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SURVEY ARTICLE

Local embeddability

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Abstract. For an arbitrary class of algebraic structures we consider a notion of a structure locally embeddable to structures of the class. This generalizes the notion of a group locally embeddable to finite groups studied by Vershik and Gordon. We give various model-theoretic characterizations of such structures. Some of them generalize known group-theoretic results.

Introduction

In [12] a notion of a group locally embeddable to finite groups had been introduced and studied. A group G is called locally embeddable into finite groups (for short, G is a LEF-group) if for any finite subset S of G there are a finite group H and a map $\phi : G \to H$ such that ϕ is injective on S, and $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in S$. It is mentioned in [12] that this notion is a special case of some very general notion in model theory [7]. Apparently, the authors had in mind the following notion considered in [7, §7.2]: an algebraic structure M is said to be locally embeddable into a class of structures $\mathcal{K}$ if any finite reduct of any finite submodel of M is embeddable into a $\mathcal{K}$-structure.

In the present paper we consider, for an arbitrary class of structures $\mathcal{K}$, a notion of a structure locally embeddable into $\mathcal{K}$ (a LE$\mathcal{K}$-structure, for short) in the spirit of the definition of a LEF-group in [12]. This is a version of the above definition from [7] but looks more algebraic because does not

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refer to the relational structure associated with a given algebraic structure. We present a proof that the LE$\mathcal{K}$-structures are exactly the models of the universal theory of the class $\mathcal{K}$, or, equivalently, the substructures of ultraproducts of $\mathcal{K}$-structures. Implicitely, these results are contained in [7]. In fact, the characterization of LE$\mathcal{K}$-structures as models of the universal theory of $\mathcal{K}$ has its roots in Maltsev's idea of using compactness theorem for obtaining local theorems in group theory [6], which was the first application of logic to algebra.

The notions of sofic and hyperlinear groups, which are being actively studied in the last years (see [10, 11]), are based on the same idea, but with ultraproducts of metric structures instead of classical ones.

Under the assumption that there exists a so called universal locally $\mathcal{K}$-structure, we show that the LE$\mathcal{K}$-structures can be characterized as substructures of ultrapowers of locally $\mathcal{K}$-structures. In particular, the LEF-groups are exactly the subgroups of ultrapowers of locally finite groups, and the groups locally embeddable into free groups are exactly the subgroups of ultrapowers of the free group of rank two.

Any locally residually finite group is a LEF-group; for finitely presented groups the converse is also true [8, 12]. We show that these results can be generalized from the subclass of finite groups of the variety of all groups to any subclass of any quasi-variety of algebraic structures.

In [3, 1, 2] a metric space of groups with a fixed set of n generators was considered. We show that this is a special case of a general construction of the Stone topology on the set of ultrafilters of a Boolean algebra. Since for any language L the ultrafilters of the Boolean algebra of all quantifier-free L-formulas in n free variables can be identified with the isomorphism types of marked n-generated L-structures, one can define a Stone topology on the class of these isomorphism types, and this topology is induced by a natural metric.

We show that the marked n-generated models of the universal theory of the class $\mathcal{K}$ are exactly the so called $\mathcal{K}$-limit marked n-generated structures, the members of the closure of the marked n-generated $\mathcal{K}$-structures. Versions of this result are known for the class of finite groups (the n-generated LEF-groups are exactly the limits of n-generated finite groups [12]), and for the class of free groups (the limit groups are exactly the finitely generated subgroups of ultraproducts of free groups [2]).

All definitions and facts from model theory that we use in the paper can be found in [4, 7].

1. Local embeddability into a class of structures

We call an algebraic structure in a language L an L-structure, and members of a class $\mathcal{K}$ of L-structures $\mathcal{K}$-structures.

Let M and N be L-structures, S a subset of M, and L' a sublanguage of L. We call a map $\phi : M \to N$ an (L', S)-embedding of M into N if

- ϕ is injective on S,
- $\phi(c^M) = c^N$,
- $\phi(f^M(a_1, \ldots, a_n)) = f^N(\phi(a_1), \ldots, \phi(a_n))$,
- $(a_1, \ldots, a_n) \in P^M$ iff $(\phi(a_1), \ldots, \phi(a_n)) \in P^N$,

for $a_1, \ldots, a_n \in S$, and any L'-constant symbol c, function L'-symbol f, and relation L'-symbol P of arity $n \geq 1$. Clearly, an (L, M)-embedding of M into N is just an ordinary embedding of M into N.

For a class of L-structures $\mathcal{K}$, an L-structure M is said to be locally embeddable into $\mathcal{K}$ if, for any finite subset S of M and any finite sublanguage L' of L, there is an (L', S)-embedding of M into a $\mathcal{K}$-structure N (depending on L' and S). For short, we call such structures M LE$\mathcal{K}$-structures.

For example, let L be the language of groupoids $\{\cdot\}$, and $\mathcal{K}$ be a class of groupoids. Then a groupoid M is a LE$\mathcal{K}$-groupoid iff for every finite subset S of M there is a groupoid $N \in \mathcal{K}$ and a map $\phi : M \to N$ such that ϕ is injective on S, and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in S$.

Note that if $\mathcal{K}$ is a class of semigroups then any LE$\mathcal{K}$-groupoid M is a semigroup. Indeed, let $a, b, c \in M$. For

$$S = \{a, b, c, ab, bc, (ab)c, a(bc)\}$$

there is an S-embedding ϕ of M into a semigroup N. We have

$$\begin{aligned} \phi(a(bc)) = \phi(ab)\phi(c) &= (\phi(a)\phi(b))\phi(c) \\ &= \phi(a)(\phi(b)\phi(c)) = \phi(a)\phi(bc) = \phi(a(bc)) \end{aligned}$$

in N. Then $(ab)c = a(bc)$ in M because ϕ is injective on S.

However, if $\mathcal{K}$ is a class of groups in the language of grouppoids, a LE$\mathcal{K}$-groupoid can be not a group: an example is any subgroupoid of a $\mathcal{K}$-group which is not a group.

Let $\mathcal{K}$ be a class of groups in the language of groups $L = \{\cdot, ^{-1}, e\}$. Then, by definition, an L-structure G is a LE$\mathcal{K}$-structure iff for every finite subset S of G there is a $\mathcal{K}$-group H and an injective on S map

$\phi : G \to H$ such that $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a^{-1}) = \phi(a)^{-1}$ for all a, $b \in S$, and $\phi(e^G) = e^H$.

In this case G is a group (and therefore in the present context we always consider groups as $\{\cdot, ^{-1}, e\}$-structures but not as $\{\cdot\}$)-structures). As above, in G the operation $\cdot$ is associative. For any $a \in G$ we have $ae^G = e^G a = a$ and $aa^{-1} = a^{-1}a = e^G$ in G. Indeed, for $S = \{a, a^{-1}, e^G\}$ there is an S-embedding ϕ of G into a group H. We have in H

$$\phi(ae^G) = \phi(a)\phi(e^G) = \phi(a)e^H = \phi(a),$$

and so $ae^G = a$ in G because ϕ is injective on S. Similarly, $e^G a = a$ in G. Also, in H

$$\phi(aa^{-1}) = \phi(a)\phi(a^{-1}) = \phi(a)\phi(a)^{-1} = e^H = \phi(e^G),$$

and so $aa^{-1} = e^G$ in G because ϕ is injective on S. Similarly, $a^{-1}a = e^G$.

A finite LE$\mathcal{K}$-structure in a finite language is obviously embeddable into a $\mathcal{K}$-structure. Here is an example of a finite LE$\mathcal{K}$-structure in an infinite language which is not embeddable into a $\mathcal{K}$-structure.

Let L consist of unary functional symbols f_i, where $i < \omega$. Let M be a finite L-structure of size > 1 such that f_i^M is the identity map for all i. For $n < \omega$ let M_n be an L-structure with the same universe, and such that $f_i^{M_n}$ is the identity map iff $i \leq n$. Let $\mathcal{K} = \{M_n : n < \omega\}$. For $L_n = \{f_0, \dots, f_n\}$, the identity map on M is a (L_n, M)-embedding of M into M_n; hence M is a LE$\mathcal{K}$-structure. But, obviously, there is no embedding of M into any of M_n.

2. Model-theoretic characterization of LE$\mathcal{K}$-structures

First order formulas of the form $\forall x_1 \dots x_n \theta$, where θ is a quantifier-free formula, are called universal; formulas of the form $\exists x_1 \dots x_n \theta$ are called existential. A formula without free variables is said to be a sentence. For a class $\mathcal{K}$ of L-structures the set of all sentences that hold in all $\mathcal{K}$-structures is called the theory of $\mathcal{K}$; we denote it by $\mathrm{Th}(\mathcal{K})$. The set of all universal sentences that hold in all $\mathcal{K}$-structures is called the universal theory of $\mathcal{K}$; it is denoted by $\mathrm{Th}_\forall(\mathcal{K})$.

A set of L-formulas Φ with free variables $\{x_i : i \in I\}$ is called satisfiable in an L-structure M if there is a family $\{a_i : i \in I\}$ of elements of M such that all formulas in Φ are true in M when x_i takes the value a_i for all $i \in I$. We say that Φ is satisfiable if it is satisfiable in some L-structure M. The fundamental compactness theorem asserts that a set of formulas is satisfiable if and only if every finite subset of it is satisfiable.

Theorem 1. *Let K be a class of L-structures, and M an L-structure. The following are equivalent:*

(i) *M is a model of* $\mathrm{Th}_\forall(\mathcal{K})$*;*

(ii) *M is a LE$\mathcal{K}$-structure;*

(iii) *M is embeddable into a model of* $\mathrm{Th}(\mathcal{K})$.

Proof. Let M be an L-structure. Let $\Delta(M)$ be the set of all L-formulas in variables x_a, where $a \in M$, of the forms $x_{a_1} \neq x_{a_2}$, $P(x_{a_1}, \ldots, x_{a_n})$, $\neg P(x_{a_1}, \ldots, x_{a_n})$, $c = x_a$, $f(x_{a_1}, \ldots, x_{a_n}) = x_a$ that hold in M when each x_a takes the value a.

For a subset S of M and a sublanguage L' of L, let $\Delta_{L',S}(M)$ be the set of all formulas from $\Delta(M)$ with P, f, $c \in L'$ and $a_i \in S$ (but with a not necessarily in S).

Lemma 1. *$\Delta_{L',S}(M)$ is satisfiable in an L-structure N iff there is an (L', S)-embedding of M into N. In particular, $\Delta(M)$ is satisfiable in N iff M is embeddable into N.*

Proof. For a map $\phi : M \to N$, we say that an L-formula in variables x_a, where $a \in M$, is valid in N if it holds in N when each x_a takes the value $\phi(a)$. Let $\Delta = \Delta_{L',S}(M)$. It suffises to check that $\phi : M \to N$ is an (L', S)-embedding iff Δ holds in N. Injectivity of ϕ on S means validity in N of all formulas $v_{a_1} \neq v_{a_2}$ from Δ. For $P \in L'$ and $a_1, \ldots, a_n \in S$, the property

$$(a_1, \ldots, a_n) \in P^M \Leftrightarrow (\phi(a_1), \ldots, \phi(a_n)) \in P^N$$

means that if $P(v_{a_1}, \ldots, v_{a_n})$ or $\neg P(v_{a_1}, \ldots, v_{a_n})$ belongs to Δ then it is valid in N. For $f \in L'$ and $a_1, \ldots, a_n \in S$, the property

$$\phi(f^M(a_1, \ldots, a_n)) = f^N(\phi(a_1), \ldots, \phi(a_n))$$

means the validity in N of the formula $f(x_{a_1}, \ldots, x_{a_n}) = x_a$, where $a = f^M(a_1, \ldots, a_n)$. For $c \in L'$, the property $\phi(c^M) = c^N$ means the validity in N of the formula $c = x_a$, where $a = c^M$. □

Lemma 2. *Let $\Delta = \Delta_{L',S}(M)$, where S and L' are finite. The following are equivalent:*

(a) $\mathrm{Th}(\mathcal{K}) \cup \Delta$ *is satisfiable;*

(b) $\mathrm{Th}_\forall(\mathcal{K}) \cup \Delta$ *is satisfiable;*

(c) *there is an (L', S)-embedding of M into a $\mathcal{K}$-structure.*

Proof. Let $\bar{x}$ be all variables that occur in Δ. By Lemma 1, (c) holds iff Δ is satisfiable in a $\mathcal{K}$-structure iff $\exists\bar{x} \bigwedge \Delta$ holds in some $\mathcal{K}$-structure iff $\mathrm{Th}(\mathcal{K}) \cup \{\exists\bar{x} \bigwedge \Delta\}$ has a model iff $\mathrm{Th}_\forall(\mathcal{K}) \cup \{\exists\bar{x} \bigwedge \Delta\}$ has a model. The latter iff holds because $\neg\exists\bar{x} \bigwedge \Delta$ is equivalent to a universal formula. □

(i)⇒(ii). If M is a model of $\mathrm{Th}_\forall(\mathcal{K})$ then $\mathrm{Th}_\forall(\mathcal{K}) \cup \Delta_{L',S}(M)$ is obviously satisfiable in M for all finite $S \subseteq M$ and $L' \subseteq L$, and hence M is a LE$\mathcal{K}$-structure, by Lemma 1 and Lemma 2, (b)⇒(c).

(ii)⇒(iii). If M is a LE$\mathcal{K}$-structure then $\mathrm{Th}(\mathcal{K}) \cup \Delta_{L',S}(M)$ is satisfiable for all finite $S \subseteq M$ and $L' \subseteq L$, by Lemma 1 and Lemma 2, (c)⇒(a). Then, by compactness theorem, $\mathrm{Th}(\mathcal{K}) \cup \Delta(M)$ is satisfiable, and hence M is embeddable into a model of $\mathrm{Th}(\mathcal{K})$.

(iii)⇒(i) because validity of universal sentences is preserved in substructures. □

A class $\mathcal{K}$ of L-structures is called axiomatizable if $\mathcal{K}$ is the class of all models of some set of L-sentences, or, equivalently, $\mathcal{K}$ is the class of all models of $\mathrm{Th}(\mathcal{K})$. The class $\mathcal{K}$ is called universally axiomatizable if $\mathcal{K}$ is the class of all models of some set of universal L-sentences, or, equivalently, $\mathcal{K}$ is the class of all models of $\mathrm{Th}_\forall(\mathcal{K})$.

Theorem 1 explains, for example, why if $L = \{\cdot, ^{-1}, e\}$, and $\mathcal{K}$ is any class of groups in the language L, then any LE$\mathcal{K}$-structure is a group. The reason is that the class of all groups in the language L (in contrast with the language $\{\cdot\}$) is universally axiomatizable.

Corollary 1. *If a class $\mathcal{K}$ of L-structures is axiomatizable then the LE$\mathcal{K}$-structures are exactly the substructures of $\mathcal{K}$-structures.*

Now we give another characterization of LE$\mathcal{K}$-structures in terms of ultraproducts.

Let $\{M_i : i \in I\}$ be a family of L-structures, and $\bar{M}$ its cartesian product. For an ultrafilter D on I, elements g and h of $\bar{M}$ are called D-equivalent if $\{i \in I : g(i) = h(i)\} \in D$. This is an equivalence relation on $\bar{M}$; we denote the D-equivalence class of g by g/D. Moreover, it is easy to check that the D-equivalence is a congruence relation on $\bar{M}$; the quotient of $\bar{M}$ modulo this congruence is called the ultraproduct of $\{M_i : i \in I\}$ over D and is denoted by $\prod_{i\in I} M_i/D$. For example, when all M_i are groups, the ultraproduct is nothing but the quotient group $\bar{M}/N_D$, where N_D is the normal subgroup of the elements of $\bar{M}$ that are D-equivalent to the indentity.

The Łoś theorem claims that $\theta(g_1/D, \ldots, g_n/D)$ holds in $\prod_{i \in I} M_i/D$ iff the set $\{i \in I : \theta(g_1(i), \ldots, g_n(i))$ holds in $M_i\}$ belongs to D, for any L-formula $\theta(x_1, \ldots, x_n)$ and $g_1, \ldots, g_n \in \bar{M}$. When θ is an atomic formula of the form $P(x_1, \ldots, x_n)$ or $f(x_1, \ldots, x_{n-1}) = x_n$, this holds just by definition of ultraproduct; for an arbitrary θ it can be proved by induction on the complexity of θ (see [4, 7]).

An immediate consequence of the Łoś theorem is that any ultraproduct of $\mathcal{K}$-structures is a model of $\mathrm{Th}(\mathcal{K})$. Hence any axiomatizable class is closed under ultraproducts.

Theorem 2. *An L-structure is a model of* $\mathrm{Th}_\forall(\mathcal{K})$ *iff it is embeddable into an ultraproduct of $\mathcal{K}$-structures.*

Proof. Since ultraproducts of $\mathcal{K}$-structures are models of $\mathrm{Th}(\mathcal{K})$, and the validity of unversal sentences is preserved in substructures, any substructure of an ultraproduct of $\mathcal{K}$-structures is a model of $\mathrm{Th}_\forall(\mathcal{K})$.

Let M be a model of $\mathrm{Th}_\forall(\mathcal{K})$. Then any finite subset Δ of $\Delta(M)$ is satisfiable in a $\mathcal{K}$-structure. Indeed, suppose not. Let $\bar{x}$ be all variables that occur in Δ. Then $\neg\exists\bar{x} \bigwedge \Delta$ holds in any $\mathcal{K}$-structure. Hence the universal sentence $\forall\bar{x}\neg \bigwedge \Delta$ belongs to $\mathrm{Th}_\forall(\mathcal{K})$, and so holds in M. But $\exists\bar{x} \bigwedge \Delta$ holds in M because $\Delta \subseteq \Delta(M)$. Contradiction.

Thus for any finite subset Δ of $\Delta(M)$ there are a $\mathcal{K}$-structure M_Δ and a map $\phi : M \to M_\Delta$ such that Δ holds in M_Δ when the variable x_a takes the value $\phi_\Delta(a)$ for all $a \in M$.

Denote by I the collection of all finite subsets of $\Delta(M)$. For $\theta \in \Delta(M)$ let I_θ be the set of all $\Delta \in I$ such that θ holds in M_Δ when x_a takes the value $\phi_\Delta(a)$ for all $a \in M$. The set $\{I_\theta : \theta \in \Delta(M)\}$ has the finite intersection property because for any nonempty $\Delta \in I$ we have $\Delta \in \bigcap_{\theta \in \Delta} I_\theta$. Therefore there exists an ultrafilter D on I containing $\{I_\theta : \theta \in \Delta(M)\}$.

Let $N = \prod_{\Delta \in I} M_\Delta/D$. Define $\phi : M \to N$ by the rule

$$\phi(a) = (\phi_\Delta(a) : \Delta \in I)/D.$$

We show that ϕ is an embedding of M into N, that is, $\theta \in \Delta(M)$ holds in N when x_a takes the value $\phi(a)$ for all $a \in M$. By definition of ultraproduct, this holds iff $I_\theta \in D$, which is true by the choice of D. Thus M is embeddable into N which is an ultraproduct of $\mathcal{K}$-structures. □

Thus Theorems 1 and 2 imply

Corollary 2. *Up to isomorphism, the LE$\mathcal{K}$-structures are exactly the substructures of ultraproducts of $\mathcal{K}$-structures. In particular, if $\mathcal{K}$ is closed under ultraproducts then the LE$\mathcal{K}$-structures are exactly the substructures of $\mathcal{K}$-structures.*

For example, the LEF-groups are exactly the subgroups of ultraproducts of finite groups.

It is not difficult to show that the class $\mathcal{K}$ of all linear groups of a fixed degree is closed under ultraproducts; so in this case the LE$\mathcal{K}$-groups are just the $\mathcal{K}$-groups.

Clearly, for an axiomatizable class $\mathcal{K}'$ of L'-structures with $L' \supseteq L$, the class $\mathcal{K}$ of L-reducts of $\mathcal{K}'$-structures is closed under ultraproducts. So Corrolary 2 is applicable to such classes $\mathcal{K}$.

For instance, let $\mathcal{K}$ be the class of orderable groups in the group language L, that is, the class of L-reducts of $\mathcal{K}'$-structures, where $\mathcal{K}'$ is the class of all ordered groups in the language $L' = \{\cdot, ^{-1}, e, <\}$. Clearly, $\mathcal{K}'$ is axiomatizable. Hence a group is a LE$\mathcal{K}$-group iff it is a $\mathcal{K}$-group.

3. Universal locally $\mathcal{K}$-structures

Let $\mathcal{K}$ be a class of L-structures. An L-structure M is called a locally $\mathcal{K}$-structure if every finite subset of M is contained in a $\mathcal{K}$-substructure; when $\mathcal{K}$ is closed under finitely generated substructures this means that any finitely generated substructure of M belongs to $\mathcal{K}$. Clearly, every $\mathcal{K}$-structure is a locally $\mathcal{K}$-structure, for any class $\mathcal{K}$. Obviously, any locally LE$\mathcal{K}$-structure is a LE$\mathcal{K}$-structure.

Since a universal sentence which fails in a locally $\mathcal{K}$-structure fails in some $\mathcal{K}$-structure, we have $\mathrm{Th}_\forall(\mathcal{K}) = \mathrm{Th}_\forall(LE\mathcal{K})$.

A locally $\mathcal{K}$-structure U is called a universal locally $\mathcal{K}$-structure if any finitely generated substructure of any $\mathcal{K}$-structure is embeddable into U. For many classes of structures $\mathcal{K}$ a universal locally $\mathcal{K}$-structure does exist.

For instanse, if a class of groups $\mathcal{K}$ is closed under finite direct products then there exists a universal locally $\mathcal{K}$-group. Indeed, let $\{G_i : i \in I\}$ be a family of K-groups such that any finitely generated subgroup of a $\mathcal{K}$-group is embeddable to one of G_i; then the direct sum of all G_i is a universal locally $\mathcal{K}$-group. Examples of classes of groups closed under finite direct products are any quasi-variety of groups, the classes of finite groups, torsion groups, divisible groups, amenable groups. New examples can be obtained using the observation that the intersection of any family of such classes and the union of a chain of such classes is such a class. For example, by this observation, the classes of finite solvable groups and torsion-free nilpotent groups are such classes.

Also, there are classes of groups $\mathcal{K}$ with a universal locally $\mathcal{K}$-group which are not closed under finite direct products. Examples of such classes $\mathcal{K}$ are the class of free groups (here F_2 is a universal locally $\mathcal{K}$-group) and the class of finite cyclic groups (here $\bigoplus_p \mathbb{C}(p^\infty)$ is a universal locally $\mathcal{K}$-group).

Note that for the class of all cyclic groups there is no universal locally cyclic group because it would contain an infinite cyclic group and a finite cyclic group and so could not be locally cyclic.

We call L-structures M and N elementarily equivalent if they satisfy the same L-sentences. Clearly, M and N are elementarily equivalent iff M is a model of $\mathrm{Th}(N)$. By Theorems 1 and 2 applied to the class $\{N\}$, a structure M is a model of $\mathrm{Th}_\forall(N)$ iff M is embeddable into a structure elementarily equivalent to N iff M is embeddable into an ultrapower of N.

Theorem 3. *Let $\mathcal{K}$ be a class of L-structures such that there exists a universal locally $\mathcal{K}$-structure U. For an L-structure M the following are equivalent:*

(1) *M is a LE$\mathcal{K}$-structure;*

(2) *M is embeddable into a structure elementarily equivalent to a locally $\mathcal{K}$-structure;*

(3) *M is embeddable into an ultrapower of a locally $\mathcal{K}$-structure.*

(4) *M is embeddable into a structure elementarily equivalent to U;*

(5) *M is embeddable into an ultrapower of U.*

Proof. Since, by Theorem 1, M is a LE$\mathcal{K}$-structure iff M is a model of $\mathrm{Th}_\forall(\mathcal{K})$, it suffices to prove the following claims:

(i) $\mathrm{Th}_\forall(U) = \mathrm{Th}_\forall(\mathcal{K})$,

(ii) M is a model of $\mathrm{Th}_\forall(\mathcal{K})$ iff M is a model of $\mathrm{Th}_\forall(N)$ for some locally $\mathcal{K}$-structure N.

A universal sentence which fails in some $\mathcal{K}$-structure fails in some finitely generated substructure of it, and hence fails in U. Therefore

$$\mathrm{Th}_\forall(U) \supseteq \mathrm{Th}_\forall(LE\mathcal{K}) = \mathrm{Th}_\forall(\mathcal{K}) \supseteq \mathrm{Th}_\forall(U).$$

Therefore (i) holds. Now we prove (ii). If M is a model of $\mathrm{Th}_\forall(\mathcal{K})$ then M a model of $\mathrm{Th}_\forall(U)$. Suppose M is a model of $\mathrm{Th}_\forall(N)$, where N is a locally $\mathcal{K}$-structure. Since $\mathrm{Th}_\forall(\mathcal{K}) = \mathrm{Th}_\forall(LE\mathcal{K}) \subseteq \mathrm{Th}_\forall(N)$, the structure M is a model of $\mathrm{Th}_\forall(\mathcal{K})$. □

For example, Theorem 3 says that the LEF-groups can be characterized as the subgroups of ultrapowers of locally finite groups. Since the group $S_\omega(\mathbb{N})$ of all finitary permutations of $\mathbb{N}$ is a universal locally finite group, the LEF-groups are exactly the subgroups of ultrapowers of $S_\omega(\mathbb{N})$. Another application: the groups locally embeddable to free groups are exactly the subgroups of ultrapowers of F_2.

Now we show that although there is no universal locally cyclic group, the conclusion of Theorem 3 still holds for the class of cyclic groups.

Theorem 4. *Let $\mathcal{K}$ be the class of all cyclic groups. Then for a group G the following are equivalent:*

(1) *G is a $LE\mathcal{K}$-group,*

(2) *G is embeddable into a group elementarily equivalent to a locally cyclic group,*

(3) *G is embeddable into an ultrapower of a locally cyclic group.*

Proof. As it was explained above, the equivalence of (2) and (3) is a special case of a general fact. If (2) then G is a model of $\mathrm{Th}_\forall(H)$ for some locally cyclic group H, and hence a model of $\mathrm{Th}_\forall(\mathcal{K})$; therefore (1) holds, by Theorem 1.

Suppose (1); we need to show (2). Clearly, G is abelian. We may assume that $G \neq 0$. Let $\hat{G}$ be the divisible hull of G; then the group $\hat{G}$ is isomorphic to $\bigoplus_p \mathbb{C}(p^\infty)^{(\kappa_p)} \oplus \mathbb{Q}^{(\kappa)}$, for some cardinals κ, κ_p not all of which are equal to 0.

It suffices to show that all $\kappa_p \leq 1$. Indeed, it is known (see [4]) that if A is an abelian group of unbounded exponent then $A \oplus \mathbb{Q}^{(\kappa)}$ is elementarily equivalent to A. If $\kappa_p = 0$ for all p then $\kappa > 0$, and so $\hat{G}$ is isomorphic to $\mathbb{Q}^{(\kappa)}$; hence $\hat{G}$ is elementarily equivalent to the locally cyclic group $\mathbb{Q}$. If some $\kappa_p \neq 0$ then $\hat{G}$ is elementarily equivalent to the group $\bigoplus_p \mathbb{C}(p^\infty)^{(\kappa_p)}$ which is locally cyclic when all $\kappa_p \leq 1$.

For any nontrivial divisible subgroup B of $\hat{G}$ we have $G \cap B \neq 0$. Indeed, otherwise $\hat{G} = B \oplus C$ for some subgroup C containing G. Clearly, C is divisible, and $G \leq C < \hat{G}$, contrary to minimality of the divisible hull. It follows that if $\kappa_p \neq 0$, then G contains a subgroup which is a direct sum of κ_p copies of a cyclic group of order p. If $\kappa_p \geq 2$ then G

contains a finite subgroup A which is a direct sum of two copies of cyclic groups of order p. This A is not embeddable into a $\mathcal{K}$-group, which is impossible because G is a LE$\mathcal{K}$-group. $\square$

Remark. The characterization of LE$\mathcal{K}$-groups given in Theorem 3 is essentially different from the ones in Theorems 1 and 2, because a model of the theory of $\mathcal{K}$ and a model of the theory of a locally $\mathcal{K}$-structure are not the same. To demonstrate this for the class $\mathcal{K}$ of all finite groups, we show that

(i) the additive group of reals $\mathbb{R}$ is a model of the theory of finite groups but is not elementarily equivalent to a locally finite group,

(ii) the group $\mathbb{C}(p^\infty)$ is locally finite but is not a model of the theory of finite groups.

First, $\mathbb{R}$ is torsion-free and so cannot be elementarily equivalent to a locally finite group. Second, $\mathbb{R}$ is isomorphic to an ultraproduct of finite groups. Indeed, let D be a non-principal ultrafilter on the set of all primes, and $G = \prod_{p \in P} \mathbb{Z}_p / D$. It is easy to show that $|G| = 2^{\aleph_0}$. Using the Łoś theorem one can check that G is a torsion-free, divisible abelian group. Therefore $G \simeq \mathbb{R}$. Thus (i) is proven.

Since the first order sentence *"if there is an element of order p then there is a non-p-divisible element"* holds in any finite group but not in $\mathbb{C}(p^\infty)$, we have (ii).

4. Fully residually $\mathcal{K}$-structures

Let $\mathcal{K}$ be a class of L-structures. We call an L-structure M a residually $\mathcal{K}$-structure if for any two different $a, b \in M$ there are a $\mathcal{K}$-structure N and a homomorphism $\tau : M \to N$ such that $\tau(a) \neq \tau(b)$. We call M a fully residually $\mathcal{K}$-structure if for any finite subset S of M there are a $\mathcal{K}$-structure N and a homomorphism $\tau : M \to N$ which is injective on S.

Clearly, any fully residually $\mathcal{K}$-structure is a residually $\mathcal{K}$-structure.

Theorem 5. *If $\mathcal{K}$ is closed under finite direct products then any residually $\mathcal{K}$-structure is a fully residually $\mathcal{K}$-structure.*

Proof. Let M be a residually-$\mathcal{K}$ structure, and S a finite subset of M. For any different $a, b \in M$ choose $N_{\{a,b\}} \in \mathcal{K}$ and a homomorphism $\tau_{\{a,b\}} : M \to N_{\{a,b\}}$ injective on $\{a, b\}$. Let N be the direct product of all $N_{\{a,b\}}$. Since $\mathcal{K}$ is closed under finite direct products, $N \in \mathcal{K}$. Let

$\tau : M \to N$ be defined by the rule $\tau(x)_{\{a,b\}} = \tau_{\{a,b\}}(x)$. Then τ is a homomorphism which is injective on S. □

Obviously, any locally fully residually $\mathcal{K}$-structure is a LE$\mathcal{K}$-structure. In general, the converse is not true: for example, there exist finitely generated LEF-froups which are not residually finite [9].

Let $\mathcal{K}$ be any of the classes of finite groups, nilpotent groups, amenable groups. Then $\mathcal{K}$ is closed under finite direct products. Since any nonabelian free group F is a residually $\mathcal{K}$-group, it is a fully residually $\mathcal{K}$-group, and hence a LE$\mathcal{K}$-group but not a locally $\mathcal{K}$-group. Since F is not embeddable into a $\mathcal{K}$-group, this, together with Corollary 1, proves that these classes are not axiomatizable.

It was proven in [8, 12] that a finitely presented group is a LEF-group iff it is residually finite. The following result generalizes that fact.

Theorem 6. *Let $\mathcal{V}$ be a quasi-variety of L-structures, and M a finitely presented member of $\mathcal{V}$. Let $\mathcal{K}$ be a subclass of $\mathcal{V}$. Then M is a LE$\mathcal{K}$-structure iff M is a fully residually $\mathcal{K}$-structure iff M is a residually $\mathcal{K}$-structure.*

Proof. It suffices to show that if M is a LE$\mathcal{K}$-structure then M is a fully residually $\mathcal{K}$-structure. Let $\langle \bar{c}\,;\, \rho_1(\bar{c}), \ldots, \rho_m(\bar{c})\rangle$ be a finite presentation of M in $\mathcal{V}$, where ρ_i are atomic L-formulas. Let S be a finite subset of M of size n. Then $S = \{t_i^M(\bar{c}) : 1 \le i \le n\}$ for some L-terms $t_i(\bar{x})$. Since the sentence

$$\phi := \exists \bar{x}(\bigwedge_j \rho_j(\bar{x}) \wedge \bigwedge_{i \neq k} t_i(\bar{x}) \neq t_k(\bar{x}))$$

holds in M, and M is a model of the universal theory of $\mathcal{K}$ by Theorem 1, ϕ holds in some $\mathcal{K}$-structure N. Let $\bar{b}$ be a tuple in N such that $\rho_j(\bar{b})$ holds in N for all j, and all $t_i^N(\bar{b})$ are distinct. Since $N \in \mathcal{V}$ and $\rho_j(\bar{b})$ holds in N for all j, there is a homomorphism $\tau : M \to N$ which takes $\bar{c}$ to $\bar{b}$. Then τ takes $t_i^M(\bar{c})$ to $t_i^N(\bar{b})$ for all i. Since all $t_i^N(\bar{b})$ are distinct, τ is injective on S. □

5. $\mathcal{K}$-limit structures

Each Boolean algebra B has an associated topological space called the Stone space of B and denoted by $\mathrm{St}(B)$ (see [5]). Its points are the ultrafilters on B, and a base of its topology consists of the sets

$$U_b = \{p \in \mathrm{St}(B) : b \in p\},$$

where $b \in B$. Clearly, the complement of U_b is $U_{\bar{b}}$; hence the sets U_b are clopen. The topological space $\mathrm{St}(B)$ is compact and totally disconnected.

Let QF_n denote the set of all quantifier-free L-formulas in variables $\bar{x} = (x_1, \ldots, x_n)$. For an L-structure N and a tuple $\bar{a} = (a_1, \ldots, a_n)$ in N, the set of all formulas in QF_n such that $\theta(\bar{a})$ holds in N is called the quantifier-free type of the tuple $\bar{a}$ in N, and denoted by $\mathrm{qftp}_N(\bar{a})$. It is easy to see that $\mathrm{qftp}_N(\bar{a})$ is a maximal finitely satisfiable subset of QF_n. Moreover, the compactness theorem implies that any maximal finitely satisfiable subset q of QF_n is the quantifier-free type of an n-tuple $\bar{a}$ in an L-structure N; we say that $\bar{a}$ realizes q in N. We denote the set of all quantifier-free types of n-tuples in L-structures by $\mathcal{Q}_n$.

Clearly, $\mathrm{qftp}_N(\bar{a}) = \mathrm{qftp}_{\langle \bar{a} \rangle}(\bar{a})$, where $\langle a \rangle$ denotes the substructure generated by $\bar{a}$. For n-tuples $\bar{a}$ and $\bar{b}$ in L-structures N and M, we have $\mathrm{qftp}_N(\bar{a}) = \mathrm{qftp}_M(\bar{b})$ if and only if there is an isomorphism between $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ which takes $\bar{a}$ to $\bar{b}$. Therefore the quantifier-free types of n-tuples in L-structures can be identified with the isomorphism types of pairs $(N, \bar{a})$, where N is an L-structure generated by $\bar{a}$. We call such pairs marked (or, more precisely, n-marked) L-structures. This generalizes the notion of a marked group (see [2]); so the quantifier-free types of tuples in groups are nothing but the isomorphism types of marked groups.

For a formula θ in QF_n let $[\theta]$ denote the equivalence class of θ. Let $[QF_n]$ be the Boolean algebra of equivalence classes of formulas in QF_n with the operations induced by $\wedge, \vee, \neg$. It is easy to see that the map $p \mapsto \{\theta : [\theta] \in p\}$ is a bijection from $\mathrm{St}([QF_n])$ to $\mathcal{Q}_n$. This bijection induces a topology on $\mathcal{Q}_n$, a clopen base of which is the set of all $U_\theta = \{q \in \mathcal{Q}_n : \theta \in q\}$, where $\theta \in QF_n$. Clearly, the set of all U_θ with atomic or negated atomic θ is a subbase of the topology. We call this topology the Stone topology on $\mathcal{Q}_n$. Since the elements of $\mathcal{Q}_n$ can be identified with n-marked L-structures, the Stone topology can be considered as a topology on the set of n-marked L-structures.

Define an ultrametric d on $\mathcal{Q}_n$ as follows: $d(p,p) = 0$, and if $p \neq q$ then $d(p,q) = 1/v(p,q)$, where $v(p,q)$ is the maximal m such that $\theta \in p$ iff $\theta \in q$ for all $\theta \in QF_n$ of length $< m$. Note that $v(p,q)$ is the maximal m such that $\theta \in p$ iff $\theta \in q$ for all atomic $\theta \in QF_n$ of length $< m$.

Theorem 7. *If L is finite then the Stone topology on $\mathcal{Q}_n$ is induced by the metric d.*

Proof. First we show that for any point p and $m \geq 1$ the open ball $B(p, 1/m)$ is open in the Stone topology; then any d-open set is open in the Stone topology. We have $B(p, 1/m) = \{q : v(q,p) > m\}$, and the condition $v(q,p) > m$ means that $\theta \in q$ iff $\theta \in p$ for any $\theta \in QF_n$ of the length $\le m$. Therefore $B(p, 1/m)$ is the intersection of all U_θ and all $U_{\neg\psi}$ such that $\theta, \neg\psi \in p$ and θ, ψ are of length $\le m$. Since L is finite, there are only finitely many formulas in QF_n of length $\le m$, and so this is a finite intersection. Hence $B(p, 1/m)$ is open in the Stone topology.

To prove that any subset of $\mathcal{Q}_n$ which is open in the Stone topology is d-open, it suffices to show that for any consistent $\theta \in QF_n$ the set U_θ is d-open. Let $p \in U_\theta$. If the length of θ is m, then $B(p, 1/m) \subseteq U_\theta$. Indeed, if $q \in B(p, 1/m)$ then $v(q,p) > m$, and hence $q \in U_\theta$, because $\theta \in q$ iff $\theta \in p$. □

Let $\mathcal{K}$ be a class of L-structures. We call an n-marked L-structure $(N, \bar{a})$ an n-marked $\mathcal{K}$-structure if $N \in \mathcal{K}$. Similarly, $(N, \bar{a})$ is said to be a model of a theory T if N is a model of T. We call any member of the closure of the set of n-marked K-structures in the Stone topology a $\mathcal{K}$-limit n-marked L-structure.

Theorem 8. *Let $\mathcal{K}$ be a class of L-structures closed under substructures. Then an n-marked L-structure is $\mathcal{K}$-limit iff it is a model of the universal theory of $\mathcal{K}$.*

Proof. Let $(N, \bar{a})$ be an n-marked structure which is a model of $\mathrm{Th}_\forall(\mathcal{K})$; we show that $(N, \bar{a})$ is $\mathcal{K}$-limit. Denote $q = \mathrm{qftp}_N(\bar{a})$. Suppose $q \in U_\theta$, where $\theta \in QF_n$. Then $\exists \bar{x}\theta(\bar{x})$ holds in N, and hence in some $\mathcal{K}$-structure M. Let $\theta(\bar{b})$ holds in M. Then $\langle \bar{b} \rangle \in \mathcal{K}$, and $\mathrm{qfpt}_{\langle \bar{b} \rangle}(\bar{b})$ belongs to U_θ.

Suppose $(N, \bar{a})$ is $\mathcal{K}$-limit; we show that N is a model of the universal theory of $\mathcal{K}$. The latter means that, whenever an existential L-sentence ψ holds in N, it holds in some $\mathcal{K}$-structure. Let ψ be $\exists y_1 \ldots y_k\, \theta(y_1, \ldots, y_k)$, where θ is quantifier-free. Since $\bar{a}$ generates N, there are L-terms $t_i(\bar{x})$ such that $\rho(\bar{a})$ holds in N, where ρ is $\theta(t_1(\bar{x}), \ldots, t_k(\bar{x}))$. Since $(N, \bar{a})$ is $\mathcal{K}$-limit, and $\rho(\bar{a})$ holds in N, there is an n-marked $\mathcal{K}$-structure $(M, \bar{b})$ such that $\rho(\bar{b})$ holds in M. Then ψ holds in the $\mathcal{K}$-structure M. □

If $\mathcal{K}$ is the class of finite groups or the class of free groups, we have that the marked n-generated LE$\mathcal{K}$-groups are exactly the K-limit marked n-generated groups. These results were proven in [12] for the finite groups, and in [2] for the free groups.

Corollary. *If $\mathcal{K}$ is universally axiomatizable then the n-marked $\mathcal{K}$-limit structures are exactly the n-marked LE$\mathcal{K}$-structures.*

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