H and W, i.e. $H \subseteq V \subseteq W$.

Definition 1.2. A function $y \in V$ is a V-solution of the problem (1.2)–(1.3), or its variational solution, if the integral identity (1.7) holds for any test function $\varphi \in V$.

Using the strict monotonicity of the nonlinear operator $\mathcal{A} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$ (see (1.8)), it is easy to show that a V-solution exists and it is unique. Moreover, since in the case of V-solutions, the test function φ in (1.7) can be taken equal to the solution, it leads us to the energy equality

$$\int_\Omega |\nabla y|^{p(x)}\, dx = \int_\Omega (f, \nabla y)_{\mathbb{R}^N}\, dx. \tag{1.10}$$

Theorem 1.2. *Let V be an arbitrary intermediate space between H and W. Then for any $f \in L^\infty(\Omega)^N$ there exists a unique V-solution to the boundary value problem* (1.2)–(1.3) *and it satisfies the energy equality* (1.10).

The converse statement is also true.

Proposition 1.1. A weak solution in the sense of Definition 1.1 is variational if and only if the energy equality (1.10) holds.

Indeed, in this case we can take V as the smallest closed subspace containing $C_0^\infty(\Omega)$ and the solution itself. For $V = H$, we speak of H-solutions.

Another definition of a weak solution to (1.2)–(1.3) can be stated as follows.

Definition 1.3. A function $y \in W_0^{1,p(\cdot)}(\Omega)$ is said to be a weak solution in the sense of Minty to the boundary value problem (1.2)–(1.3), if the integral inequality

$$\int_\Omega |\nabla \varphi|^{p(x)-2} (\nabla \varphi, \nabla \varphi - \nabla y)\, dx \geq \int_\Omega (f, \nabla \varphi - \nabla y)\, dx \tag{1.11}$$

holds true for all $\varphi \in C_0^\infty(\Omega)$.

It follows from this definition that the set of weak solutions in the sense of Minty is convex and closed. However, the relations between Definitions 1.1 and 1.3 are very intricate for a general exponent $p(\cdot) \in \mathbb{P}_{ad}$. At least we can not assert that each of the Minty's weak solutions satisfies the integral identity (1.7) or vise versa. In Section 3 we describe the case where the three concepts of the weak solutions are given before coincide.

As a result, the variational formulation of the optimal control problem (1.1)–(1.4) can be stated in different forms and this depends on the choice of the corresponding set of solutions. In view of this, we indicate the following sets of admissible pairs to the problem (1.1)–(1.4):

$$\Xi_w = \left\{ (p,y) \in \mathbb{P}_{ad} \times W_0^{1,p(\cdot)}(\Omega) \;\middle|\; \begin{array}{c} y \text{ and } p \text{ are related by integral} \\ \text{identity (1.7) for all } \varphi \in C_0^\infty(\Omega) \end{array} \right\}, \quad (1.12)$$

$$\Xi_V = \left\{ (p,y) \in \mathbb{P}_{ad} \times V \;\middle|\; \begin{array}{c} \exists \text{ an intermediate space } V \text{ such that} \\ H_0^{1,p(\cdot)}(\Omega) \subseteq V \subseteq W_0^{1,p(\cdot)}(\Omega) \\ \text{and } y \text{ and } p \text{ are related by integral} \\ \text{identity (1.7) for all } \varphi \in V \end{array} \right\}, \quad (1.13)$$

$$\Xi_M = \left\{ (p,y) \in \mathbb{P}_{ad} \times W_0^{1,p(\cdot)}(\Omega) \;\middle|\; \begin{array}{c} y \text{ and } p \text{ are related by Minty} \\ \text{inequality (1.11)} \\ \text{for all } \varphi \in C_0^\infty(\Omega) \end{array} \right\} \quad (1.14)$$

However, because of the Lavrentieff effect, it may happen that the corresponding minimization problems

$$\left\langle \inf_{(p,y)\in\Xi_w} J(p,y) \right\rangle, \ \left\langle \inf_{(p,y)\in\Xi_V} J(p,y) \right\rangle, \text{ and } \left\langle \inf_{(p,y)\in\Xi_M} J(p,y) \right\rangle \quad (1.15)$$

are essentially different, in general. In particular, it means that optimal pairs to the problems (1.15) can be different as well.

Thus, the main question we are going to answer in this paper is about solvability of optimal control problem (1.1)–(1.4) with respect to the different choice of the set of admissible solutions. To the best knowledge of the authors, the existence of optimal pairs to the problems (1.15) has not been studied in the literature.

2. Preliminaries On Orlicz and Sobolev – Orlicz Spaces

To begin with, we note that class $L^{p(\cdot)}(\Omega)^N$ is a reflexive separable Banach space with respect to both the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \inf\left\{\lambda > 0 \ : \ \rho_p(\lambda^{-1} f) \le 1\right\} \quad (2.1)$$

and the Orlicz norm

$$\|f\|_{O^{p(\cdot)}(\Omega)^N} = \sup\left\{ \int_\Omega (f,g)\;dx \ : \ \int_\Omega |g|^{p'(x)}\,dx \le 1 \right\}, \quad (2.2)$$

where $p'(x) = \frac{p(x)}{p(x)-1}$ is the conjugate exponent. In view of (1.4), it is clear that

$$1 \le \underbrace{\frac{\beta}{\beta-1}}_{\beta'} \le p'(x) \le \underbrace{\frac{\alpha}{\alpha-1}}_{\alpha'} \text{ a.e. in } \Omega.$$

As for the infimum in (2.1), we have the following result (for reader's convenience, we furnish it with the proof).

Proposition 2.1. The infimum in (2.1) is attained if $\rho_p(f) > 0$. Moreover

$$\text{if} \quad \lambda_* := \|f\|_{L^{p(\cdot)}(\Omega)^N} > 0, \quad \text{then} \quad \rho_p(\lambda_*^{-1} f) = 1. \tag{2.3}$$

Proof. Indeed, as follows from (2.1), $\|f\|_{L^{p(\cdot)}(\Omega)^N} = 0$ if and only if $f(x) = 0$ a.e. in Ω, i.e. $\rho_p(f) = 0$. Assume that $\rho_p(f) > 0$. We define a function $\psi : [0, \infty) \to \mathbb{R}$ as

$$\psi(s) := \rho_p(\lambda^{-1} f) = \int_\Omega |s f(x)|^{p(x)}\, dx.$$

Since $\psi(0) = 0$, $\lim_{s\to\infty} \psi(s) = +\infty$, and

$$\frac{d}{ds}\psi(s) = \int_\Omega p(x) s^{p(x)-1} |f(x)|^{p(x)}\, dx > 0,$$

it follows that $\psi = \psi(s)$ is a monotonically increasing function. Hence, there exists a positive value $\lambda_* > 0$ such that

$$\psi(\lambda_*^{-1}) = 1 \quad \text{and} \quad \psi(\lambda^{-1}) = \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \quad \forall\, \lambda \in [\lambda_*, +\infty).$$

Therefore,

$$\inf \left\{ \lambda > 0 \ : \ \psi(\lambda^{-1}) \le 1 \right\} = \lambda_* \quad \text{and} \quad \int_\Omega \left| \frac{f(x)}{\lambda_*} \right|^{p(x)} dx = 1. \tag{2.4}$$

As a result, we deduce from (2.4) and (2.1) that $\lambda_* = \|f\|_{L^{p(\cdot)}(\Omega)^N}$. □

Taking this result and condition $1 < \alpha \le p(x) \le \beta$ into account, we see that

$$\frac{1}{\lambda_*^\beta} \int_\Omega |f(x)|^{p(x)}\, dx \le \int_\Omega \left| \frac{f(x)}{\lambda_*} \right|^{p(x)} dx \le \frac{1}{\lambda_*^\alpha} \int_\Omega |f(x)|^{p(x)}\, dx,$$
$$\frac{1}{\lambda_*^\beta} \int_\Omega |f(x)|^{p(x)}\, dx \le 1 \le \frac{1}{\lambda_*^\alpha} \int_\Omega |f(x)|^{p(x)}\, dx.$$

Hence,

$$\begin{aligned} \|f\|^\alpha_{L^{p(\cdot)}(\Omega)^N} &\le \int_\Omega |f(x)|^{p(x)}\, dx \le \|f\|^\beta_{L^{p(\cdot)}(\Omega)^N}, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)^N} > 1, \\ \|f\|^\beta_{L^{p(\cdot)}(\Omega)^N} &\le \int_\Omega |f(x)|^{p(x)}\, dx \le \|f\|^\alpha_{L^{p(\cdot)}(\Omega)^N}, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)^N} < 1. \end{aligned} \tag{2.5}$$

As a result, we arrive at the estimates

$$\|f\|^\alpha_{L^{p(\cdot)}(\Omega)^N} - 1 \le \int_\Omega |f(x)|^{p(x)}\, dx \le \|f\|^\beta_{L^{p(\cdot)}(\Omega)^N} + 1, \quad \forall\, f \in L^{p(\cdot)}(\Omega)^N. \tag{2.6}$$

The next result gives us the analog of the Hölder inequality.

Proposition 2.2. If $f \in L^{p(\cdot)}(\Omega)^N$ and $g \in L^{p'(\cdot)}(\Omega)^N$, then $(f,g) \in L^1(\Omega)$ and

$$\int_\Omega (f,g)\, dx \le 2\|f\|_{L^{p(\cdot)}(\Omega)^N} \|g\|_{L^{p'(\cdot)}(\Omega)^N}. \tag{2.7}$$

Proof. To begin with, we note that the following inequality holds

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} \le \|f\|_{O^{p(\cdot)}(\Omega)^N} \le \left(\frac{1}{\alpha} + \frac{1}{\beta'}\right) \|f\|_{L^{p(\cdot)}(\Omega)^N} \le 2\|f\|_{L^{p(\cdot)}(\Omega)^N}. \tag{2.8}$$

Indeed, setting $\lambda_* := \|f\|_{L^{p(\cdot)}(\Omega)^N}$ and $g(x) = \left|\frac{f(x)}{\lambda_*}\right|^{p(x)-2} \frac{f(x)}{\lambda_*}$, we see that

$$|g|^{p'(x)} = \left|\frac{f}{\lambda_*}\right|^{p(x)} \quad \text{and} \quad \int_\Omega |g|^{p'(x)}\, dx \overset{\text{by (2.4)}}{=} 1. \tag{2.9}$$

Hence,

$$\begin{aligned} \|f\|_{O^{p(\cdot)}(\Omega)^N} &\overset{\text{by (2.2)}}{\ge} \int_\Omega (f,g)\, dx = \lambda_* \int_\Omega \left|\frac{f}{\lambda_*}\right|^{p(x)} dx \\ &\overset{\text{by (2.9)}}{=} \lambda_* = \|f\|_{L^{p(\cdot)}(\Omega)^N}. \end{aligned}$$

Thus, the left hand side of (2.8) is proven. To prove the rest part of this inequality, we make use of the Young inequality

$$(f,g) \le \frac{|f|^{p(x)}}{p(x)} + \frac{|g|^{p'(x)}}{p'(x)} \le \frac{|f|^{p(x)}}{\alpha} + \frac{|g|^{p'(x)}}{\beta'}.$$

Let $\lambda_* = \|f\|_{L^{p(\cdot)}(\Omega)^N}$. Then

$$\begin{aligned} \|\lambda_*^{-1} f\|_{O^{p(\cdot)}(\Omega)^N} &\overset{\text{by (2.2)}}{=} \sup\left\{ \int_\Omega \left(\lambda_*^{-1} f, g\right) dx \ : \ \int_\Omega |g|^{p'(x)}\, dx \le 1 \right\} \\ &\le \sup\left\{ \frac{1}{\alpha} \int_\Omega \left|\frac{f}{\lambda_*}\right|^{p(x)} dx + \frac{1}{\beta'} \int_\Omega |g|^{p'(x)}\, dx \ : \ \int_\Omega |g|^{p'(x)}\, dx \le 1 \right\} \\ &\overset{\text{by (2.4)}}{\le} \frac{1}{\alpha} + \frac{1}{\beta'}. \end{aligned}$$

As a result, we obtain

$$\|f\|_{O^{p(\cdot)}(\Omega)^N} \le \left(\frac{1}{\alpha} + \frac{1}{\beta'}\right) \lambda_* = \left(\frac{1}{\alpha} + \frac{1}{\beta'}\right) \|f\|_{L^{p(\cdot)}(\Omega)^N}.$$

Thus, the inequality (2.8) holds true. In order to establish the Hölder inequality (2.7), it is sufficient to set $t = \|g\|_{L^{p'(\cdot)}(\Omega)^N}$ and observe that

$$\int_\Omega \left|\frac{g(x)}{t}\right|^{p'(x)} dx = 1. \tag{2.10}$$

As a result, we finally arrive at the following relation

$$\int_\Omega t^{-1}(f,g)\,dx \overset{\text{by (2.2)}}{\le} \|f\|_{O^{p(\cdot)}(\Omega)^N} \overset{\text{by (2.2) and (2.10)}}{\le} 2\|f\|_{L^{p(\cdot)}(\Omega)^N}$$

which immediately implies the inequality (2.7). □

Proposition 2.3. The following estimates

$$\|f\|_{L^\alpha(\Omega)^N} \le (1+|\Omega|)^{1/\alpha}\,\|f\|_{L^{p(\cdot)}(\Omega)^N}, \quad \forall\, f\in L^p(\cdot)(\Omega)^N, \tag{2.11}$$

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} \le (1+|\Omega|)^{1/\beta'}\,\|f\|_{L^\beta(\Omega)^N}, \quad \beta'=\frac{\beta}{\beta-1}, \quad \forall\, f\in L^\beta(\Omega)^N. \tag{2.12}$$

are valid.

Proof. We note that

$$\int_\Omega |\lambda_*^{-1}f|^\alpha\,dx \le \int_\Omega \left(|\lambda_*^{-1}f|^{p(x)}+1\right)\,dx \overset{\text{by (2.3)}}{\le} 1+|\Omega|,$$

with $\lambda_*=\|f\|_{L^{p(\cdot)}(\Omega)^N}$ for each $f\in L^{p(\cdot)}(\Omega)^N$. Hence,

$$\int_\Omega |f|^\alpha\,dx \le \lambda_*^\alpha(1+|\Omega|)$$

and this estimate immediately implies (2.11). In order to establish the estimate (2.12), we use the inequality (2.8). Since

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} \le \|f\|_{O^{p(\cdot)}(\Omega)^N} = \sup\left\{\int_\Omega (f,g)\,dx \;:\; \int_\Omega |g|^{p'(x)}\,dx \le 1\right\}$$

and

$$\int_\Omega |g|^{\beta'}\,dx \overset{\text{by (2.11)}}{\le} (1+|\Omega|)^{1/\beta'}\int_\Omega |g|^{p'(x)}\,dx \le (1+|\Omega|)^{1/\beta'} \tag{2.13}$$

for $\int_\Omega |g|^{p'(x)}\,dx\le 1$, it follows from (2.13) that

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} \le \sup\left\{\int_\Omega (f,g)\,dx \;:\; \int_\Omega |g|^{\beta'}\,dx \le (1+|\Omega|)^{1/\beta'}\right\}.$$

It remains to apply the classical Hölder inequality to $\int_\Omega (f,g)\,dx$ with $f\in L^\beta(\Omega)^N$. □

For our further analysis, we make use of the following results.

Lemma 2.1 ([15], p.536)**.** *If a sequence $\{f_k\}_{k\in\mathbb{N}}$ is bounded in $L^{p(\cdot)}(\Omega)^N$ and $f_k\rightharpoonup f$ in $L^\alpha(\Omega)^N$ as $k\to\infty$, then $f\in L^{p(\cdot)}(\Omega)^N$ and $f_k\rightharpoonup f$ in $L^{p(\cdot)}(\Omega)^N$, i.e.*

$$\lim_{k\to\infty}\int_\Omega (f_k,\varphi)\,dx = \int_\Omega (f_k,\varphi)\,dx, \quad \forall\,\varphi\in L^{p'(\cdot)}(\Omega)^N \text{ with } p'(x)=\frac{p(x)}{p(x)-1}.$$

Lemma 2.2 ([8], p.428)**.** *Let $f, f_k \in L^{p(\cdot)}(\Omega)^N$ for $k = 1, 2, \dots$ Then the following statements are equivalent to each other:*

(i) $\lim_{k\to\infty} \|f_k - f\|_{L^{p(\cdot)}(\Omega)^N} = 0$;

(ii) $\lim_{k\to\infty} \int_\Omega |f_k - f|^{p(x)}\, dx = 0$;

(iii) $f_k \to f$ *in* Ω *in measure and* $\lim_{k\to\infty} \int_\Omega |f_k|^{p(x)}\, dx = \int_\Omega |f|^{p(x)}\, dx.$

To end of this subsection, we discuss a couple of results that will be useful later on.

Proposition 2.4. Let $p(\cdot) \in \mathbb{P}_{ad}$ and $y(\cdot) \in W_0^{1,p(x)}(\Omega)$ be a given distributions. Let $F = F(\varphi)$, where

$$F(\varphi) = |\nabla\varphi|^{p(x)-2} (\nabla\varphi, \nabla\varphi - \nabla y).$$

Then $\varphi \overset{\mathcal{F}:}{\mapsto} F(\varphi)$ is the mapping $W_0^{1,p(x)}(\Omega) \mapsto L^1(\Omega)$.

Proof. Let $\varphi \in W_0^{1,p(x)}(\Omega)$ be a fixed distribution. We have to show that

$$\|F(\varphi)\|_{L^1(\Omega)} := \int_\Omega |\nabla\varphi|^{p(x)-2} |(\nabla\varphi, \nabla\varphi - \nabla y)|\, dx < +\infty.$$

With that in mind, it is enough to make use of the following estimates

$$\begin{aligned}
\int_\Omega |\nabla\varphi|^{p(x)-2} |(\nabla\varphi, \nabla\varphi - \nabla y)|\, dx &\le \int_\Omega |\nabla\varphi|^{p(x)-1} |\nabla\varphi - \nabla y|\, dx \\
&\le \int_\Omega |\nabla\varphi|^{p(x)}\, dx + \int_\Omega |\nabla\varphi|^{p(x)-1} |\nabla y|\, dx \\
&\overset{\text{by the Young inequality}}{\le} \int_\Omega |\nabla\varphi|^{p(x)}\, dx + \int_\Omega \frac{1}{p'(x)} |\nabla\varphi|^{p(x)}\, dx + \int_\Omega \frac{1}{p(x)} |\nabla y|^{p(x)}\, dx \\
&\overset{\text{by the properties of } \mathbb{P}}{\le} 2\int_\Omega |\nabla\varphi|^{p(x)}\, dx + \int_\Omega |\nabla y|^{p(x)}\, dx \\
&\overset{\text{by (2.6)}}{\le} 2\|\varphi\|^\beta_{W_0^{1,p(x)}(\Omega)} + \|y\|^\beta_{W_0^{1,p(x)}(\Omega)} + 3 < +\infty.
\end{aligned}$$

□

Proposition 2.5. $\mathcal{F} : W_0^{1,p(x)}(\Omega) \mapsto L^1(\Omega)$ is a continuous mapping.

Proof. Let $\varphi \in W_0^{1,p(x)}(\Omega)$ and $\{\varphi_k\}_{k=1}^{+\infty} \subset W_0^{1,p(x)}(\Omega)$ be such that

$$\varphi_k \to \varphi \ \text{ in } W_0^{1,p(x)}(\Omega) \text{ as } k \text{ tends to } +\infty. \tag{2.14}$$

Our aim is to establish the equality

$$\lim_{k\to\infty} \int_\Omega |F(\varphi_k) - F(\varphi)|\, dx = 0. \tag{2.15}$$

In view of the property (iii) of Lemma 2.2, we can suppose that, up to a subsequence,

$$\begin{aligned} &\nabla\varphi_k(x) \to \nabla\varphi(x) \ \text{ a.e. in } \ \Omega, \quad \text{and} \\ &\lim_{k\to\infty}\int_\Omega |\nabla\varphi_k|^{p(x)}\,dx = \int_\Omega |\nabla\varphi|^{p(x)}\,dx. \end{aligned} \tag{2.16}$$

Then

$$\begin{aligned} \lim_{k\to\infty}\int_\Omega F(\varphi_k)\,dx &= \lim_{k\to\infty}\int_\Omega |\nabla\varphi_k|^{p(x)-2}\left(\nabla\varphi_k, \nabla\varphi_k - \nabla y\right)dx \\ &= \lim_{k\to\infty}\int_\Omega |\nabla\varphi_k|^{p(x)}\ dx - \lim_{k\to\infty}\int_\Omega |\nabla\varphi_k|^{p(x)-2}\left(\nabla\varphi_k, \nabla y\right)dx \\ &\overset{\text{by } (1.11)_2}{=} \int_\Omega |\nabla\varphi|^{p(x)}\ dx - \lim_{k\to\infty}\int_\Omega |\nabla\varphi_k|^{p(x)-2}\left(\nabla\varphi_k, \nabla y\right)dx. \end{aligned} \tag{2.17}$$

Hence, to conclude the proof, it remains to show that

$$\lim_{k\to\infty}\int_\Omega |\nabla\varphi_k|^{p(x)-2}\left(\nabla\varphi_k, \nabla y\right)dx = \int_\Omega |\nabla\varphi|^{p(x)-2}\left(\nabla\varphi, \nabla y\right)dx. \tag{2.18}$$

With that in mind, we note that (1.11) implies the pointwise convergence

$$\begin{aligned} &|\nabla\varphi_k(x)|^{p(x)-2}\left(\nabla\varphi_k(x), \nabla y(x)\right) \\ &\qquad\to |\nabla\varphi(x)|^{p(x)-2}\left(\nabla\varphi(x), \nabla y(x)\right) \ \text{ a.e. in } \ \Omega. \end{aligned} \tag{2.19}$$

Let us show that the set

$$\Phi = \left\{\psi_k := |\nabla\varphi_k|^{p(x)-2}\left(\nabla\varphi_k, \nabla y\right)\right\}_{k=1}^\infty$$

is uniformly integrable, i.e. for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\int_E f(x)\,dx\right| < \varepsilon \quad \text{whenever} \quad f\in\Phi \quad \text{and} \quad E\subset\Omega \quad \text{with} \quad |E| < \delta.$$

Since

$$\begin{aligned} \|\,|\nabla\varphi_k|^{p(x)-2}\,\nabla\varphi_k\|^\alpha_{L^{p'(\cdot)}(\Omega)^N} &\overset{\text{by } (2.6)}{\leq} 1 + \int_\Omega |\nabla\varphi_k|^{p(x)}\ dx \\ &\overset{\text{by } (2.6)}{\leq} \|\nabla\varphi_k\|^\beta_{L^{p(\cdot)}(\Omega)^N} + 2 \\ &\leq \left(\sup_{k\in\mathbb{N}}\|\varphi_k\|_{W_0^{1,p(\cdot)}(\Omega)}\right)^\beta + 2 \overset{\text{by } (2.14)}{=} C < +\infty, \end{aligned} \tag{2.20}$$

it follows that the sequence $\left\{|\nabla\varphi_k|^{p(x)-2}\,\nabla\varphi_k\right\}_{k=1}^\infty$ is bounded in $L^{p'(\cdot)}(\Omega)^N$. Then, for every $E\subset\Omega$, we have

$$\begin{aligned} \|\psi_k\|_{L^1(E)^N} &\overset{\text{by } (2.7)}{\leq} 2\|\,|\nabla\varphi_k|^{p(x)-2}\,\nabla\varphi_k\|_{L^{p'(\cdot)}(E)^N}\|\nabla y\|_{L^{p(\cdot)}(E)^N} \\ &\leq 2\|\,|\nabla\varphi_k|^{p(x)-2}\,\nabla\varphi_k\|_{L^{p'(\cdot)}(\Omega)^N}\|\nabla y\|_{L^{p(\cdot)}(E)^N} \\ &\overset{\text{by } (2.20)}{\leq} 2C\|\nabla y\|_{L^{p(\cdot)}(E)^N} \overset{\text{by } (2.5)}{\leq} 2C\left(\int_E |\nabla y|^{p(x)}\,dx\right)^\zeta, \end{aligned} \tag{2.21}$$

where

$$\zeta = \alpha^{-1} \ \text{ if } \ \|\nabla y\|_{L^{p(\cdot)}(E)^N} > 1, \ \text{ and } \ \zeta = \beta^{-1} \ \text{ if } \ \|\nabla y\|_{L^{p(\cdot)}(E)^N} < 1.$$

Let us show that for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that if the Lebesgue measure of E is less than δ, the integral of $|\nabla y|^{p(x)}$ over E is less than $\left(\frac{\varepsilon}{2C}\right)^{\beta}$. Indeed, by the Lebesgue dominated theorem, we have

$$\lim_{\lambda\to+\infty} \int_{\left\{|\nabla y|^{p(x)}>\lambda\right\}} |\nabla y|^{p(x)}\, dx = 0.$$

This fact follows easily since

$$\chi_{\left\{|\nabla y|^{p(x)}>\lambda\right\}}\, |\nabla y|^{p(x)} \le |\nabla y|^{p(x)} \in L^1(\Omega)$$

and

$$\chi_{\left\{|\nabla y|^{p(x)}>\lambda\right\}} \to 0 \ \text{ as } \ \lambda \to +\infty$$

since $|\nabla y|^{p(x)}$, being integrable, is finite almost everywhere. Then, for a given ε, there exists $\lambda > 0$ such that

$$\int_{\left\{|\nabla y|^{p(x)}>\lambda\right\}} |\nabla y|^{p(x)}\, dx < \frac{1}{2}\left(\frac{\varepsilon}{2C}\right)^{\beta}.$$

Choosing $\delta \le \frac{1}{2\lambda}\left(\frac{\varepsilon}{2C}\right)^{\beta}$ and taking any measurable set $E \subset \Omega$ such that $|E| < \delta$, we get

$$\begin{aligned}\int_E |\nabla y|^{p(x)}\, dx &= \int_{E\cap\left\{|\nabla y|^{p(x)}>\lambda\right\}} |\nabla y|^{p(x)}\, dx + \int_{\left\{E\cap|\nabla y|^{p(x)}\le\lambda\right\}} |\nabla y|^{p(x)}\, dx\\ &\le \int_{\left\{|\nabla y|^{p(x)}>\lambda\right\}} |\nabla y|^{p(x)}\, dx + \int_{\left\{E\cap|\nabla y|^{p(x)}\le\lambda\right\}} \lambda\, dx = I_1 + I_2.\end{aligned}$$

As a result, we obtain

$$\int_E |\nabla y|^{p(x)}\, dx \le I_1 + I_2 \le \frac{1}{2}\left(\frac{\varepsilon}{2C}\right)^{\beta} + \lambda\delta \le \left(\frac{\varepsilon}{2C}\right)^{\beta}.$$

In view of the estimates (2.5), it is plausible to suppose that $\|\nabla y\|_{L^{p(\cdot)}(E)^N} < 1$ for ε small enough. Hence, $\zeta = \beta^{-1}$ in (2.21), and, therefore,

$$\begin{aligned}\|\psi_k\|_{L^1(E)^N} &:= \left\| |\nabla\varphi_k|^{p(x)-2}\, (\nabla\varphi_k, \nabla y)\right\|_{L^1(E)} \le 2C\left(\int_E |\nabla y|^{p(x)}\, dx\right)^{\zeta}\\ &\le 2C\left(\left(\frac{\varepsilon}{2C}\right)^{\beta}\right)^{\beta^{-1}} = \varepsilon. \qquad (2.22)\end{aligned}$$

Since the choice of $\delta > 0$ in estimate (2.22) does not depend on ψ_k, it follows that the set Φ is uniformly integrable [14].

To conclude the proof, it remains to apply the Vitali convergence theorem [12] which asserts that if the sequence $\{\psi_k\}_{k=1}^{\infty}$ is uniformly integrable and $\psi_k \to \psi := |\nabla\varphi|^{p(x)-2}(\nabla\varphi, \nabla y)$ almost everywhere in Ω as $k \to \infty$ (see (2.19)), then

$$\psi \in L^1(\Omega) \quad \text{and} \quad \psi_k \to \psi \quad \text{in} \quad L^1(\Omega). \tag{2.23}$$

Taking into account that

$$\left| \int_\Omega |\nabla\varphi_k|^{p(x)-2}(\nabla\varphi_k, \nabla y)\, dx - \int_\Omega |\nabla\varphi|^{p(x)-2}(\nabla\varphi, \nabla y)\, dx \right| \\ \leq \int_\Omega |\psi_k - \psi|\, dx \to 0 \quad \text{as} \quad k \to \infty$$

and combining this fact with (2.23) and (2.17), we arrive at the equality (2.15). The proof is complete. □

3. Some Auxiliary Results and Properties of the Sets of Admissible Solutions

Due to the well-know results (see, for instance, [17]), there can be indicated an admissible control $p \in \mathbb{P}_{ad}$ such that the corresponding boundary value problem (1.2)–(1.3) admits infinitely many weak solutions. Hence, it is plausible to expect that the sets of admissible solutions Ξ_w, Ξ_V, and Ξ_M, which are defined in (1.12)–(1.14), possess drastically different properties in general.

We begin this section with the case when the sets Ξ_w, Ξ_V, and Ξ_M describe the same collection of admissible pairs to the OCP (1.1)–(1.4).

Proposition 3.1. Assume that the set of admissible controls $\mathbb{P}_{ad}$ is specified as follows: $p \in \mathbb{P}_{ad}$ if and only if the following conditions

$$\begin{gathered} |p(x) - p(y)| \leq \omega(|x-y|),\ \forall\, x, y, \in \Omega,\ |x-y| \leq 1/2, \\ \omega(t) = k_0/\ln(|t|^{-1}),\ 1 < \alpha \leq p(x) \leq \beta \text{ in } \Omega \end{gathered} \tag{3.1}$$

hold true with a given constant $k_0 > 0$. Then the sets Ξ_w, Ξ_V, and Ξ_M coincide.

Proof. First of all, we note that if $p = p(x)$ is an admissible exponent, then $p = p(x)$ is a continuous function in $\overline{\Omega}$ with the same logarithmic modulus of continuity $\omega(t) = \frac{k_0}{\ln(1/t)}$. As a result we have: the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ for each $p \in \mathbb{P}_{ad}$ (see [15, Theorem 13.10]).

We divide the proof into three steps. Step 1. Let us show that $\Xi_w \subseteq \Xi_V$. Let $(p, y) \in \Xi_w$ be an arbitrary pair. It is worth to notice that such choice is always possible because Ξ_w is a nonempty set. Indeed, if we set $p(x) = \beta$, then the boundary value problem

$$-\operatorname{div}\left(|\nabla y|^{\beta-2}\nabla y\right) = -\operatorname{div} f, \quad x \in \Omega, \tag{3.2}$$

$$y = 0 \quad \text{on} \quad \partial\Omega \tag{3.3}$$

is well-posed and it admits a unique weak solution $y_\beta \in W_0^{1,\beta}(\Omega)$ satisfying the integral identity (1.7) for all $\varphi \in C_0^\infty(\Omega)$ (see [10]). Hence, $(\beta, y_\beta) \in \Xi_w$ and $\Xi_w \neq \emptyset$ follows.

By definition of the set Ξ_w and the arguments of the density, the validity of the integral identity (1.7), which is written down for the chosen pair (p, y), can be extended to the test functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$. Then, putting $\varphi = y$ in (1.7), we immediately arrive at the energy equality (1.10). Hence, $(p, y) \in \Xi_V$ and, therefore, $\Xi_w \subseteq \Xi_V$.

Step 2. At this step we show that $\Xi_V \subseteq \Xi_M$. Let $(\widehat{p}, \widehat{y}) \in \Xi_V$ be an arbitrary pair. Let V be the smallest closed subspace of $W_0^{1,\widehat{p}(\cdot)}(\Omega)$ containing $C_0^\infty(\Omega)$ and the solution $\widehat{y}$ itself. By density of $C_0^\infty(\Omega)$ in $W_0^{1,\widehat{p}(\cdot)}(\Omega)$, it follows that $V = W_0^{1,\widehat{p}(\cdot)}(\Omega)$. As a result, (1.13) implies that

$$\int_\Omega |\nabla \widehat{y}|^{\widehat{p}(x)-2} (\nabla \widehat{y}, \nabla \varphi)\, dx = \int_\Omega (f, \nabla \varphi)\, dx, \quad \forall\, \varphi \in W_0^{1,\widehat{p}(\cdot)}(\Omega). \tag{3.4}$$

Using the strict monotonicity of operator $\mathcal{A} : W_0^{1,\widehat{p}(\cdot)}(\Omega) \to \left(W_0^{1,\widehat{p}(\cdot)}(\Omega)\right)^*$, given by the equality (1.8), we have

$$\begin{aligned} 0 &\leq \int_\Omega \left(|\nabla v|^{\widehat{p}(x)-2} \nabla v - |\nabla \widehat{y}|^{\widehat{p}(x)-2} \nabla \widehat{y}, \nabla v - \nabla \widehat{y} \right) dx \\ &= \int_\Omega |\nabla v|^{\widehat{p}(x)-2} (\nabla v, \nabla v - \nabla \widehat{y})\, dx - \int_\Omega |\nabla \widehat{y}|^{\widehat{p}(x)-2} (\nabla \widehat{y}, \nabla v - \nabla \widehat{y})\, dx \\ &\overset{\text{by (3.4)}}{=} \int_\Omega |\nabla v|^{\widehat{p}(x)-2} (\nabla v, \nabla v - \nabla \widehat{y})\, dx - \int_\Omega (f, \nabla v - \nabla \widehat{y})\, dx, \end{aligned} \tag{3.5}$$

where $\varphi = v - \widehat{y}$ and v is an arbitrary element of $W_0^{1,\widehat{p}(\cdot)}(\Omega)$. Hence,

$$\int_\Omega |\nabla v|^{\widehat{p}(x)-2} (\nabla v, \nabla v - \nabla \widehat{y})\, dx \geq \int_\Omega (f, \nabla v - \nabla \widehat{y})\, dx, \quad \forall\, v \in W_0^{1,\widehat{p}(\cdot)}(\Omega)$$

and we arrive at the Minty relation (1.11). Thus, $(\widehat{p}, \widehat{y}) \in \Xi_M$.

Step 3. It remains to show that $\Xi_M \subseteq \Xi_w$. Let $(\widetilde{p}, \widetilde{y}) \in \Xi_M$ be a fixed pair. In view of Propositions 2.4 and 2.5, we can apply the so-called Minty trick. Namely, we can take any $\varphi \in W_0^{1,\widehat{p}(\cdot)}(\Omega)$ as a test function in the Minty inequality

$$\int_\Omega |\nabla \varphi|^{\widetilde{p}(x)-2} (\nabla \varphi, \nabla \varphi - \nabla \widetilde{y})\, dx \geq \int_\Omega (f, \nabla \varphi - \nabla \widetilde{y})\, dx, \tag{3.6}$$

and, after taking $\varphi = \widetilde{y} \pm v$ with $v \in C_0^\infty(\Omega)$ and $t > 0$ in (3.6), we can pass to the limit in this relation as $t \to 0$. This yields

$$\pm \int_\Omega |\nabla \widetilde{y} \pm t \nabla v|^{\widetilde{p}(x)-2} (\nabla \widetilde{y} \pm t \nabla v, \nabla v)\, dx \geq \pm \int_\Omega (f, \nabla v)\, dx, \quad \forall\, t > 0 \tag{3.7}$$

and, therefore, after the limit passage as $t \to 0$, we finally obtain

$$\int_\Omega |\nabla \widetilde{y}|^{\widetilde{p}(x)-2} (\nabla \widetilde{y}, \nabla v)\, dx = \int_\Omega (f, \nabla v)\, dx, \quad \forall\, v \in C_0^\infty(\Omega).$$

Thus, $(\widetilde{p}, \widetilde{y}) \in \Xi_w$, and this concludes the proof. □

As follows from this result, the sets Ξ_w, Ξ_V, and Ξ_M coincide under rather restrictive assumptions on the class of admissible exponents which exclude the appearance of the Lavrentieff effect.

Definition 3.1. We say that the Lavrentieff phenomenon is inherent in the OCP (1.1)–(1.4) if there is a gap between two constrained minimization problems

$$\left\langle \inf_{(p,y)\in\Xi_w} J(p,y) \right\rangle \quad \text{and} \quad \left\langle \inf_{(p,y)\in\Xi_V} J(p,y) \right\rangle, \tag{3.8}$$

namely, there exist two pairs $(\widetilde{p}^0, \widetilde{y}^0) \in \Xi_w$ and $(\widehat{p}^0, \widehat{y}^0) \in \Xi_V$ such that

$$J\left(\widetilde{p}^0, \widetilde{y}^0\right) = \inf_{(p,y)\in\Xi_w} J(p,y) < \inf_{(p,y)\in\Xi_V} J(p,y) = J\left(\widehat{p}^0, \widehat{y}^0\right). \tag{3.9}$$

It is interesting to note that solutions of the problems (3.8), in general, are different in the sense of smoothness provided Lavrentieff effect takes a place. In particular, the optimal state $\widetilde{y}^0$ cannot belong to the space V and, hence, to $H_0^{1,p(\cdot)}(\Omega)$. In view of this, we can indicate a few characteristic properties of the sets Ξ_w, Ξ_V, and Ξ_M that will be useful later on.

Proposition 3.2. For a given set of admissible controls $\mathbb{P}_{ad}$ the following statements hold:

(i) the inclusions $\Xi_V \subset \Xi_w$ and $\Xi_V \subset \Xi_M$ are valid;

(ii) the sets Ξ_w, Ξ_V, and Ξ_M are nonempty;

(iii) Ξ_M is a convex set with respect to y;

(iv) the set Ξ_M is sequentially closed in the following sense:
if $\{(p_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi_M$ is a sequence of pairs such that

$$\begin{gathered} p_k(x) \to p(x) \ \text{ a.e. in } \ \Omega, \quad y_k \rightharpoonup y \quad \text{in } \ W_0^{1,\alpha}(\Omega) \ \text{ as } k \to \infty, \\ \text{and } y \in W_0^{1,p(\cdot)}(\Omega), \end{gathered} \tag{3.10}$$

then $(p, y) \in \Xi_M$.

Proof. The validity of assertions (i)–(ii) can be easily established following the similar arguments as in the proof of Proposition 3.1 Let us show that Ξ_M is a convex set with respect to y. Let (p, y_1) and (p, y_2) be arbitrary pairs of Ξ_M. Then, for each $\lambda \in [0,1]$ and $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \left(|\varphi|^{p(x)-2}\,\nabla\varphi, \nabla\varphi - (\lambda\nabla y_1 + (1-\lambda)\nabla y_2) \right) dx \\
&= \lambda \int_\Omega \left(|\varphi|^{p(x)-2}\,\nabla\varphi, \nabla\varphi - \nabla y_1 \right) dx + (1-\lambda) \int_\Omega \left(|\varphi|^{p(x)-2}\,\nabla\varphi, \nabla\varphi - \nabla y_2 \right) dx \\
&\overset{\text{by (1.11)}}{\geq} \lambda \int_\Omega (f, \nabla\varphi - \nabla y_1)\, dx + (1-\lambda) \int_\Omega (f, \nabla\varphi - \nabla y_2)\, dx \\
&= \int_\Omega (f, \nabla\varphi - (\lambda\nabla y_1 + (1-\lambda)\nabla y_2))\, dx.
\end{aligned}$$

Hence, $(p, \lambda y_1 + (1-\lambda) y_2) \in \Xi_M$, i.e. $\lambda y_1 + (1-\lambda) y_2$ is a weak solution in the sense of Minty of the boundary value problem (1.2)–(1.3).

It remains to show that Ξ_M is a closed set in the sense of convergence (3.10). Let $\{(p_k, y_k)\}_{k\in\mathbb{N}}$ be a sequence such that $(p_k, y_k) \in \Xi_M$ for all $k \in \mathbb{N}$ and properties (3.10) hold with some distributions $p \in \mathbb{P}_{ad}$ and $y \in W_0^{1,p(\cdot)}(\Omega)$. Our aim is to show that $(p, y) \in \Xi_M$. With that in mind, we note that, in view of the estimate

$$\begin{aligned}
\int_E |\nabla y_k|\, dx &\leq |E|^{1/\alpha'} \left(\int_E |\nabla y_k|^\alpha\, dx \right)^{1/\alpha} \leq |E|^{1/\alpha'} \left(\int_\Omega |\nabla y_k|^\alpha\, dx \right)^{1/\alpha} \\
&\leq |E|^{1/\alpha'} \sup_{k\in\mathbb{N}} \|y_k\|_{W_0^{1,\alpha}(\Omega)} \overset{\text{by (3.10)}}{\leq} C|E|^{1/\alpha'}
\end{aligned}$$

where $E \neq \emptyset$ is an arbitrary measurable subset of Ω, the sequence $\{\nabla y_k\}_{k\in\mathbb{N}}$ is equi-integrable, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_B |\nabla y_k|\, dx < \varepsilon$$

holds for all $k \in \mathbb{N}$ and all Borel sets $B \subset \Omega$ with $|B| < \delta$. Let $\varphi \in C_0^\infty(\Omega)$ be a test function. Then, setting up

$$\psi_k := |\nabla\varphi|^{p_k(x)-2}\,\nabla\varphi \quad \text{and} \quad \psi := |\nabla\varphi|^{p(x)-2}\,\nabla\varphi = \psi,$$

we see that $\psi_k \to \psi$ almost everywhere in Ω as $k \to \infty$ and

$$|\psi_k| = |\nabla\varphi|^{p_k(x)-1} \leq \|\varphi\|_{C(\overline{\Omega})}^{\beta-1} + 1 = C^* \quad \text{a.e. in } \Omega \qquad \forall\, k \in \mathbb{N}.$$

Therefore, $\psi_k \to \psi$ in $L^1(\Omega)$ by Lebesgue dominated theorem and

$$\begin{aligned}
\int_\Omega \left(|\varphi|^{p_k(x)-2}\,\nabla\varphi, \nabla\varphi - \nabla y_k \right) dx &= \int_\Omega (\psi, \nabla\varphi - \nabla y_k)\, dx \\
&\quad + \int_\Omega (\psi_k - \psi, \nabla\varphi - \nabla y_k)\, dx = I_1 + I_2,
\end{aligned}$$

where, by definition of the weak convergence in $L^\alpha(\Omega)^N$, we have

$$\lim_{k\to\infty} I_1 = \lim_{k\to\infty}\int_\Omega (\psi, \nabla\varphi - \nabla y_k)\,dx = \int_\Omega (\psi, \nabla\varphi - \nabla y)\,dx,$$

and

$$\begin{aligned} I_2 &\le \int_\Omega |(\psi_k - \psi, \nabla\varphi - \nabla y_k)|\;dx \\ &= \int_{\{|\nabla\varphi - \nabla y_k|\ge n\}} |(\psi_k - \psi, \nabla\varphi - \nabla y_k)|\;dx \\ &\quad + \int_{\{|\nabla\varphi - \nabla y_k|< n\}} |(\psi_k - \psi, \nabla\varphi - \nabla y_k)|\;dx \\ &\le 2C^* \int_{\{|\nabla\varphi - \nabla y_k|\ge n\}} |\nabla\varphi - \nabla y_k|\;dx + n\int_\Omega |\psi_k - \psi|\;dx \end{aligned}$$

for any fixed $n \in \mathbb{R}_+$. Hence, for a given $\varepsilon > 0$ there exist indices $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that for all $k > k_0$ and $n > n_0$ we have

$$\int_{\{|\nabla\varphi - \nabla y_k|\ge n\}} |\nabla\varphi - \nabla y_k|\;dx < \frac{\varepsilon}{4C^*} \text{ by equi-integrability of } \{\nabla\varphi - \nabla y_k\}_{k\in\mathbb{N}},$$

$$\int_\Omega |\psi_k - \psi|\;dx < \frac{\varepsilon}{2n} \text{ by the strong convergence } \psi_k \to \psi \text{ in } L^1(\Omega).$$

Thus, $I_2 < \varepsilon$ for k large enough. Hence,

$$\lim_{k\to\infty}\int_\Omega \left(|\varphi|^{p_k(x)-2}\,\nabla\varphi, \nabla\varphi - \nabla y_k\right)\,dx = \int_\Omega \left(|\varphi|^{p(x)-2}\,\nabla\varphi, \nabla\varphi - \nabla y\right)\,dx.$$

Taking into account this fact and the weak convergence $\nabla y_k \rightharpoonup \nabla y$ in $L^\alpha(\Omega)^N$, we can pass to the limit in the integral inequality

$$\int_\Omega |\nabla\varphi|^{p_k(x)-2}\,(\nabla\varphi, \nabla\varphi - \nabla y_k)\,dx \ge \int_\Omega (f, \nabla\varphi - \nabla y_k)\,dx$$

as k tends to ∞. As a result, we get the following: the limit pair (p, y) belongs to the set $\mathbb{P}_{ad} \times W_0^{1,p(\cdot)}(\Omega)$ and satisfies the inequality (1.11). Hence, $(p, y) \in \Xi_M$. The proof is complete. □

Proposition 3.3. Assume that the set $\mathbb{P}_{ad}$ is given as in (1.4). Then

$$\emptyset \ne \Xi_V \subset \Xi_M \quad\text{and}\quad \Xi_M \setminus \Xi_V \ne \emptyset. \tag{3.11}$$

Proof. Let $p \in \mathbb{P}_{ad}$ be an arbitrary admissible exponent. Then we can come across with two different situations: either $H_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega)$ or $H_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$. In the first case, as Proposition 3.1 indicates, there is a unique weak solution $y = y(p)$ to the boundary value problem (1.1)–(1.3) such that this solutions satisfies both Definitions 1.2 and 1.3. Hence, $(p,y) \in \Xi_M$, $(p,y) \in \Xi_V$, and, therefore, $\Xi_V \neq \emptyset$.

As for the second case, we have a more delicate situation. Indeed, in this case, for any intermediate space V, $H \subseteq V \subseteq W$, the following assertions are well-known: V is a reflexive separable Banach space [8] and the operator $\mathcal{A} : V \to V^*$, given by the equality

$$(\mathcal{A}u, v) = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u, \nabla \varphi \right) dx, \quad \forall\, v \in V,$$

is bounded, semicontinuous, coercive, and strictly monotone [16]. Hence, the equation $\mathcal{A}y = f$ is uniquely solvable for each $f \in V^*$ [11]. In other words, there exists a unique distribution $y_V \in V$ such that $(p, y_V) \in \Xi_V$ and, therefore, $(p, y_V) \in \Xi_M$ by property (i) of Proposition 3.2. Since $H_0^{1,p(\cdot)}(\Omega) \neq W_0^{1,p(\cdot)}(\Omega)$, it follows that there exist at least two subspaces V_1 and V_2 of $W_0^{1,p(\cdot)}(\Omega)$ such that the corresponding variational solutions y_{V_1} and y_{V_2} do not coincide. Hence,

$$(p, y_{V_1}) \in \Xi_V, \quad (p, y_{V_2}) \in \Xi_V, \quad \left(p, \frac{1}{2}(y_{V_1} + y_{V_2}) \right) \overset{\text{by Proposition 3.2}}{\in} \Xi_M, \quad (3.12)$$

and

$$\left(p, \frac{1}{2}(y_{V_1} + y_{V_2}) \right) \notin \Xi_V. \quad (3.13)$$

Since the inclusions (3.12) are obvious, we focus on the proof of (3.13). To do so, it is enough to show that we loss the energy equality for the pair $\left(p, \frac{1}{2}(y_{V_1} + y_{V_2})\right)$ and have the energy inequality instead. Indeed, by the energy equalities for (p, y_{V_1}) and (p, y_{V_2}), we obtain

$$\int_\Omega \frac{1}{2} \left(|\nabla y_{V_1}|^{p(x)} + |\nabla y_{V_2}|^{p(x)} \right) dx = \int_\Omega \left(f, \nabla \left(\frac{y_{V_1} + y_{V_2}}{2} \right) \right) dx.$$

However, in view of the Clarkson's inequalities (see [2])

$$\frac{1}{2}\left(|\varphi|^p + |\psi|^p \right) \geq \left| \frac{\varphi - \psi}{2} \right|^p + \left| \frac{\varphi + \psi}{2} \right|^p, \quad p \geq 2,$$

$$\left(\frac{1}{2}\left(|\varphi|^p + |\psi|^p \right) \right)^{\frac{1}{p-1}} \geq \left(\left| \frac{\varphi - \psi}{2} \right|^p \right)^{\frac{1}{p-1}} + \left(\left| \frac{\varphi + \psi}{2} \right|^p \right)^{\frac{1}{p-1}}, \quad 1 < p < 2,$$

and the fact that $y_{V_1} \neq y_{V_2}$, it is easy to derive the relation

$$\int_\Omega \frac{1}{2} \left(|\nabla y_{V_1}|^{p(x)} + |\nabla y_{V_2}|^{p(x)} \right) dx > \int_\Omega \left| \nabla \left(\frac{y_{V_1} + y_{V_2}}{2} \right) \right|^p dx.$$

Thus, the pair $\left(p, \frac{1}{2}(y_{V_1} + y_{V_2})\right)$ satisfies the strict energy inequality

$$\int_\Omega \left| \nabla \left(\frac{y_{V_1} + y_{V_2}}{2} \right) \right|^p dx < \int_\Omega \left(f, \nabla \left(\frac{y_{V_1} + y_{V_2}}{2} \right) \right) dx$$

and, therefore, $\left(p, \frac{1}{2}(y_{V_1} + y_{V_2})\right) \notin \Xi_V$. To conclude the proof, it is enough to recall that $\left(p, \frac{1}{2}(y_{V_1} + y_{V_2})\right) \in \Xi_M$ by Proposition 3.2. Hence, $\Xi_M \setminus \Xi_V \neq \emptyset$. □

At the end of this section, it is worth to notice that the relationship between Ξ_w, Ξ_V, and Ξ_M is very intricate problem, in general. In particular, it is unknown whether $\Xi_M \subseteq \Xi_w$ or vice versa. We even can not assert that if $(p, y) \in \Xi_M$ and this pair is related by the energy equality (1.10), then $(p, y) \in \Xi_V$. It also remains an open question about the standard topological properties of Ξ_w such as compactness, closedness and etc. Moreover, as was shown in [16, pp.107-112], the set Ξ_w is not convex, in general. Thus, in contrast to the standard situation, where non-uniqueness is possible in classical monotone problems, it usually comes from the missing of strict convexity of the corresponding operator, whereas the solution set is convex and closed. In the case of boundary value problem (1.2)–(1.3), the corresponding operator $\mathcal{A} : W_0^{1,\widehat{p}(\cdot)}(\Omega) \to \left(W_0^{1,\widehat{p}(\cdot)}(\Omega)\right)^*$ is strictly monotone. So, non-uniqueness and non-convexity are of completely different nature.

4. On Solvability of Optimal Control Problem

In what follows, we discuss the following optimal control problem

$$\text{Minimize} \quad \Big\{ J(p,y) = \int_\Omega |y(x) - y_d(x)|^\alpha \, dx + \gamma \int_\Omega |\nabla y(x)|^{p(x)} \, dx \\ + \int_\Omega |p(x) - p_d(x)|^2 \, dx + \int_\Omega |Dp| \Big\} \tag{4.1}$$

subject to the constraints

$$-\operatorname{div}\left(|\nabla y|^{p(x)-2} \nabla y\right) = -\operatorname{div} f, \quad x \in \Omega, \tag{4.2}$$

$$y = 0 \ \text{ on } \ \partial\Omega, \tag{4.3}$$

$$p \in \widehat{\mathbb{P}}_{ad} = \{ p \in BV(\Omega) \ : \ 1 < \alpha \le p(x) \le \beta, \ \text{ a.e. in } \ \Omega \}, \tag{4.4}$$

where by $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ for which the norm

$$\|p\|_{BV(\Omega)} = \|p\|_{L^1(\Omega)} + \int_\Omega |Dp| = \|p\|_{L^1(\Omega)} \\ + \sup \Big\{ \int_\Omega p \operatorname{div} \varphi \, dx \ : \ \varphi \in C_0^1(\Omega; \mathbb{R}^N), \ |\varphi(x)| \le 1 \text{ for } x \in \Omega \Big\}$$

is finite.

We introduce the set of admissible solutions to the OCP (4.1)–(4.4) as follows:

$$\widehat{\Xi}_M = \left\{ (p,y) \ \middle| \ \begin{array}{c} p \in \widehat{\mathbb{P}}_{ad}, \ y \in W_0^{1,p(\cdot)}(\Omega), \\ (p,y) \text{ is related by Minty inequality (1.10).} \end{array} \right\} \tag{4.5}$$

It is clear that $J(p, y) < +\infty$ for all $(p, y) \in \widehat{\Xi}_M$.

Remark 4.1. We recall that a sequence $\{p_k\}_{k=1}^{\infty}$ converges weakly* to p in $BV(\Omega)$ if and only if the two following conditions hold (see [1]): $p_k \to p$ strongly in $L^1(\Omega)$ and $Dp_k \overset{*}{\rightharpoonup} Dp$ weakly* in the space of Radon measures $\mathcal{M}(\Omega;\mathbb{R}^N)$. Moreover, if $\{p_k\}_{k=1}^{\infty} \subset BV(\Omega)$ converges strongly to some p in $L^1(\Omega)$ and satisfies $\sup_{k\in\mathbb{N}} \int_\Omega |Dp_k| < +\infty$, then (see, for instance, [1] and [9])

$$\begin{aligned} &(i)\ p \in BV(\Omega) \ \text{ and } \int_\Omega |Dp| \le \liminf_{k\to\infty} \int_\Omega |Dp_k|;\\ &(ii)\ p_k \overset{*}{\rightharpoonup} p \ \text{ in } \ BV(\Omega). \end{aligned} \tag{4.6}$$

We say that $(p^0, y^0) \in BV(\Omega) \times W_0^{1,p^0(\cdot)}(\Omega)$ is a Minty optimal solution to the problem (4.1)–(4.4) if

$$(p^0, y^0) \in \widehat{\Xi}_M \ \text{ and } \ J(p^0, y^0) = \inf_{(p,y)\in\widehat{\Xi}_M} J(p, y).$$

Our main intention in this section is to show that the set of Minty optimal pairs is nonempty for the problem (4.1)–(4.4). With that in mind we make use of the direct method of Calculus of Variations.

To begin with, we note that the set of admissible controls $\widehat{\mathbb{P}}_{ad}$, given by (4.4), is nonempty, convex, it has an empty topological interior, and satisfies the inclusion $\widehat{\mathbb{P}}_{ad} \subset L^2(\Omega)$. Hence, all results of Section 3, concerning topological and algebraic properties of the sets Ξ_w, Ξ_V, and Ξ_M, remain valid. Moreover, it is worth to emphasize that $\widehat{\mathbb{P}}_{ad}$ is a sequentially closed set with respect to the weak* convergence in $BV(\Omega)$.

In what follows, we make use of a couple of auxiliary results which are crucial for our further analysis.

Lemma 4.1. *Let* $\{p_k\}_{k\in\mathbb{N}} \subset \widehat{\mathbb{P}}_{ad}$ *and* $\left\{y_k \in W_0^{1,p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ *be sequences such that* $p_k \overset{*}{\rightharpoonup} p$ *in* $BV(\Omega)$, *and* $y_k \rightharpoonup y$ *in* $W_0^{1,\alpha}(\Omega)$. *Then*

$$\liminf_{k\to\infty} \int_\Omega |\nabla y_k|^{p_k(x)}\, dx \ge \int_\Omega |\nabla y|^{p(x)}\, dx. \tag{4.7}$$

Proof. By the definition of the weak* convergence in $BV(\Omega)$, we have: $p_k \to p$ strongly in $L^1(\Omega)$. Hence, within a subsequence, we may suppose that $p_k(x) \to p(x)$ almost everywhere in Ω. As a result, for any $\xi \in \mathbb{R}^N$, the following pointwise convergence holds:

$$\lim_{k\to\infty} |\xi|^{p_k(x)} = |\xi|^{p(x)} \ \text{ a.e. in } \ \Omega, \tag{4.8}$$

$$\lim_{k\to\infty} \left(|\xi|^{p_k(x)}\right)^* (\eta) = \left(|\xi|^{p(x)}\right)^* (\eta) \ \text{ a.e. in } \ \Omega, \tag{4.9}$$

where $\left(|\xi|^{p_k(x)}\right)^* (\eta) = (p_k(x) - 1) \left|\frac{\eta}{p_k(x)}\right|^{p_k'(x)}$ is the conjugate function, i.e.

$$\left(|\xi|^{p_k(x)}\right)^* (\eta) = \sup_{\xi\in\mathbb{R}^N} \left((\xi, \eta) - |\xi|^{p_k(x)}\right). \tag{4.10}$$

As follows from (4.10), for each $z \in L^\infty(\Omega)^N = \left(L^1(\Omega)^N\right)^*$, we have the Young-Fenchel inequality

$$\int_\Omega |\nabla y_k|^{p_k(x)}\,dx \ge \int_\Omega (z, \nabla y_k)\;dx - \int_\Omega f_k^*(x, z(x))\,dx,$$

where it has been denoted $f_k^*(x,\eta) = \left(|\xi|^{p_k(x)}\right)^*(\eta)$.

Then, using (4.9) and the continuity of the embedding $L^\alpha(\Omega)^N \hookrightarrow L^1(\Omega)^N$, we obtain $\nabla y_k \rightharpoonup \nabla y$ in $L^1(\Omega)^N$ and, therefore,

$$\liminf_{k\to\infty} \int_\Omega |\nabla y_k|^{p_k(x)}\,dx \ge \int_\Omega (z, \nabla y)\;dx - \int_\Omega f^*(x, z(x))\,dx.$$

Since this relation holds for each $z \in L^\infty(\Omega)^N$, it follows that

$$\liminf_{k\to\infty} \int_\Omega |\nabla y_k|^{p_k(x)}\,dx \ge \sup_{z\in L^\infty(\Omega)^N} \left[\int_\Omega (z, \nabla y)\;dx - \int_\Omega f^*(x, z(x))\,dx\right].$$

To conclude the proof, it remains to apply the following well-known result on conjugate functionals [5, Chapter IX, Section 2.1]:

> Let f be an integrant such that $|\xi|^\alpha \le f(x,\xi) \le |\xi|^\beta$, and let f^* be its conjugate integrant. Then for any $v \in L^1(\Omega)^N$
>
> $$\begin{aligned}\int_\Omega f(x, v(x))\,dx &= \sup_{h\in L^\infty(\Omega)^N} \left[\int_\Omega (h, v)\;dx - \int_\Omega f^*(x, h(x))\,dx\right]\\ &= \sup_{h\in L^1(\Omega)^N} \left[\int_\Omega (h, v)\;dx - \int_\Omega f^*(x, h(x))\,dx\right].\end{aligned}$$

□

Lemma 4.2. *Let $\{(p_k, y_k)\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_M$ be a sequence such that*

$$\sup_{k\in\mathbb{N}} \left[\|p_k\|_{BV(\Omega)} + \int_\Omega |\nabla y_k|^{p_k(x)}\,dx\right] < +\infty. \tag{4.11}$$

Then, there is a pair $(p, y) \in \widehat{\Xi}_M$ such that, up to a subsequence, $p_k \overset{*}{\rightharpoonup} p$ in $BV(\Omega)$, $p_k(x) \to p(x)$ almost everywhere in Ω, and $y_k \rightharpoonup y$ in $W_0^{1,\alpha}(\Omega)$.

Proof. Taking into account condition (4.11) and estimate (2.6), we see that the sequence $\{y_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,\alpha}(\Omega)$. Hence, by Remark 4.1 and compactness properties of $BV(\Omega) \times W_0^{1,\alpha}(\Omega)$, there exists a subsequence of the sequence $\{(p_k, y_k)\}_{k\in\mathbb{N}}$, still denoted by the same indices, and functions $p \in BV(\Omega)$ and $y \in W_0^{1,\alpha}(\Omega)$ such that

$$p_k \overset{*}{\rightharpoonup} p \text{ in } BV(\Omega), \quad y_k \rightharpoonup y \;\text{ in }\; W_0^{1,\alpha}(\Omega), \tag{4.12}$$

$$\text{and }\; p_k(x) \to p(x) \text{ a.e. in } \;\Omega. \tag{4.13}$$

Then by Lemma 4.1, we have

$$+\infty > \sup_{k\in\mathbb{N}} \int_\Omega |\nabla y_k|^{p_k(x)}\, dx \geq \liminf_{k\to\infty} \int_\Omega |\nabla y_k|^{p_k(x)}\, dx$$
$$\geq \int_\Omega |\nabla y|^{p(x)}\, dx \overset{\text{by (2.6)}}{\geq} \|y\|^\alpha_{W_0^{1,p(\cdot)}(\Omega)} - 1.$$

This estimate implies that $y \in W_0^{1,p(\cdot)}(\Omega)$. To conclude the proof, i.e. to show that the limit pair is related by the Minty inequality (1.11), it remains to use the property (iv) of Proposition 3.2. □

We are now in a position to give the existence result for optimal pairs to the OCP (4.1)–(4.4).

Theorem 4.1. *Let $p_d \in L^2(\Omega)$, $y_d \in L^\alpha(\Omega)$, and $f \in L^\infty(\Omega)^N$ be given functions. Then the optimal control problem* (4.1)–(4.4) *admits at least one solution in the sense of Minty.*

Proof. Since the set $\widehat{\Xi}_M$ is nonempty and the cost functional is bounded from below on $\widehat{\Xi}_M$, it follows that there exists a minimizing sequence $\{(p_k, y_k)\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_M$ to the problem (4.1)–(4.4), i.e.

$$\inf_{(p,y)\in\widehat{\Xi}_M} J(p,y) = \lim_{k\to\infty} \left[\int_\Omega |y_k(x) - y_d(x)|^\alpha \, dx + \gamma \int_\Omega |\nabla y_k(x)|^{p_k(x)}\, dx \right.$$
$$\left. + \int_\Omega |p_k(x) - p_d(x)|^2 \, dx + \int_\Omega |Dp_k| \right] < +\infty.$$

Hence, in view of estimate (2.6) and definition of the set $\widehat{\mathbb{P}}_{ad}$, the sequence $\{(p_k, y_k)\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega) \times W_0^{1,\alpha}(\Omega)$. From Lemma 4.2 we deduce the existence of a subsequence, which is denoted in the same way, and a pair $(p^*, y^*) \in \widehat{\Xi}_M$ such that $p_k \overset{*}{\rightharpoonup} p^*$ in $BV(\Omega)$, $p_k(x) \to p^*(x)$ almost everywhere in Ω, and $y_k \rightharpoonup y^*$ in $W_0^{1,\alpha}(\Omega)$. From these convergences and Sobolev embedding theorem, we infer that

$$\lim_{k\to\infty} \int_\Omega |y_k - y_d|^\alpha \, dx = \int_\Omega |y^* - y_d|^\alpha \, dx$$
$$\Big(\text{by compactness of the embedding } W_0^{1,\alpha}(\Omega) \hookrightarrow L^\alpha(\Omega)\Big),$$
$$\liminf_{k\to\infty} \int_\Omega |Dp_k| \geq \int_\Omega |Dp^*| \text{ by (4.6)},$$
$$\liminf_{k\to\infty} \int_\Omega |\nabla y_k(x)|^{p_k(x)}\, dx \geq \int_\Omega |\nabla y^*(x)|^{p^*(x)}\, dx \text{ by (4.7), and}$$
$$\lim_{k\to\infty} \int_\Omega |p_k - p_d|^2 \, dx = \int_\Omega |p^* - p_d|^2 \, dx,$$

where the last assertion is a direct consequence of the strong convergence $p_k \to p^*$ in $L^1(\Omega)$ and boundedness of this sequence in $L^\infty(\Omega)$. So,

$$J(p^*, y^*) \le \inf_{(p,y)\in\widehat{\Xi}_M} J(p, y)$$

and, consequently, (p^*, y^*) is a Minty optimal solution of the OCP (4.1)–(4.4). □

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