Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 14 **(2012)**. Number 1. pp. 84 – 106

Groups with many pronormal and transitively normal subgroups

L. A. Kurdachenko, N. N. Semko (Jr.), I. Ya. Subbotin

To outstanding Ukrainian algebraist
Professor V. V. Kirichenko
on occasion of his 70-th birthday

Abstract. A subgroup H of a group G is said to be *transitively normal in* G, if H is normal in every subgroup $K \geqslant H$ such that H is subnormal in K. We described some infinite groups, whose non–finitely generated subgroups are transitively normal.

Introduction

We say that a subgroup H of a group G is **pronormal in** G if for each element $g \in G$ there exists an element $u \in \langle H, H^g \rangle$ such that $H^g = H^u$. Pronormal subgroups have naturally appeared in the process of investigation of such important subgroups of finite (soluble) group as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups. The term "a pronormal subgroup" belongs to P. Hall. The pronormal subgroups possess the following essential property. Let G be an arbitrary group and K be a pronormal subgroup of G. If L is a subgroup of G such that $K \leq L$ and K is subnormal in L, then K is normal in L. Observe that not only pronormal subgroups, but some its generalizations have this property also. Quite recently in the paper [1], the concept of transitively normal subgroup has been introduced. A subgroup H of a group G is

2010 MSC: 20E15, 20F19.
Key words and phrases: soluble group, radical group, locally nilpotent group, transitively normal subgroup, non finitely generated subgroup.

said to be **transitively normal in** G, if H is normal in every subgroup $K \geq H$ such that H is subnormal in K. As we can see, every pronormal subgroup is transitively normal, but converse is not true. Indeed, every self-normalizing subgroup H (that is $H = N_G(H)$) is transitively normal. Note that a pronormal and self-normalizing subgroup is abnormal (see, for example, [2, § 6, Theorem 7]). Recall that a subgroup A of a group G is called **abnormal in** G, if $x \in \langle A, A^x \rangle$ for each element $x \in G$. We note that every subgroup which includes an abnormal subgroup is abnormal and therefore is self-normalizing too (see, for example, [2, § 6, Theorem 1]). However, not every self-normalizing subgroup is abnormal.

The property "to be transitively normal subgroups" is connected with another important property. A group G is said to be a T**-group** if every subnormal subgroup of G is normal. In other words, G is a T-group if the property "to be normal subgroup" is transitive in G. Abelian groups and groups whose all subgroups are normal (the **Dedekind groups**) are trivial examples of T-groups. R. Baer [3] proved that every Dedekind group G either is abelian or $G = Q \times D \times B$ where Q is a quaternion group, D is an elementary abelian 2-subgroup and B is a periodic abelian subgroup such that $2 \notin \Pi(B)$. The structure of finite soluble T-groups has been established by W. Gaschütz [4]. We note that in general case, not every subgroup of T-group is a T-group itself; so we come to the following type of groups. A group G is said to be a $\overline{T}$**-group** if every subgroup of G is a T-group. It turns out that a finite $\overline{T}$-group is metabelian. W. Gaschütz proved that every finite soluble T-group is a $\overline{T}$-group although this is no longer true for infinite soluble groups. In finite soluble T-groups and $\overline{T}$-groups have been studied by D.J.S. Robinson [5]. We observed that T-groups have many pronormal subgroups, actually every finitely generated subgroup of a T-group is pronormal. Converse is also true. T.A. Peng [6] proved that a finite group G is a T-group if and only if every cyclic subgroup of G is pronormal. This result was extended to infinite locally soluble groups by N.F. Kuzennyj and I.Ya. Subbotin [7]. In particular, we obtain the following characterization of locally soluble$\overline{T}$-group: a locally soluble group G is a $\overline{T}$-group if and only if every finitely generated subgroup of G is pronormal. In this connection, it is natural to consider an opposite case, that is the groups in which every non-finitely generated subgroup is pronormal. In this paper, we will consider more general situation, more precisely we will study the groups whose non-finitely generated subgroups are transitively normal. Of course, we will consider these groups under some natural restrictions. As the results of A.Yu. Olshanskij (see the book [8]) show, in many cases it is not realistic to expect meaningful results on groups beyond the class of generalized soluble groups. We will study our groups under the following restriction. Recall that a group G is said to be **radical** if it has an ascending series whose factors are locally nilpotent. Some standard properties of radical groups can be found in [9]. In this paper we start the consideration of radical groups, whose non-finitely generated subgroups are transitively normal. Recall that a periodic radical group is locally soluble. But converse is not true.

The fist main result of this paper is the following description of the locally soluble groups, whose non-finitely generated subgroups are transitively normal.

Theorem A. *Let G be a locally soluble periodic group whose non-finitely generated subgroups are transitively normal. If G is not a Chernikov group, then G is a $\overline{T}$-group. If G is a Chernikov group, then either G is a Dedekind group or the divisible part Y of G is a quasicyclic group and G/Y is a finite $\overline{T}$-group.*

We observe that the conditions of Theorem A are also sufficient. Indeed, let G be a Chernikov group, Y be the divisible part of G, and suppose that Y is quasicyclic and G/Y is a finite $\overline{T}$-group. If H is a subgroup of G such that H is non-finitely generated, then clearly $Y \leq H$. Since G/Y is a $\overline{T}$-group, H/Y is transitively normal in G/Y. Then H is transitively normal in G.

First natural step in every study of radical group is a consideration of locally nilpotent case. For these groups we proved

Theorem B. *Let G be a locally nilpotent group whose non-finitely generated subgroups are transitively normal. Then every subgroup of G, which is non-finitely generated, is normal in G.*

The groups, whose non-finitely generated subgroups are normal, have been studied by L.A. Kurdachenko and V.V. Pylaev [10], G. Cutolo [11], G. Cutolo and L.A. Kurdachenko [12]. In particular, locally nilpotent groups, whose non-finitely generated subgroups are normal, were described quite completely.

The last main result of this paper is following

Theorem C. *Let G be a radical group whose non-finitely generated subgroups are transitively normal. Suppose that G is not periodic. If a locally nilpotent radical of G is non minimax group, then G is abelian.*

The case of minimax groups, whose non-finitely generated subgroups are transitively normal, requires separate consideration.

1. Preliminary results

1.1. Lemma. *Let G be a group whose non-finitely generated subgroups are transitively normal. Let L be an ascendant locally nilpotent subgroup of G and suppose that $L = \mathbf{Dr}_{\lambda\in\Lambda} L_\lambda$ where the subgroups L_λ is non-identity for each. If the index set is infinite, then L is normal in G, moreover, every subgroup L_λ is G-invariant for each $\lambda \in \Lambda$.*

Proof. Since the set Λ is infinite, the subgroup L cannot be finitely generated. Hence it is transitively normal in G. Being ascendant in G, it is normal in G [1, Lemma 1.1].

Let λ be an arbitrary index, put $M = \Lambda\backslash\{\lambda\}$. Then the subset M is infinite, so that there are two infinite subset M_1, M_2 of M such that $M = M_1 \cup M_2$, $M_1 \cap M_2 = \varnothing$. The subgroup $K_j = L_\lambda \times \langle L_\mu | \mu \in M_j\rangle$ is non-finitely generated, because the set M_j is infinite, so that K_j is transitively normal in $G, j \in \{1, 2\}$. Since K_j is normal in L, it is ascendant in G, which follows that K_j is normal in G[1, *Lemma* 1.1], $j \in \{1, 2\}$. The equation $L_\lambda = K_1 \cap K_2$ implies that L_λ is normal in G. □

1.2. Lemma. *Let G be a locally nilpotent group. If a cyclic subgroup $\langle g\rangle$ is transitively normal in G, then $\langle g\rangle$ is normal in G.*

Proof. Let F be an arbitrary finitely generated subgroup of G, containing an element g. Then F is nilpotent, which follows that $\langle g\rangle$ is subnormal in F. Being transitively normal, a subgroup $\langle g\rangle$ is normal in F. Since it is true for every finitely generated subgroup of $G, \langle g\rangle$ is normal in G. □

1.3. Lemma. *Let G be a group and T be a normal subgroup of G. Suppose that $T = \mathbf{Dr}_{p\in\Pi(T)} T_p$ where L_p is a Sylow p-subgroup of T. If a set $\Pi(T)$ is infinite, then the subgroup $\langle g, T\rangle$ is non-finitely generated for each element $g \in G$.*

Proof. Suppose the contrary, let $K = \langle g, T\rangle$ is finitely generated. If g has finite order, then the index $|K : T|$ is finite. But in this case, the subgroup T must be finitely generated (see, for example, [13, Corollary 7.2.1]), and we obtain a contradiction. Let g has infinite order. Then T has the elements $x_1, \ldots, x_n$ such that $T = \langle x_1\rangle^K \ldots \langle x_n\rangle^K$ [14, p. 421 and 426]. There exists a finite subset π of $\Pi(T)$ such that $x_1, \ldots, x_n \in \mathbf{Dr}_{p\in\pi} T_p = T_\pi$. Clearly, every subgroup T_p is G-invariant, so that T_π is normal in G. Then T_π includes the normal subgroup $\langle x_1\rangle^K \ldots \langle x_n\rangle^K$. Since $\Pi(T)$ is infinite, $T_\pi \neq T$, and we obtain a contradiction. This contradiction shows that the subgroup K is non-finitely generated. □

1.4. Lemma. *Let G be a group whose non-finitely generated subgroups are transitively normal. Let L be a periodic locally nilpotent subgroup of G and suppose that the set $\Pi(L)$ is infinite. If K is a locally nilpotent subgroup of $N_G(L)$, then K is a Dedekind group; moreover, if K is not periodic, then it is abelian.*

Proof. Since L is locally nilpotent, $L = \mathbf{Dr}_{p\in\Pi(L)} L_p$, where L_p is a Sylow p-subgroup of $L, p \in \Pi(L)$. Clearly, every subgroup L_p is K-invariant. Let h be an arbitrary element of K. If h has infinite order, then $\langle h\rangle \cap L = \langle 1\rangle$. If the order of h is finite, then there exists an infinite subset M of $\Pi(L)$ such that $\langle h\rangle \cap \mathbf{Dr}_{p\in M} L_p = \langle 1\rangle$. Thus in every case, there exists an infinite subset M of $\Pi(L)$ with the property $\langle h\rangle \cap \mathbf{Dr}_{p\in M} L_p = \langle 1\rangle$. Being infinite, M includes two infinite subset Θ, Ξ such that $M = \Xi \cup \Theta$ and $\Xi \cap \Theta = \varnothing$. Let $U = \mathbf{Dr}_{p\in\Theta} L_p$ and $V = \mathbf{Dr}_{p\in\Xi} L_p$, then both subgroups U, V are K — invariant and $\langle 1\rangle = U \cap V$. By Lemma 1.3, a subgroup $\langle h, U\rangle$ respectively $\langle h, V\rangle$) is non-finitely generated. It follows that $\langle h, U\rangle$ (respectively $\langle h, V\rangle$) is transitively normal in G. By Lemma 1.2, $\langle h\rangle U/U$ (respectively $\langle h\rangle V/V$) is normal in K/U (respectively in K/V), so that $\langle h, U\rangle$ and $\langle h, V\rangle$ are normal in K. Then $\langle h\rangle = \langle h\rangle U \cap \langle h\rangle V$ is normal in K. In other words, every cyclic subgroup of K is normal in K. This means that K is a Dedekind group. □

1.5. Lemma. *Let G be a group whose non-finitely generated subgroups are transitively normal. Let p be a prime and P be a locally finite p-subgroup of G. If P is not a Chernikov group, then P is a Dedekind group.*

Proof. Let x be an arbitrary element of P and F be an arbitrary finite subgroup of P, contains x. Since P is not a Chernikov group, P includes an infinite elementary abelian p-subgroup A, which is F-invariant [15]. Let $A = \mathbf{Dr}_{\lambda\in\Lambda} A_\lambda$, where $A_\lambda = \langle a_\lambda\rangle$ is a cyclic subgroup of order $p, \lambda \in \Lambda$. By Lemma 1.1 every subgroup A_λ is F-invariant, $\lambda \in \Lambda$. There exists an infinite subset M of such that $\langle x\rangle \cap \mathbf{Dr}_{\lambda\in M} A_\lambda = \langle 1\rangle$. Being infinite, M includes two infinite subset $\Theta\Xi$, such that $M = \Xi \cup \Theta$ and $\Xi \cap \Theta = \varnothing$. Let $U = \mathbf{Dr}_{\lambda\in\Theta} A_\lambda$ and $V = \mathbf{Dr}_{\lambda\in\Xi} A_\lambda$, then both subgroups U, V are F-invariant and $U \cap V = \langle 1\rangle$. By Lemma 1.3, the subgroup $\langle x, U\rangle$ (respectively $\langle x, V\rangle$) is non-finitely generated. It follows that $\langle x, U\rangle$ (respectively $\langle x, V\rangle$) is transitively normal in G. By Lemma 1.2, $\langle x\rangle U/U$ (respectively $\langle x\rangle V/V$) is normal in FA/U (respectively in FA/V), so that $\langle x, U\rangle$ and $\langle x, V\rangle$ are normal in FA. Then $\langle x\rangle = \langle x\rangle U \cap \langle x\rangle V$ is normal in FA. In particular, $\langle x\rangle$ is normal in F. Since it is true for each finite subgroup F of $P, \langle x\rangle$ is normal in P. Thus every cyclic subgroup of P is normal in P. This means that P is a Dedekind group. □

1.6. Lemma. *Let G be a group whose non-finitely generated subgroups are transitively normal. Let P be a Chernikov subgroup of G. If P is not a Dedekind group, then its divisible part D is quasicyclic; moreover, if P is locally nilpotent, then P/D is a Dedekind group.*

Proof. Suppose that D is not quasicyclic. Let x be an arbitrary element of D. Since D is divisible, there exists a quasicyclic subgroup X of D, which contains x. Being quasicyclic, the subgroup X is non-finitely generated, so that it is transitively normal. Since X is subnormal in P, X is normal in P [1, Lemma 1.1]. Since X is quasicyclic, every subgroup of X is P-invariant. In particular, a subgroup $\langle x\rangle$ is normal in P. It follows that every subgroup of D is normal in P. Let g be an arbitrary element of P. Without loss of generality we may suppose that g is a p-element for some prime p. Then $\langle y\rangle = \langle g\rangle \cap D$ is a cyclic p-subgroup. In this case D includes a quasicyclic p-subgroup Y such that $\langle y\rangle \leq Y$. Since Y is divisible, there exists a subgroup Z such that $D = Y \times Z$ [16, Theorem 21.2]. As we proved above, this subgroup Z is P-invariant. The subgroups $\langle g\rangle Y$ and $\langle g\rangle Z$ are both infinite, therefore they are non-finitely generated. It follows that both subgroups $\langle g\rangle Y$ and $\langle g\rangle Z$ are transitively normal in G. By Lemma 1.2, $\langle g\rangle Y/Y$ (respectively $\langle g\rangle Z/Z$) is normal in P/Y (respectively in P/Z), so that $\langle g\rangle U$ and $\langle g\rangle Z$ are normal in P. Then the equations

$$\langle g\rangle Y \cap \langle g\rangle Z = \langle g\rangle(\langle g\rangle Y \cap Z) = \langle g\rangle(\langle g\rangle Y \cap D \cap Z) =$$

$$\langle g\rangle(\langle y\rangle \cap Z) = \langle g\rangle$$

shows that $\langle g\rangle$ is normal in P. Thus every cyclic subgroup of P is normal in P. This means that P must be a Dedekind group. This contradiction shows that D is a quasicyclic subgroup.

Assume that P is locally nilpotent. For every element $g \in P$ the subgroup $\langle g, D\rangle$ is non-finitely generated. It follows that $\langle g, D\rangle$ is are transitively normal in G. By Lemma 1.2, $\langle g\rangle D/D$ is normal in P/D. Thus every cyclic subgroup of P/D is normal in P/D. This means that P/D is a Dedekind group. □

We need now information about structure of finite groups whose cyclic subgroups are transitively normal.

1.7. Lemma. *Let G be a finite group whose cyclic subgroups are transitively normal. If G_p is a Sylow p-subgroup of G, then G_p is a Dedekind group. Moreover, $G = H \lambda G_2$; in particular, G is soluble.*

Proof. Since G_p is nilpotent, every cyclic subgroup of G_p by Lemma 1.2 is normal. It follows that every subgroup of G_p is normal in G_p. This means that G_p is a Dedekind group.

Let $V = N_G(G_2)$, then $V = G_2 \lambda S$ where S is a Hall 2'-subgroup of V. If x is an element of G_2 of order 2, then the subgroup $\langle x \rangle$ is subnormal in V. Lemma 1.2 implies that $\langle x \rangle$ is normal in V. We remark that any normal subgroup of order 2 lies in the center of a group. Put $\Omega_k(G_2) = \langle g | g \in G_2$ and $|g| \leq 2^k \rangle$. Then $\Omega_1(G_2) \leq \zeta(V)$. Using similar arguments we prove that $\Omega_{k+1}(G_2)/\Omega_k(G_2) \leq \zeta(V/\Omega_k(G_2))$ for every positive integer k. It follows that the upper hypercenter of V includes G_2. It follows that $V = G_2 \times S$. By Theorem 1 of the paper [17], $G = H \lambda G_2$. □

1.8. Lemma. *Let G be a finite group whose cyclic subgroups are transitively normal. Then G is supersoluble.*

Proof. Let U, V be normal subgroups of G such that $U \leq V$ and a factor V/U is chief. Lemma 1.7 shows that V/U is an elementary abelian p-group for some prime p.

Let $x \in V \backslash U$. The subgroup $\langle x \rangle$ is transitively normal in G, so that $\langle x \rangle U/U$ is transitively normal in G/U. Lemma 1.2 shows that $\langle x \rangle U/U$ is normal in G/U. Since the factor V/U is G-chief, $V/U = \langle x \rangle U/U$. Hence every G-chief factor of G is cyclic. It follows that a group G is supersoluble. □

Let G be a group and $\mathbf{X}$ be a class of groups. Then the intersection G_X of all normal subgroups H of G such that $G/H \in \mathbf{X}$ is called the **X-residual of a group** G.

In particular, if G is finite group and $\mathbf{X} = \mathbf{N}$ is a class of all nilpotent groups, then we will say about the **nilpotent residual** G_N **of a group** G. We observe that in this case, G/G_N is nilpotent.

If $\mathbf{X} = \mathbf{LN}$ is a class of all locally nilpotent groups, then we will say about the locally nilpotent residual G_{LN} of a group G. We observe that in general the factor-group G/G_{LN} is not locally nilpotent. However, if a group G is locally finite, then G/G_{LN} is locally nilpotent.

1.9. Corollary. *Let G be a finite group whose cyclic subgroups are transitively normal. If L is a nilpotent residual of G, then the following assertions hold:*

(i) *L is abelian;*

(ii) $2 \in \Pi(L)$*;*

(iii) *G/L is a Dedekind group;*

(iv) *every subgroup of L is G-invariant.*

Proof. By Lemma 1.7, $G = H\lambda G_2$ where G_2 is a Sylow 2-subgroup of G. In particular, G/H is nilpotent. It follows that $L \leq H$, so that $2 \notin \Pi(L)$. By Lemma 1.8, G is supersoluble, and therefore the derived subgroup $[G, G]$ is nilpotent. Clearly, the inclusion $L \leq [G, G]$ implies that L is nilpotent. Finally, using again Lemma 1.7 and take into account that $2 \notin \Pi(L)$, we obtain that the Sylow p-subgroups of L are abelian for every prime p, so that L is abelian itself. Since G/L is nilpotent, $\langle g\rangle L/L$ is normal in G/L by Lemma 1.2. It follows that G/L is a Dedekind group. Finally, Lemma 1.2 proves (iv). □

1.10. Corollary. *Let G be a finite group whose cyclic subgroups are transitively normal. Then the derived length* $\mathbf{dl}(G)$ *is at most 3.*

Proof. By Corollary 1.9, the locally nilpotent residual L of G is abelian and G/L is a Dedekind group. Then $\mathbf{dl}(G/L) \leq 2$, so that $\mathbf{dl}(G) \leq 3$. □

1.11. Lemma. *Let G be a locally soluble periodic group whose non-finitely generated subgroups are transitively normal. Suppose that G is not a Chernikov group. If F is an arbitrary finite subgroup of G, then every cyclic subgroup of F is transitively normal in F.*

Proof. Since G is not a Chernikov group, it includes an F-invariant infinite abelian subgroup $A = \mathbf{Dr}_{\lambda\in\Lambda} A_\lambda$, where $A_\lambda = \langle a_\lambda\rangle$ is a cyclic subgroup of prime order, $\lambda \in \Lambda$ [15]. By Lemma 1.1, every subgroup A_λ is F-invariant, $\lambda \in \Lambda$. Since F is finite, there exists an infinite subset M of such that $F \cap \mathbf{Dr}_{\lambda\in M} A_\lambda = \langle 1\rangle$. Let $V = \mathbf{Dr}_{\lambda\in M} A_\lambda$. We note that the subgroup V is F-invariant and $F \cap V = \langle 1\rangle$. Let x be an arbitrary element of F. Lemma 1.3 shows that the subgroup $\langle x, V\rangle$ is non-finitely generated. It follows that $\langle x, V\rangle$ is transitively normal in G. In particular, $\langle x\rangle V/V$ is transitively normal in FV/V. The isomorphism $F \cong F/(F \cap V) \cong FV/V$ shows that $\langle x\rangle$ is transitively normal in F. □

1.12. Corollary. *Let G be a locally soluble periodic group whose non-finitely generated subgroups are transitively normal. Suppose that G is not a Chernikov group. Then the following assertions hold:*

(i) *the locally nilpotent residual L is abelian;*

(ii) $2 \notin (L)$*;*

(iii) G/L *is a Dedekind group;*

(iv) *every subgroup of L is G-invariant.*

Proof. Let L be a local system of G consisting of all finite subgroups. If $F \in L$, then Lemma 1.11 shows that every cyclic subgroup of F is transitively normal in F. Since G/L is locally nilpotent, the isomorphism $F/(F \cap L) \cong FL/L$ shows that $F/(F \cap L)$ is nilpotent. It follows that $F \cap L$ includes a nilpotent residual K_F of the subgroup F. On the other hand, if H is a finite subgroup of G such that $F \leq H$ and K_H is the nilpotent residual of the subgroup K, then using again the isomorphism $F/(F \cap K_H) \cong FK_H/K_H$, we obtain that $F/(F \cap K_H)$ is nilpotent, which implies $K_F \leq F \cap K_H$; in particular, $F \leq H$ implies that $K_F \leq K_H$. In other words, the system $\mathbf{K} = \{K_F | F \in L\}$ is also local. It follows that $\mathbf{K} = \cap_{F \in L} K_F$ is a normal subgroup of G. For every finite subgroup F we have $F \cap K = K_F$, so that $F/(F \cap K) = F/K_F$ is nilpotent. The isomorphism $F/(F \cap K) \cong FK/K$ shows that every finite subgroup of G/K is nilpotent. It follows that G/K is locally nilpotent, so that $L \leq K$. On the other hand, as we showed above, L includes the nilpotent residual K_F of each finite subgroup F of G, so that L includes $\cup_{F \in L} K_F = K$. Hence $L = \cup_{F \in L} K_F$. Using now Lemma 1.11 and Corollary 1.9 we conclude that L is abelian and $2 \notin \Pi(L)$. Let x, y be two arbitrary element of G, put $X = \langle x, y \rangle$. Then X is finite. Using again Lemma 1.11 and Corollary 1.9 we obtain that $X/K_X = X/(X \cap L) \cong XL/L$ is a Dedekind group. It follows that $(xL)^{yL} = (xL)^t$ for some positive integer $\mathbf{t}$. This means that G/L is a Dedekind group.

Similarly, let $g \in L$ and let h be an arbitrary element of G, put $F = \langle g, h \rangle$. Then F is finite and $g \in F \cap L = K_F$. Using Lemma 1.11 we obtain that every cyclic subgroup of F is transitively normal in F. Corollary 1.9 implies that $g^h = g^m$ for some positive integer m. Since it is true for an arbitrary element $h \in G$, the subgroup $\langle g \rangle$ is G-invariant. □

Proof of theorem A

Assume first that G is not a Chernikov group. Let L be the locally nilpotent residual of G. Suppose that $\Pi(L) \cap \Pi(G/L) \neq \varnothing$, and choose a prime q with the property $q \in \Pi(L) \cap \Pi(G/L)$. By Corollary 1.12, L is abelian, thus $L = \mathbf{Dr}_{p\in\Pi(L)} L_p$ where L_p is the Sylow p-subgroup of L. Let $Q = L_q$ and $R = \mathbf{Dr}_{p\neq q} L_p$. Then $Q \cong L/R$. Since $q \in \Pi(G/L)$, the Sylow q-subgroup S/L is non-identity. We recall that G/L is locally nilpotent by Corollary 1.12, so that S/L is normal in G/L. It follows that S/R is a normal Sylow q-subgroup of G/R. Let $x \in S\backslash R, g$ be an arbitrary element of G. Put $F = \langle x, g\rangle$. Lemma 1.11 shows that every cyclic subgroup of F is transitively normal in F. The subgroups $S \cap F$ and $R \in F$ are normal in F and $(S \cap F)/(R \cap F)$ is a q-group. By Lemma 1.2, every cyclic subgroup of $(S \cap F)/(R \cap F)$ is F-invariant, in particular,

$$(x(R \cap F))^{g(R\cap F)} = (x(R \cap F))^t = x^t(R \cap F)$$

for some positive integer $\mathbf{t}$. It follows that $(xR)^{gR} = x^tR = (xR)^t$. Since this is true for every element $g \in G$, the subgroup $\langle xR\rangle$ is G-invariant. If we suppose that $S/R \leq \zeta(G/R)$, then the factor-group G/R is locally nilpotent, and we obtain a contradiction with the choice of R. This contradiction shows that $G/R \neq C_{G/R}(S/R)$. Since S/R is a Sylow q-subgroup of G/R, there exists the q'-element $zR \notin C_{G/R}(S/R)$. Choose the element $y \in S\backslash R$ such that $(zR)(yR) \neq (yR)(zR)$. Let H/R be an arbitrary finite subgroup of S/R, containing yR. By proved above, every subgroup of H/R is $\langle zR\rangle$-invariant and $zR \notin C_{G/R}(H/R)$. Then H/R is abelian [18]. It follows that S/R is abelian. If we suppose that $L/R \leq \zeta(G/R)$, then the factor-group G/R is locally nilpotent, and we obtain a contradiction with the choice of R. This contradiction shows that $G/R \neq C_{G/R}(L/R)$. Since S/R is an abelian Sylow q-subgroup of G/R, there exists a q-element $vR \in C_{G/R}(L/R)$. Choose the element $d \in S\backslash L$ such that $d^p \in L$. By proved above, the subgroup $\langle dR\rangle$ is G-invariant, which implies that $\langle dL\rangle$ is also G-invariant. Since G/L is locally nilpotent, $D/L = \langle dL\rangle \leq \zeta(G/L)$. Then $[D/L, vR]L/R$, in particular, $[D/R, vR] \neq D/R$. Since S/R is an abelian q-group, Proposition 2.12 of the paper [19] shows that $[D/R, vR] \times C_{D/R}(vR) = D/R$. Then from $[D/R, vR] \neq D/R$ we obtain that $C_{D/R}(vR) \neq \langle 1\rangle$. Choose now in $C_{D/R}(vR)$ an element cR of order p. Let now aR be an element of $[D/R, vR]$ having order p. Since every subgroup of D/R is $\langle vR\rangle$-invariant and $aR \notin C_{D/R}(vR), (aR)^{xR} = (aR)^m$ where m is a p'-number; moreover, $\mathbf{m} \not\equiv 1(\mathbf{mod}p)$. We have

$$(acR)^{xR} = ((aR)(cR))^{xR} = (aR)^{xR}(cR)^{xR} = (aR)^{xR}(cR) = (aR)^m(cR).$$

On the other hand, since $acR \notin C_{D/R}(vR), (acR)^{xR} = (acR)^t$ where $\mathbf{t}$ is also p'-number such that $\mathbf{t} \not\equiv 1(\mathbf{mod}\ p)$. Hence $(aR)^m(cR) = (acR)^t =$

$(aR)^t(cR)^t$, it follows that $\mathbf{m} \equiv \mathbf{t}(\mathbf{mod}\ p)$ and $\mathbf{t} \equiv 1(\mathbf{mod}\ p)$, and we obtain a contradiction. This contradiction proves that $\Pi(L) \cap \Pi(G/L) \neq \varnothing$.

Corollary 1.12 shows that L is abelian, $2 \notin \Pi(L), G/L$ is a Dedekind group, every subgroup of L is G-invariant. In other words, all conditions of Theorem 6.1.1 of paper [5] are satisfied. By this theorem G is a $\overline{T}$-group.

Consider now the case when G is a Chernikov group. Suppose that D is not a Dedekind group. Then Lemma 1.6 shows that Y is a quasicyclic group. Let H/Y is an arbitrary subgroup of G/Y. Clearly, H is non-finitely generated, so that H is transitively normal in G. Hence every subgroup of G/Y is transitively normal. It follows that G/Y is a $\overline{T}$-group. □

1.14. Corollary. *Let G be a periodic (locally soluble)-by finite group whose non-finitely generated subgroups are transitively normal. If G is not a Chernikov group, then G is a $\overline{T}$-group.*

Proof. Let K be a normal locally soluble subgroup of G. Then K is not a Chernikov group. If $g \in G\ K$, then the subgroup $\langle g, K\rangle$ is non-finitely generated. It follows that $\langle g, K\rangle$ is transitively normal in G. Then $\langle g, K\rangle/K = \langle g\rangle K/K$ is transitively normal in G/K. In other words, every cyclic subgroup of G/K is transitively normal in G/K. Corollary 1.10 shows that G/K is soluble. Then G is locally soluble, and we can apply Proposition 1.13. □

2. The locally nilpotent groups whose non-finitely generated subgroups are transitively normal

A group G is said to have **finite 0-rank** $\mathbf{r}_0(G) = \mathbf{r}$ if G has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly $\mathbf{r}$. If G has another ascending series whose factors are either infinite cyclic or periodic, then it is not hard to see that the number of infinite cyclic factors in this new series is also $\mathbf{r}$. It shows that $\mathbf{r_0}(G)$ is an invariant of the group G.

If G has an ascending series with periodic and infinite cyclic factors and the set of infinite cyclic factors are infinite, then we will say that the group G has infinite 0-rank.

Otherwise we will say that G has no 0-rank.

2.1. Lemma. Let G be a group whose non-finitely generated subgroups are transitively normal. If G includes a normal free abelian subgroup of infinite 0-rank, then G is a Dedekind group.

Proof. Let A be a normal free abelian subgroup of infinite 0-rank, then $A = \mathbf{Dr}_{\lambda\in\Lambda} A_\lambda$ where $A_\lambda = \langle a_\lambda \rangle$ is an infinite cyclic subgroup, $\lambda \in \Lambda$. By Lemma 1.1, every subgroup A_λ is G-invariant, $\lambda \in \Lambda$. Let x be an arbitrary element of a group G. There exists an infinite subset M of Λ such that $\langle x \rangle \cap \mathbf{Dr}_{\lambda ß M} A_\lambda = \langle 1 \rangle$. Being infinite, M includes two infinite subset $\Theta\Xi$, such that $M = \Xi \cup \Theta$ and $\Xi \cap \Theta = \varnothing$. Let $U = \mathbf{Dr}_{\lambda\in\Theta} A_\lambda$ and $V = \mathbf{Dr}_{\lambda\in\Xi} A_\lambda$; then both subgroups U, V are G-invariant and $U \cap V = \langle 1 \rangle$. By Lemma 1.3, the subgroup $\langle x, U \rangle$ (respectively $\langle x, V \rangle$) is non-finitely generated. It follows that $\langle x, U \rangle$ (respectively $\langle x, V \rangle$) is transitively normal in G. By Lemma 1.2 $\langle x \rangle U/U$ (respectively $\langle x \rangle V/V$) is normal in G/U (respectively in G/V), so that $\langle x, U \rangle$ and $\langle x, V \rangle$ are normal in G. Then $\langle x \rangle = \langle x \rangle U \cap \langle x \rangle V$ is normal in G. In particular, $\langle x \rangle$ is normal in G. Thus every cyclic subgroup of G is normal in G. This means that G is a Dedekind group. □

2.2. Corollary. *Let G be a group whose non-finitely generated subgroups are transitively normal. If G includes an ascendant free abelian subgroup of infinite* 0*-rank, then G is a Dedekind group.*

Proof. Let A be an ascendant free abelian subgroup of infinite 0-rank. By Lemma 1.1, the subgroup A is normal in G, so we may apply Lemma 2.1. □

If G is a group, then by $\mathbf{Tor}(G)$ we will denote the maximal normal periodic subgroup of G. We recall that if G is a locally nilpotent group, then $\mathbf{Tor}(G)$ is a (characteristic) subgroup of G and $G/\mathbf{Tor}(G)$ is torsion-free.

2.3. Lemma. *Let G be a group whose non-finitely generated subgroups are transitively normal. Suppose that L is a locally nilpotent radical of G. If its periodic part* $\mathbf{Tor}(L)$ *is not a Chernikov group, then L is a Dedekind group.*

Proof. Let $T = \mathbf{Tor}(L)$. Then $T = \mathbf{Dr}_{p\in\Pi(L)} T_p$ where T_p is a Sylow p-subgroup of $L, p \in \Pi(L)$. Clearly every subgroup L_p is G-invariant. The fact $p \in \Pi(L)$ means that $T_p \neq \langle 1 \rangle$. Thus if $\Pi(L)$ is infinite, then the decomposition $T = \mathbf{Dr}_{p\in\Pi(L)} T_p$ is infinite. Suppose now that the set $\Pi(L)$ is finite. Then it includes a prime p such that the Sylow p-subgroup T_p of T is not a Chernikov group. By Lemma 1.5, the subgroup T_p is a Dedekind group. If $p \neq 2$, then T_p is abelian. If $p = 2$, then either T_p is abelian or $T_p = Q \times B$ where Q is a quaternion group and B is abelian. In both cases, T_p includes the G-invariant elementary abelian p-subgroup

A. We have $A = \mathbf{Dr}_{\lambda\in\Lambda}A_\lambda$, where$A_\lambda = \langle a_\lambda\rangle$ is a cyclic subgroup of order $p, \lambda \in \Lambda$. By Lemma 1.1, every subgroup A_λ is G-invariant. Thus in every case, L includes the subgroup $D = \mathbf{Dr}_{\lambda\in\Lambda}D_\lambda$, where D_λ is a non-identity G-invariant subgroup for each $\lambda \in \Lambda$ and F the index set Λ is infinite. Let g be an arbitrary element of L and F be an arbitraty finitely generated subgroup of L, which contains g. Since F is finitely generated and nilpotent, it satisfies the maximal condition. It follows that $F \cap D \leq \mathbf{Dr}_{\lambda\in K}D_\lambda$ for some finite subset K. Then $F \cap \mathbf{Dr}_{\lambda\in\Lambda\ K}D_\lambda = \langle 1\rangle$. Since the subset $\Lambda\ K$ is infinite, the subgroup $\langle g\rangle(\mathbf{Dr}_{\lambda\in\Lambda\ K}D_\lambda)$ is non-finitely generated. It follows that $\langle g\rangle(\mathbf{Dr}_{\lambda\in\Lambda\ K}D_\lambda)$ is transitively normal in G, in particular, it is transitively normal in $F(\mathbf{Dr}_{\lambda\in\Lambda\ K}D_\lambda)$. On the other hand, $F \cong F/(F \cap \mathbf{Dr}_{\lambda\in\Lambda\setminus K}D_\lambda) \cong F(\mathbf{Dr}_{\lambda\in\Lambda\setminus K}D_\lambda)/(\mathbf{Dr}_{\lambda\in\Lambda\setminus K}D_\lambda)$. From this isomorphism we obtain that $\langle g\rangle$ is transitively normal in F. Since F is nilpotent, **Lemma 1.2**, shows that $\langle g\rangle$ is normal in F. Since it is true for every finitely generated subgroup of L, $\langle g\rangle$ is normal in L. It follows that L is a Dedekind group. □

2.4. Lemma. *Let G be a nilpotent torsion-free group whose non-finitely generated subgroups are transitively normal. If its center $\zeta(G)$ is non-finitely generated, then G is abelian.*

Proof. Put $Z = \zeta(G)$. Choose in Z a free abelian subgroup C such that Z/C is periodic. If $\mathbf{r}_0(C)$ is infinite, then G is a Dedekind group by Corollary 2.2. Suppose now that C is finitely generated. Since G is torsion-free, G/Z is also torsion-free (see, for example, [20, § 66]). Put $D_n = C^n, n \in N$. If $\Pi(Z/C)$ is infinite, then $\Pi(Z/D_n)$ is also infinite for each $n \in N$. Lemma 2.3 shows that in this case, the factor-group G/D_n is a Dedekind group, $n \in N$. Since G/Z is torsion-free, G/D_n is not periodic and, being a Dedekind group, is abelian. It is true for each $n \in N$, so that $[G, G] \leq D_n$ for each $n \in N$. Hence $[G, G] \leq \cap_{n\in N}D_n = \langle 1\rangle$.

Assume now that a set $\Pi(Z/C)$ is finite. Let $\pi = P\setminus(Z/C)$ (here P is the set of all prime numbers). Then π is infinite. Since C is free abelian, $\cap_{p\in\pi}D_p = \langle 1\rangle$. By the choice of $\pi, C/D_p$ is a Sylow p-subgroup of Z/D_p. It follows that $Z/D_p = C/Dp \times Q_p/D_p$ where Q_p/D_p is a Sylow p'-subgroup of Z/D_p. In particular, $C \cap Q_p = D_p$. The finiteness of C/D_p implies that the subgroup Q_p is non-finitely generated. Since G is nilpotent, the subgroup $\langle g, Q_p\rangle$ is non-finitely generated. Hence $\langle g, Q_p\rangle$ is transitively normal in G. We recall that in a nilpotent group every subgroup is subnormal. Hence $\langle g, Q_p\rangle$ being transitively normal and subnormal in G is normal in G [1, Lemma 1.1]. In other words, every cyclic subgroup of G/Q_p is normal. It follows that G/Q_p is a Dedekind group. Since it is not periodic, it is abelian. This is true for each $p \in \pi$, so that $[G, G] \leq D_p$ for each $p \in \pi$. Hence $[G, G] \leq \cap_{p\in\pi} Q_p = Q$. We have

$$Q \cap C = (\cap_{p\in\pi} Q_p) \cap C = \cap_{p\in\pi}(Q_p \cap C) = \cap_{p\in\pi} D_p = \langle 1 \rangle.$$

Thus $Q \cong Q/(Q \cap C) \cong QC/C \leq Z/C$. We remark that Z/C is a periodic group, so that Q must be periodic. On the other hand, G is torsion-free, hence every its periodic subgroup is identity. Consequently $Q = \langle 1 \rangle$, which follows that $[G, G] = \langle 1 \rangle$ and G is abelian. □

2.5. Proposition. *Let G be a nilpotent torsion-free group whose non-finitely generated subgroups are transitively normal. If G is non-finitely generated, then G is abelian.*

Proof. Suppose that G has an infinite 0-rank. Then G includes a free abelian subgroup A having infinite 0-rank [21, Theorem 5]. Since G is nilpotent, A is subnormal in G. Then Corollary 2.2 implies that G is a Dedekind group. Being not periodic, G is abelian. Assume now that G has finite 0-rank. If we suppose that $\zeta(G)$ is finitely generated, then G is finitely generated [22, Corollary 1 of Lemma 2.6]. Thus $\zeta(G)$ is non-finitely generated, and we can apply Lemma 2.4. □

2.6. Proposition. *Let G be a locally nilpotent torsion-free group and A a free abelian subgroup of infinite countable 0-rank. If F is a finitely generated subgroup of G, then either $\langle F, A\rangle$ includes the F-invariant subgroups K, L such that $K \leq L, K$ of finite 0-rank and L/K is an abelian torsion-free group of infinite 0-rank, or $\langle F, A\rangle$ has an infinite central series*

$$\langle 1 \rangle = D_0 \leq D_1 \leq D_2 \leq \ldots \leq D_n \leq D_{n+1} \leq \ldots$$

whose factors are torsion free and of finite 0-rank.

Proof. We have $A = \mathbf{Dr}_{n\in N} A_n$, where $A_n = \langle a_n \rangle$ is an infinite cyclic subgroup, $n \in N$. Put $L_n = \langle F, a_1, \ldots, a_n\rangle$, $n \in N$. By this choice, L_n is an infinite finitely generated subgroup for each $n \in N$. We remark that the center $Z_n = \zeta(L_n)$ is a pure subgroup of L_n for each $n \in N$ (see, for example, [20, § 66, 67]). The inclusion $F \leq L_n$ implies that Z_n is an F-invariant subgroup for each $n \in N$. The subgroup Z_1 is abelian, therefore Z_1Z_2 is also abelian. Since L_2/Z_2 is torsion-free, then either $Z_1 \leq Z_2$ or $\mathbf{r}_0(Z_1Z_2)\rangle\mathbf{r}_0(Z_)1)$. By the same reason a product

$(Z_1Z_2)Z_3$ is abelian and either $(Z_1Z_2) \leq Z_3$ or $\mathbf{r}_0(Z_1Z_2Z_3)\rangle\mathbf{r}_0(Z_1Z_2)$. Using the similarly arguments, we obtain that the subgroups $Z_1 \dots Z_n$ are abelian for each $n \in N$. Then and its union $C_1 = \cup_{n\in N} Z_1 \dots Z_n$ is abelian. We note that every subgroup $Z_1 \dots Z_n$ is F-invariant for each $n \in N$, therefore C_1 is also F-invariant. We have the following two possibilities: C_1 has an infinite 0-rank, or $\mathbf{r}_0(C_1)$ is finite. In the first case, put $K = \langle 1 \rangle, L = C_1$. In the second case, there exists a positive integer t such that $\mathbf{r}_0(Z_1 \dots Z_t) = \mathbf{r}_0(Z_1 \dots Z_{t+n})$ for all $n \in N$. As we have seen above, this means that $Z_1 \dots Z_{t-1} \leq Z_t \leq Z_{t+1} \leq \dots$. It follows that $C_1 \leq \zeta(\langle F, A\rangle)$, in particular, $\zeta(\langle F, A\rangle) = D_1 \neq \langle 1 \rangle$. We remark that $\mathbf{r}_0(D_1) = \mathbf{r}_0(C_1)$. In this case, we consider the factor-group $\langle F, A\rangle/D_1$. This factor-group is torsion-free (see, for example, [20, § 66, 67]), therefore we can repeat all previous arguments. Then either $\langle F, A\rangle/D_1$ includes an F-invariant abelian subgroup C_2/D_1 having infinite 0-rank or $D_2/D_1 = \zeta(\langle F, A\rangle/D_1)$ is a non-identity subgroup having finite 0-rank. In the first case, put $K = D_1, L = C_2$. In the second case, we will consider the factor-group $\langle F, A\rangle/D_2$. Similar arguments prove the result. □

2.7. Proposition. *Let G be a locally nilpotent torsion-free group whose non-finitely generated subgroups are transitively normal. If G is non-finitely generated, then G is abelian.*

Proof. Suppose that G has finite 0-rank. Then G is nilpotent [21, Theorem 5], and we can apply Proposition 2.5. Suppose now that G has infinite 0-rank. Then G includes a free abelian subgroup A having infinite countable 0-rank [21, Theorem 5]. Let F be an arbitrary finitely generated subgroup of G. Consider the subgroup $\langle F, A\rangle$. Proposition 2.6 shows that either $\langle F, A\rangle$ includes F-invariant subgroups K, L such that $K \leq L, K$ has finite 0-rank and L/K is an abelian torsion-free group of infinite 0-rank, or $\langle F, A\rangle$ has an infinite central series $\langle 1 \rangle = D_0 \leq D_1 \leq D_2 \leq \dots \leq D_n \leq D_{n+1} \leq \dots$, whose factors are torsion free and have finite 0-rank.

Consider the first case. Put $S = \langle F, L\rangle$. There exists a hypercenter C of S with a natural number including the subgroup K [23]. By Lemma 2.1, S/K is a Dedekind group, in particular, it is nilpotent. Then S is likewise nilpotent. Proposition 2.5 shows that S is abelian. It follows that F is abelian.

Consider now the second case. Put $D = \cup_{n\in N} D_n$ and $V = \langle F, D\rangle$. Since $V/D = FD/D \cong F/(F \cap D)$ is nilpotent, V is hypercentral. By the construction of $D, \mathbf{r}_0(D)$ is infinite. Then V includes a free abelian subgroup B having infinite countable 0-rank [21, Theorem 5]. We recall that the subgroup V satisfies the normalizer condition (see, for example, [20, § 63]). It follows that the subgroup B is ascending. Using Corollary 2.2, we obtain that V is a Dedekind group. Being non-periodic, V is abelian; in particular, its subgroup F is abelian.

Consequently, every finitely generated subgroup of G is abelian. It follows that G is abelian. □

Proof of theorem B

Let $T = \mathbf{Tor}(G)$. If T is a non-Chernikov subgroup, then Lemma 2.3 shows that G is a Dedekind group. In particular, every subgroup of G, which is non-finitely generated, is normal in G. Suppose now that T is a Chernikov subgroup. Since G is locally nilpotent, some hypercenter of G includes T. By Proposition 2.7, G/T is abelian or finitely generated. In the second case G/T is nilpotent. Hence in both cases G is hypercentral. We note that a hypercentral group satisfies the normalizer condition (see, for example, [20, § 63]). In other words, every subgroup of G is ascendant. In particular, every subgroup K of G, which is non-finitely generated, is ascendant in G. Being transitively normal in G, K is normal in G [1, Lemma 1.1]. □

3. The radical groups whose non-finitely generated subgroups are transitively normal

Following A.I. Maltsev [21] we say that a group G is a **soluble A_3-group** if G has a finite subnormal series $\langle 1 \rangle = K_0 \triangleleft K_1 \triangleleft \ldots \triangleleft K_n = G$ whose factors either are abelian Chernikov group or torsion-free abelian group of finite 0-rank.

3.1. Lemma. *Let G be a group whose non-finitely generated subgroups are transitively normal. Suppose that L is a locally nilpotent radical of G. If L is a non-A_3-group, then every subgroup of L is G-invariant.*

Proof. Taking into account the results of papers [10, 11, 12] we can see that Theorem 2.8 implies that L is a Dedekind group. If L is not periodic, then L is abelian. If L is periodic, then L is nilpotent and abelian-by-finite. Hence in every case, L includes a subgroup $D = \mathbf{Dr}_{\lambda\in\Lambda} D_\lambda$ where D is a non-identity cyclic subgroup for each and the index set is infinite. Let g be an arbitrary element of L. Then $\langle g\rangle \cap D \leq \mathbf{Dr}_{\lambda\in K} D_\lambda$ for some finite subset K, so that $\langle g\rangle \cap \mathbf{Dr}_{\lambda\in\Lambda\setminus K} D_\lambda = \langle 1\rangle$. Since subset $\Lambda\setminus K$ is infinite, it includes two infinite subsets Θ, Ξ such that $\Lambda\setminus K = \Xi \cup \Theta$ and $\Xi \cap \Theta = \varnothing$.

Let $U = \mathbf{Dr}_{\lambda\in\Theta} D_\lambda$ and $V = \mathbf{Dr}_{\lambda\in\Xi} D_\lambda$, then $U \cap V = \langle 1\rangle$. Clearly the subgroup $\langle g, U\rangle$ (respectively $\langle g, V\rangle$) is non-finitely generated, therefore $\langle g, U\rangle$ (respectively $\langle g, V\rangle$) is transitively normal in G. Being subnormal in G, $\langle g, U\rangle$ (respectively $\langle g, V\rangle$) is normal in G. Then $\langle g\rangle = \langle g\rangle U \cap \langle g\rangle V$ is normal in G. Thus every cyclic subgroup of L is normal in G. This means that every subgroup of L is G-invariant. □

3.2. Corollary. *Let G be a radical group whose non-finitely generated subgroups are transitively normal. Suppose that L is a locally nilpotent radical of G. If L is a non-A_3-group, then G/L is abelian.*

Proof. By Lemma 3.1 every subgroup of L is G-invariant. Then $G/C_G(L)$ is abelian (see, for example [24, Theorem 1.5.1]). We note that in a radical group, the locally nilpotent radical includes its centralizer [9, Lemma 4]. It follows that G/L is abelian. □

3.3. Proposition. *Let G be a radical group whose non-finitely generated subgroups are transitively normal. Suppose that G is not periodic. If the locally nilpotent radical of G is a non-A_3-group, then G is abelian.*

Proof. Let L be the locally nilpotent radical of G. By Lemma 3.1, every subgroup of L is G-invariant, and Corollary 3.2 shows that G/L is abelian. It follows that G is an FC-nilpotent group. Then every finitely generated subgroup of G is nilpotent-by-finite [25, Corollary 3.20], in particular, it satisfies the maximal condition for all subgroups. As in the proof of Lemma 3.1 we can obtain that L includes the subgroup $D = \mathbf{Dr}_{n\in N} D_n$, where D_n is a non-identity cyclic subgroup for each $n \in N$. Since G is non-periodic, G contains an element g of infinite order. Let F be an arbitrary finitely generated subgroup of G containing element g. There exists a positive integer k such that $\langle g\rangle \cap D \le \mathbf{Dr}_{1\le n\le k} D_n$. Put $K_j = \mathbf{Dr}_{n\ge k+j+1} D_n$, $j \in N$, so that $\langle g\rangle \cap K_j = \langle 1\rangle$ for each $j \in N$. Furthermore, $\cap_{j\in N} K_j = \langle 1\rangle$. Consider the subgroup FK_j. If we suppose that it is finitely generated, then by proved above it satisfies the maximal condition for all subgroups. In particular, its subgroup K_j satisfies the maximal condition for all subgroups. On the other hand, $K_j = \mathbf{Dr}_{n\ge k+j+1} D_n$, and we obtain a contradiction. This contradiction shows that FK_j cannot be finitely generated. Then FK_j is transitively normal in G. By Lemma 3.1 the subgroup K_j is normal in G, so that FK_j/K_j is transitively normal in G/K_j. Thus every finitely generated subgroup of G/K_j is transitively normal in G/K_j. Furthermore, every subgroup of G/K_j, which is non-finitely generated, is transitively normal in G/K_j. Hence each subgroup of

G/K_j is transitively normal in G/K_j. In other words, G/K_j is a $\overline{T}$-group. The equation $\langle g\rangle \cap K_j = \langle 1\rangle$ shows that G/K_j is non-periodic. Then G/K_j is abelian [5, Theorem 6.1.1]. It follows that $[G,G] \le K_j$. Since it is true for each $j \in N, [G,G] \le \cap_{j\in N} K_j = \langle 1\rangle$ and G is abelian. □

Proof of theorem C

Let L be a locally nilpotent radical of G. If G is a non-A3-group, then G is abelian by Proposition 3.3. Therefore suppose that L is a soluble A_3-group. Let $T = \mathbf{Tor}(G)$, then T is a Chernikov subgroup. Let D be a divisible part of T. Denote by L_1/D the locally nilpotent radical of G/D and put $T_1/D = \mathbf{Tor}(L_1/D)$. Suppose that T_1/D is a not a Chernikov group. Then G/D is abelian by Proposition 3.3. We recall that a periodic automorphism group of a Chernikov group is also Chernikov [26]. It follows that $T_2 = C_G(D) \cap T_1$ is a not a Chernikov group. Since T_2 is nilpotent, $T_2 \le \mathbf{Tor}(L)$, and we obtain a contradiction. This contradiction shows that T_1/D is a Chernikov group. Let D_1/D be a divisible part of T_1/D. Then D_1 is a Chernikov group, and being divisible, it is abelian. It follows that $D_1 \le \mathbf{Tor}(L)$, so that $D_1 = D$. Thus T_1/D is finite.

Since L/T is non-finitely generated, Proposition 2.5 shows that L/T is abelian. Since T/D is finite, L/D is nilpotent. Proposition 2 of the paper [27] shows that L/D includes a normal torsion-free subgroup H/D such that L/H is bounded. In particular, the isomorphisms

$$H/D \cong (H/D)/(H/D \cap T/D) \cong (H/D)(T/D)/(T/D) = \\ = (HT/D)/(T/D) \cong HT/T \le L/T$$

shows that H/D is abelian. Let t be a positive integer such that the orders of elements of L/H divide t, then $(L/D)^t = K/D \le H/D$. We note that K/D is a characteristic subgroup of L/D, so that K/D is G-invariant. The inclusion $K/D \le H/D$ shows that K/D is torsion-free.

Suppose that G/K is periodic. Being radical, it is locally finite. Let F be an arbitrary finitely generated subgroup of G. Then FK/K is finite, so that $F_1 = F\cap K$ has finite index in F. But in this case, the subgroup $F\cap K$ must be finitely generated (see, for example, [13, Corollary 7.2.1]). Then $F_1/(F_1 \cap D)$ is finitely generated and nilpotent. Being finitely generated and abelian-by-nilpotent, F_1 satisfies the maximal condition for normal subgroups [14, Theorem 3]. It follows that its normal abelian subgroup $F_1 \cap D$ is bounded. Being a Chernikov group, it is finite. Hence F_1 is finite-by-nilpotent and F is polycyclic. It follows that the subgroup FD cannot be finitely generated. Then FD is transitively normal in G. In other words, every finitely generated subgroup of G/D is transitively normal. On the other hand, every subgroup, which is non-finitely generated, is also transitively normal. Hence every subgroup of G/D is transitively normal. This means that G/D is a $\overline{T}$-group. Since G/D is not periodic, it is abelian [5, Theorem 6.1.1].

Suppose now that G/K is not periodic. Denote by C/D a free abelian subgroup of K/D such that K/C is periodic. Since L/K is a bounded $\mathbf{A}_3$-group, it is finite. It follows that K/D not minimax, which follows that the set $\Pi(K/C)$ is infinite. Thus $K/C = \mathbf{Dr}_{p\in\Pi(L/T)} K_p/C$ where K_p/C is a Sylow p-subgroup of K/C. Since the set $\Pi(K/C)$ is infinite, it includes two infinite subset Θ, Ξ such that $\Pi(K/C) = \Xi \cup \Theta$ and $\Xi \cap \Theta = \varnothing$. Let $U/C = \mathbf{Dr}_{\lambda\in\Theta} K_p/C$ and $V/C = \mathbf{Dr} K_p/C$, then $U \cap V = C$. Clearly, both subgroups U and V are non-finitely generated, and therefore they are transitively normal in G. Being subnormal in G, the subgroups U and V are normal in G. Then $C = U \cap V$ is normal in G. Let $\{p_n | n \in N\}$ be an infinite set of primes. Put $C_1/D = (C/D)^p 1$, $C_2/D = (C_1/D)^p 2, C_{n+1}/D = (C_n/D)^p n + 1, n \in N$. Then every subgroup C_n is G-invariant, and the set $\Pi(K/C_n)$ is infinite, $n \in N$. Thus the factor-group G/C_n includes the periodic normal abelian subgroup K/C_n such that the set $\Pi(K/C_n)$ is infinite and G/K is not periodic. By Proposition 3.3, G/C_n is abelian, so that $[G, G] \leq C_n$. Since it is true for each $n \in N, [G, G] \leq \cap_{n\in N} C_n = D$. It follows that G/D is abelian. Hence in every case, the factor-group G/D is abelian.

Suppose that D is not quasicyclic. In this case, D includes a quasicyclic p-subgroup Y such that $D \neq Y$. Since Y is divisible, there exists a subgroup Z such that $D = Y \times Z$ [16, Theorem 21.2]. Both subgroups Y, Z are non-finitely generated, and therefore they are transitively normal in G. Being subnormal, Y, Z are normal in G [1, Lemma 1.1]. Let F be an arbitrary finitely generated subgroup of G. Being finitely generated and metabelian, F satisfies the maximal condition for normal subgroups [14, Theorem 3]. It follows that its normal abelian subgroup $F \cap D$ is bounded. Being a Chernikov group, it is finite. Hence F is finite-by-abelian, in particular, F is polycyclic. It follows that FY is non-finitely generated, so that FY/Y is transitively normal in G/Y. We remark that every subgroup, which is non-finitely generated, also is transitively normal. Hence every subgroup of G/Y is transitively normal. This means that G/Y is a $\overline{T}$-group. Since G/Y is non-periodic, it is abelian [5, Theorem 6.1.1]. By the same reasons, G/Z is abelian. Then $[G, G] \leq Y \cap Z = \langle 1 \rangle$, which shows that G is abelian.

Hence, if we suppose that G is non-abelian, then D is quasicyclic. Let $g \in G \backslash D$ and consider the mapping $\Phi_g : d \longrightarrow [g, d], d \in D$. Since G/D is abelian, this mapping is a ZG-endomorphism of D. We have $\mathbf{Ker}(\mathbf{\Phi}_g) = C_D(g)$, $\mathbf{Im}(\mathbf{\Phi}_g) = [g, D]$, so that

$$D/\mathbf{Ker}(\mathbf{\Phi}_g) = D/C_D(g) = D/\mathbf{Ker}(\mathbf{\Phi}_g) \cong ZG\ \mathbf{Im}(\mathbf{\Phi}_g) = [g, D].$$

If $C_D(g)$ is a proper subgroup of D, then it is finite, and the above isomorphisms show that $[g, D]$ must be infinite. Since D is quasicyclic, $[g, D] = D$. Let x be another element of G. Then $g^x = gd$ for some element $d \in D$. The equation $[g, D] = D$ implies that there exists an element $b \in D$ such that $d = [g, b]$. Then $g^x = gd = g[g, b] = gg^{-1}b^{-1}gb = g^b$. It follows that $xb^{-1} \in C_G(g)$. In other words, $G = DC_G(g)$. The intersection $C_G(g) \cap D = C_D(g) = B$ is finite. Being finite-by-abelian and locally nilpotent, B is nilpotent. We have already proved that T/D is finite. It follows that $\mathbf{Tor}(B)$ is finite. Using again Proposition 2 of the paper [27], we obtain that B includes a normal torsion-free subgroup B_1 such that B/B_1 is bounded. Let $\mathbf{m}$ be a positive integer such that the orders of elements of B/B_1 divide $\mathbf{m}$, then $B^m = B_2 \leq B_1$. We note that B^m is a characteristic subgroup of B, which implies that B^m is G-invariant. An inclusion $B_2 \leq B_1$ shows that B_2 is torsion-free. In particular, $B_2 \cap D = \langle 1 \rangle$. Since B/B_2 is bounded $\mathbf{A}_3$-group, it is finite.

Since G/D is abelian, $[G, G] \leq D$, so that $B_2 \cap [G, G] = \langle 1 \rangle$. This means that $B_2 \leq \zeta(G)$. In particular, the subgroup $A = DB_2$ is a normal abelian subgroup of G. Every subgroup of A, which is non-finitely generated, is transitively normal in G. Being subnormal, it is normal in G. Let F be a finitely generated subgroup of A. Then $F \cap D$ is finite, so that $DF/F \cong D/(D \cap F)$ is infinite and hence quasicyclic. Being divisible, DF/F has a complement in abelian subgroup A/F (see, for example, [16, Theorem 21.2]). In other words, there exists a subgroup A_1 such that $A/F = A_1/F \times DF/F$. Since A/D is not minimax, A/DF is also not minimax. It follows that A_1/F is not minimax. Let $T_1/F = \mathbf{Tor}(A_1/F)$. Suppose first that the set $\pi = \Pi(T_1/F)$ is infinite. Thus $T_1/F = \mathbf{Dr}_{p\in\pi} S_p/F$ where S_p/F is a Sylow p-subgroup of T_1/F. Since a set π is infinite, it includes two infinite subset Θ, Ξ such that $\pi = \Xi \cup \Theta$ and $\Xi \cap \Theta = \varnothing$. Let $A_2/F = \mathbf{Dr}_{p\in\Theta} S_p/F$ and $A_3/F = \mathbf{Dr}_{p\in\Xi} S_p/F$, then $A_2 \cap A_3 = F$. Clearly, both subgroups A_2 and A_3 are non-finitely generated, therefore they are transitively normal in G. Being subnormal in G, A_2 and A_3 are normal in G. Then $A_2 \cap A_3 = F$ is normal in G.

Consider now the case when the set $\Pi(T_1/F)$ is finite. Then T_1/F is a Chernikov subgroup, so that $A_1/F = T_1/F \times A_4/F$. We noted that the torsion-free subgroup A_4/F is not minimax. Let E/F be a finitely generated subgroup of A_4/F such that A_4/E is periodic. Let $\{p_n | n \in N\}$ be an infinite set of primes. Put $E_1/F = (E/F)^p 1$. Since the set $\Pi(A_4/E_1)$ is infinite, a Sylow p_1'-subgroup R_1/E_1 of A_4/E_1 is non-finitely generated. Hence it is transitively normal and, being subnormal, is normal. By its construction, $R_1 \cap E = E_1$. Put $E_2/F = (E_1/F)^p 2$. Again the Sylow p_2'-subgroup R_2/E_2 of R_1/E_2 is non-finitely generated. As above it follows that R_2 is normal in G. In a similar way, we define inductively $E_{n+1}/F = (E_n/F)^p n + 1$ and denote by R_{n+1}/E_{n+1} the Sylow p_{n+1}-subgroup of $R_n/E_{n+1}, n \in N$. Then R_{n+1} is normal in G and $R_{n+1} \cap E = E_{n+1}, n \in N$. Since E/F is free abelian, $\cap_{n \in N} E_n/F = \langle 1 \rangle$, that is $\cap_{n \in N} E_n = F$. Put $R = \cap_{n \in N} R_n$. Then $F \leq R$ and

$$R/F \cap E/F = (\cap_{n \in N} R_n/F) \cap E/F = \\ = \cap_{n \in N}(R_n/F \cap E/F) = \cap_{n \in N} E_n/F = \langle 1 \rangle.$$

Since A_4/E is periodic and A_4/F is torsion-free, $R/F = \langle 1 \rangle$ or $R = F$. Since every subgroup R_n is normal in $G, n \in N, F$ is likewise normal in G.

Thus every subgroup of A is G-invariant. Suppose that $G \neq C_G(A)$. Then $|G/C_G(B_2) = 2$ (see, for example [24, Theorem 1.5.7]) and if h is an element such that $G = \langle h \rangle C_G(A)$, then $h^2 \in C_G(A)$ and $a^h = a^{-1}$ for each element $a \in A$. It follows that $[a, h] = a^{-2}$ for each $a \in A$. On the other hand, if $1 \neq a \in B_2$, then the inclusion $B_2 \leq \zeta(G)$ implies that $[a, h] = 1$, so that $a^{-2} = 1$. Thus we obtain a contradiction, because B_2 is torsion-free. This contradiction shows that $G = C_G(A)$. In particular, $A \leq \zeta(G)$ and G is nilpotent. Using Theorem 2.8 we obtain that every subgroup of G, which is non-finitely generated, is normal in G. Since G is not minimax. From the results of papers [10, 11, 12] it follows that G must be abelian. □

References

[1] Kurdachenko L.A., Subbotin I.Ya. *Transitivity of normality and pronormal subgroups* // AMS Special session on infinite groups. October 8-9, 2005, Bard College. "Combinatorial group Theory, discrete groups, and number theory". American Mathematical Society, Contemporary Mathematics, 2006.– 421.– P. 201 – 212.

[2] Ba M.S., Borevich Z.I. *On arrangement of intermediate subgroups* // Rings and Linear Groups.- Kubanskij Univ.: Krasnodar, 1988.– P.14 – 41.

[3] Baer R. *Situation der Untergruppen und Struktur der Gruppe* // S.-B. Heidelberg Akad. – 1933.– Vol.2.– P. 12 – 17.

[4] Gaschütz W. *Gruppen in denen das Normalreilersein transitiv ist* // J. Reine.Angew. Math. – 1957.– Vol.198.– P. 87 – 92.

[5] Robinson, D.J.S. *Groups in which normality is a transitive relation* // Proc. Cambridge Philos. Soc. – 1964.– Vol.60.– P. 21 – 38.

[6] Peng T.A. *Finite groups with pronormal subgroups* // Proc. Amer. Math, Soc. – 1969.– Vol.20.– P. 232 – 234.

[7] Kuzennyi N.F., Subbotin I.Ya. *Locally soluble groups in which all infinite subgroups are pronormal* // Izv. Vyssh. Ucheb. Zaved., Mat. – 1988.– N. 11.– P. 77 – 79.

[8] Ol'shanskii A.Yu. *Geometry of defining relations in groups* // NAUKA: Moskow.– 1989.–447 p.

[9] Plotkin B.I. *Radical groups* // Mat. Sbornik. – 1955.– Vol.37.– P. 507 – 523.

[10] Kurdachenko L.A., Pylaev V.V. *On groups which are dual to Dedekind groups* // Reports of Academy of Sciences of Ukrain. SSR. – 1989.– N. 10.– P. 21 – 22.

[11] Cutolo G. *On groups satisfying the maximal condition on non-normal subgroups* // Rivista Mat. pura et applicata- 1991.– Vol.91.– P. 49 – 59.

[12] Cutolo G., Kurdachenko L.A. *Groups with a maximality condition for some non-normal subgroups* // Geometria Dedicata. – 1995.– Vol.55.– P. 279 – 292.

[13] Hall M. *The theory of groups* // Macmillan: New York.–1959.–172 p.

[14] Hall P. *Finiteness conditions for soluble groups* // Proc. London Math. Soc. – 1954.– Vol.4.– P. 419 – 436.

[15] Zaitsev D.I. *On locally soluble groups of infinite rank* // Doklady AN USSR. – 1978.– Vol.240, N. 2.– P. 257 – 259.

[16] Fuchs L. *Infinite abelian groups*, Vol. 1 //ACADEMIC PRESS: New York.-1970.-336 p.

[17] Ballester-Bolinches A., Esteban-Romero R. *Sylow permutable subnormal subgroups of finite groups* // Journal Algebra. – 2002.– Vol.251.– P. 727 – 738.

[18] Huppert B. *Zur Sylowstruktur auflösbarer Gruppen* //Archiv Math. – 1961.– Vol.12.– P. 161 – 169.

[19] Ballester-Bolinches A., Kurdachenko L.A., Otal J., Pedraza T. *Infinite groups with many permutable subgroups* // Revista Mat. Iberoamericana. – 2008.– Vol.24, N. 3.– P. 745 – 764.

[20] Kurosh A.G. *The theory of groups* // NAUKA: Moskow.–1967.–272 p.

[21] Maltsev A.I. *On certain classes of infinite soluble groups* // Mat. Sbornik. – 1951.– Vol.28, N. 3.– P. 567 – 588.

[22] Zaitsev D.I., Kurdachenko L.A. and Tushev A.V. *The modules over nilpotent groups of finite rank* // Algebra and logic. - 1985.- Vol.24, N. 6.- P. 412 - 436.

[23] Charin V.S. *On groups possessing ascending invariant series* // Mat. Sbornik. – 2008.– Vol.41, N. 3.– P. 297 – 316.

[24] Schmidt R. *Subgroups lattices of groups.* Walter de Gruyter, Berlin, 1994.

[25] Kurdachenko L.A., Otal J. and Subbotin I.Ya. *Artinian modules over group rings* // BIRKHÄUSER: Basel.–2007.–245 p.

[26] Chernikov S.N. *On periodic automorphism groups of extremal groups* // Math. Nortes. – 1968.– Vol.4, N. 1.– P. 543 – 545.

[27] Heineken H., Kurdachenko L.A. *Groups with subnormality for all subgroups that are non-finitely generated* //Annali di Matematica pura ed applicata. – 1995.– Vol.169.– P. 203 – 232.

Contact information

L.A. Kurdachenko Department of Algebra, National University of Dnepropetrovsk, Vul.Naukova 13, Dnepropetrovsk 50, Ukraine 49050
E-Mail: lkurdachenko@gmail.com

N.N. Semko (Jr.) Department of Mathematics, National University of The State Tax Service of Ukraine, Irpen, Ukraine
E-Mail: n_semko@mail.ru

I.Ya. Subbotin Department of Mathematics and Natural Sciences, National University, 5245 Pacific Concourse Drive, LA, CA 90045, USA
E-Mail: isubboti@nu.edu

Received by the editors: 29.05.2012
and in final form 29.05.2012.