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RESEARCH ARTICLE

Groups with many pronormal and transitively normal subgroups

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Abstract. A subgroup H of a group G is said to be transitively normal in G, if H is normal in every subgroup $K \geq H$ such that H is subnormal in K. The study of radical groups, whose not finitely generated subgroups are transitively normal, has been started by L. A. Kurdachenko, N. N. Semko (Jr.), I. Ya. Subbotin. In this paper the study of such groups is continued.

Introduction

We say that a subgroup H of a group G is *pronormal in* G if for each element $g \in G$ there exists an element $u \in < H, H^g >$ such that $H^g = H^u$. Pronormal subgroups naturally appeared in the process of investigation of such important subgroups of finite (soluble) group as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups. The term "pronormal subgroup" belongs to P. Hall. Pronormal subgroups have the following essential property. Let G be an arbitrary group and K is a pronormal subgroup of G. If L is a subgroup of G such that $K \leq L$ and K is subnormal in L, then K is normal in L. Not only pronormal subgroups have this property, some generalization of pronormal subgroups have it also.

Recently in the paper [1] has been introduced the concept of transitively normal subgroup. A subgroup H of a group G is said to be

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transitively normal in G, if H is normal in every subgroup $K \geq H$ such that H is subnormal in K. As we can see, every pronormal subgroup is transitively normal, but converse is not true. Indeed, every self-normalizing subgroup H (that is $H = N_G(H)$) is transitively normal. On the other hand, a subgroup, which is pronormal and self-normalizing, is abnormal (see, for example, [2, §6, Theorem 7]. Recall that a subgroup A of a group G is called *abnormal in* G, if $x \in < A, A^x >$ for each element $x \in G$. We noted that every subgroup, including an abnormal subgroup, is self-normalizing (see, for example, [2, §6, Theorem 1], but not every self-normalizing subgroup has this property.

The property "to be transitively normal subgroups" is connected with another important property. A group G is said to to be a T-group if every subnormal subgroup of G is normal. In other words, G is a *T-group* if in G the property "to be normal subgroup" is transitive. Trivial examples of *T-groups* are first abelian groups and second groups whose subgroups are normal or *Dedekind groups*. R. Baer [3] proved that every Dedekind group G either is abelian or $G = Q \times D \times B$ where Q is a quaternion group, D is an elementary abelian 2-subgroup and B is a periodic abelian subgroup such that $2 \notin \Pi(B)$. The structure of finite soluble *T-groups* is due to W. Gaschutz [3]. We noted that in general not every subgroup of T-group is a T-group itself, so we come to the following type of groups. A group G is said to be a *$\overline{T}$-group*, if every subgroup of G is a T-group. Thus we can see that G is a *$\overline{T}$-group* if and only if every subgroup of G is transitively normal in G. W. Gaschutz himself proved that every finite soluble *T-group* is a *$\overline{T}$-group* although this is no longer true for infinite soluble groups. Infinite soluble T-groups and *$\overline{T}$-groups* have been studied by D.J.S. Robinson [5].

We observed that *$\overline{T}$-groups* have many pronormal subgroups, every finitely generated subgroup of a *$\overline{T}$-group* is pronormal. Converse is also true. T.A. Peng proved [6] that a finite group G is a *$\overline{T}$-group* if and only if every cyclic subgroup of G is pronormal. This result was extended to infinite locally soluble groups by N.F. Kuzennyj and I.Ya. Subbotin [7]. In particular, they obtained a following characterization of locally soluble *$\overline{T}$-group*: a locally soluble group G is a *$\overline{T}$-group* if and only if every finitely generated subgroup of G is pronormal. In this connection is natural to consider an opposite situation, that is the groups in which every not finitely generated subgroup is pronormal. We want to consider in this paper more general situation, more precisely we will study the groups whose not finitely generated subgroups are transitively normal. Of course, we will consider these groups by some natural restrictions. As shows the results of A.Yu. Olshanskij (see a book [8]) the study of many types of groups is not really for groups, which are not generalized soluble. We will use the following restriction. Recall that a group G is said to be *radical* if it has an ascending series whose factors are locally nilpotent. Standard properties of radical groups can be found in [9]. The study of radical groups, whose not finitely generated subgroups are transitively normal, has been started in a paper [10]. More precisely, in this paper radical groups, whose locally nilpotent radical is not minimax, have been studied. Recall that a group G is said to be minimax, if G has a finite subnormal series

$$< 1 >= H_0 \lhd H_1 \lhd ... \lhd H_k = G,$$

whose factors satisfy minimal or maximal conditions for all subgroups. If G is a radical minimax group, then G has a finite subnormal series, whose factors either are polycyclic-by-finite or Chernikov groups.

Observe also that if G is a locally nilpotent group, whose not finitely generated subgroups are transitively normal, then every not finitely generated subgroup of G is normal in G.

The groups, whose not finitely generated subgroups are normal, have been studied by L.A. Kurdachenko, V.V. Pylaev [11], G. Cutolo [12], G. Cutolo, L.A. Kurdachenko [13]. In particular, locally nilpotent groups, whose not finitely generated subgroups are normal, were described very detail. Our purpose in this paper is the description of radical groups, whose not finitely generated subgroups are transitively normal and locally nilpotent radical is minimax. This study naturally break up into two natural case: the case, when a periodic part of locally nilpotent radical in infinite, and the case, when a periodic part of locally nilpotent radical in finite.

1. Groups whose locally nilpotene radical has infinite periodic part

Lemma 1. *Let G be a radical group and L be a locally nilpotent radical of G. Suppose that L is a Chernikov subgroup. If locally nilpotent radical of G/L is periodic, then G is a Chernikov group.*

Proof. Let K/L be a locally nilpotent radical of G/L. Then K is periodic and therefore $K/C_K(L)$ is a Chernikov group [14]. Since G is radical, $C_G(L) \leq L$ [9, Lemma 4], so that K/L is Chernikov. Suppose that K/L is infinite. Then its divisible part D/L is non-identity. Denote by P a divisible part of L. A subgroup D is Chernikov and therefore its divisible part V is abelian. Since V is G-invariant, $V \leq L$. On the other hand, D/V is finite, so that D/L must be finite, and we obtain a contradiction. This contradiction proves that K/L is finite and therefore $(G/L)/C_{G/L}(K/L)$ is finite. Since G/L is radical, $C_{G/L}(K/L) \leq K/L$ [9, Lemma 4], so that $(G/L)/(K/L) \cong G/K$ is finite. Since K is Chernikov, G is also Chernikov group. □

Lemma 2. *Let G be a radical group whose not finitely generated subgroups are transitively normal. Suppose that G includes a normal infinite Chernikov subgroup C. Then G/C is a soluble $\overline{T}$-group.*

Proof. Let L/C be the locally nilpotent radical of G/L. Denote by D a divisible part of C. Since C is infinite, $D \neq < 1 >$. Choose in L/C arbitrary finitely generated subgroup F/C. If F/C is finite, then F is an infinite Chernikov subgroup. In particular, F is not finitely generated, so that F is transitively normal in G. Suppose that F/C is not periodic. Being finitely generated and nilpotent, F/C is polycyclic. Suppose that F is finitely generated. Then F satisfies a maximal condition on normal subgroup [15, Theorem 3]. On the other hand, since D is non-identity, it has an infinite series of finite G-invariant subgroups, and we obtain a contradiction. This contradiction shows that F is not finitely generated, so that F is transitively normal in G. Then F/C is transitively normal in G/C. If H/C is a subgroup of L/C which is not finitely generated, then H is not finitely generated. It follows that H is transitively normal in G, so that H/C is transitively normal in G/C. Hence every subgroup of L/C is transitively normal in G/C. It follows that every cyclic subgroup of L/C is normal in L/C [10, Lemma 1.2], which implies that every subgroup of L/C is normal in L/C, i.e. L/C is a Dedekind group. In particular, if $G = L$, a result is proved. Suppose therefore that $G \neq L$. The facts that L/C is a Dedekind group and every subgroup of L/C is transitively normal in G/C imply that every subgroup of L/C is G-invariant. It follows that $(G/C)/C_{G/C}(L/C)$ is abelian (see, for example, [16, Theorem 1.5.1]. Since G/C is radical group $C_{G/C}(L/C) \leq L/C$ [9, Lemma 4], so that $(G/C)/(L/C)$ is abelian. It follows that G/C is hypercyclic. Choose again an arbitrary finitely generated subgroup K/C of G/C. Then K/C is nilpotent-by- finite (see, for example, [17, Corollary 3.20], in particular, K/C is polycyclic. Using again Theorem 3 of paper [15], we obtain that K is not finitely generated. It follows that K/C is transitively normal in G/C. Since it is valid for subgroups, which is not finitely generated, every subgroup of G/C is transitively normal in G/C. Hence G/C is a T-group. ☐

Corollary 1. *Let G be a radical group whose not finitely generated subgroups are transitively normal. Suppose that G includes a normal infinite Chernikov subgroup C such that G/C is not periodic. Then G/C is abelian.*

Proof. In fact, lemma 2 shows that G/C is a soluble T-group. Being non-periodic, G/C is abelian [15, Theorem 6.1.1]. ☐

Corollary 2. *Let G be a radical group whose not finitely generated subgroups are transitively normal. Suppose that G includes a normal infinite Chernikov subgroup C such that G/C is not periodic. If a divisible part D of C is not quasicyclic, then G is abelian.*

Proof. Let Q be the quasicyclic subgroup of D. Then $Q \neq D$. Since D is divisible, there exists s subgroup Z such that $D = Q \times Z$ [18, Theorem 21.2]. The both subgroup Q, Z are not finitely generated, thus Q, Z are transitively normal in G. Since D is normal in G, the subgroup Q, Z are subnormal in G. It follows that Q, Z are normal in G. Since G/C is not periodic, G/Q and G/Z are not periodic. By Corollary 1 G/Q and G/Z are abelian. Using Remak theorem, we obtain an embedding $G \to G/Q \times G/Z$ which shows that G is abelian. ☐

Lemma 3. *Let G be a group and Q be a normal quasicyclic subgroup of G. Suppose that G/Q is abelian. If the center of G does not include Q, then $G = QA$ for some subgroup A, where the intersection $Q \cap A$ is finite.*

Proof. Let g be an arbitrary element of G Q. The mapping $\tau_g : a \longrightarrow [g, a]$, $a \in Q$, is an endomorphism of Q such that $Im(\tau_g) = [g, Q]$, $Ker(\tau_g) = C_Q(g)$, so that $[g, Q] = Im(\tau_g) \cong Q/Ker(\tau_g) = Q/C_Q(g)$. Suppose that $Q \in [g, Q]$, then $[g, Q]$ is finite. It follows that $Q/C_Q(g)$ is finite. However Q does not include the proper subgroups, having finite index. It follows that $Q = C_Q(g)$ and $g \in C_G(Q)$. In other words, if $g \notin C_G(Q)$, then $Q = [g, Q]$. Since $\xi(G)$ does not include Q, we can find an element g such that $g \notin C_G(Q)$. Let x be the arbitrary element of G. Since G/Q is abelian, $g^x = gb$ for some element $b \in Q$. From equation $Q = [g, Q]$ we obtain that $b = [g, c]$ for some element $c \in Q$. Then

$$g^x = gb = gg^{-1}c^{-1}gc = c^{-1}gc,$$

which follows that $xc^{-1} \in C_G(g)$ or $x \in C_G(g)$. Since it is true for every element $x \in G$, $G = QC_G(g)$. Furthermore, $Q \cap C_G(g) = C_Q(g)$. The choice of g shows that $g \notin C_G(Q)$, i.e. $C_Q(g)$ is a proper subgroup of Q, in psrticular, it is finite. Now we can put $A = C_G(g)$. If G is a group, then by $\mathbf{Tor}(G)$ we will denote the maximal normal periodic subgroup of G. We recall that if G is a locally nilpotent group, then $\mathbf{Tor}(G)$ is a (characteristic) subgroup of G and $G/\mathbf{Tor}(G)$ is torsion-free. □

Proposition 1. *Let G be a non-periodic radical group, whose not finitely generated subgroups are transitively normal, and L be a locally nilpotent radical of G. Suppose that L is minimax and $\mathbf{Tor}(L)$ is infinite. If G is non-abelian, then G satisfies the following conditions:*

(i) L includes a normal quasicyclic q-subgroup Q such that $\mathbf{Tor}(L)/Q$ is finite and G/Q is abelian.

(ii) G/L is a finite cyclic group of order, dividing $q - 1$.

(iii) Every subgroup of L, which is not finitely generated, is G-invariant.

(iv) If $G \neq L$, then $G = QA$ for some subgroup A such that $Q \cap A$ is finite.

Proof. Let Q be the divisible part of $T = \mathbf{Tor}(L)$. By Corollary 2 Q is a quasicyclic q-subgroup for some prime q. Corollary 1 shows that G/Q is abelian, so that G satidfies (i). In particular, if $Q \leq \xi(G)$, then G is nilpotent, thus $G = L$. Hence if $G \neq L$, $\xi(G)$ does not include Q. Lemma 3 shows that G satisfies (iv). Let $K = \Omega_1(Q)$ and $C = C_G(K)$. Since $|K| = q$, G/C is a cyclic group of order, dividing $q-1$. The mapping $a \longrightarrow a^p$, $a \in \Omega_2(Q)$, is a G-endomorphism of $\Omega_2(Q)$ in $\Omega_1(Q)$ with a kernel $\Omega_1(Q)$. It follows that $C_G(\Omega_2(Q)/\Omega_1(Q))$ includes C. Using the similar arguments, we obtain that $C \leq C_G(\Omega_{n+1}(Q)/\Omega_n(Q))$ for each positive integer n. Since C/Q is abelian, C is a hypercentral subgroup, in particular, $C \leq L$, so that G satisfies (ii). Moreover, L is hypercentral. Let H be a not finitely generated subgroup of L. Then H is transitively normal in G. On the other hand, H is ascendant in L(and hence in G), because L is hypercentral. Using Lemma 1 of paper[1], we obtain that H is normal in G, so that G satisfies (iii). □

Theorem 1. *Let G be a non-periodic radical group, whose not finitely generated subgroups are transitively normal, and L be a locally nilpotent radical of G. Suppose that L is minimax and $\mathbf{Tor}(L)$ is infinite. Then either G is a minimax hypercentral group, whose not finitely generated subgroups are normal, or G satisfies the following conditions:*

(i) L includes a normal quasicyclic q-subgroup Q such that G/Q is abelian and finitely generated;

(ii) G/L is a finite cyclic group of order, dividing $q-1$.

(iii) Every subgroup of L, which is not finitely generated, is G - invariant.

(iv) If $G \neq L$, then $G = QA$ for some subgroup A such that $Q \cap A$ is finite.

Proof. Suppose first that $G = L$. Being minimax and locally nilpotent, G is hypercentral. Let H be a not finitely generated subgroup of G. Then H is transitively normal in G. On the other hand, since G is hypercentral, H is ascendant in L. An application of Lemma 1 of paper[1] shows that H is normal in G.

Suppose now that $G \neq L$. Then G satisfies all conditions of Proposition 1. Let Q be a divisible part of $\mathbf{Tor}(L)$. Then Q is quasicyclic q-subgroup for some prime q. We must show that G/Q is finitely generated. Suppose the contrary, let G/Q is not finitely generated. Then $A/(Q \cap A) \cong AQ/Q = G/Q$ is finite-by-abelian minimax group. Let $T = \mathbf{Tor}(A)$, then T is finite and A/T is a torsion-free minimax group. Then A includes a normal torsion-free subgroup B such that A/B is finite [19, Lemma 3].

Let $C = B \cap L$, since G/L is finite, B/C is finite. Then A/C is also finite. In particular, C is not finitely generated. By (iii) C is normal in G. Since C is torsion-free, $C \cap Q =< 1 >$, thus $C \cap [G, G] =< 1 >$ and $C \leq (G)$. From a description of groups, whose not finitely generated subgroups are normal (see, for example, survey [20, Theorem 2.4], we obtain that L must be Dedekind. Being not periodic, L is abelian. Then $L = Q \times K \times M$ where F is a finite subgroup, M is a torsion-free minimax subgroup. Since M is not finitely generated, M is normal in G. As above it follows that $M \leq \xi(G)$. Let $< y >= \Omega_1(Q)$. Since $G \neq L$, there exists an element g such that $y^g = y^k$ where k is a positive integer and $k \not\equiv 1(mod\ q)$. Choose in M a finitely generated subgroup V such that M/V is periodic. Then $M/V = D/V \times E/V$ where D/V is divisible and E/V is finite. We may assume that $q \in (E/V)$. Indeed, if not, then instead of V we will consider V^p. We have $M/V = E_1/V \times E_2/V \times D/V$ where E_1/V (respectively E_2/V) is a Sylow q-subgroup (respectively q-subgroup) of E/V. Let u be an element of E_1 such that $|uV| = q$. Clearly V is normal in G. In a factor-group G/V consider a subgroup $< yuV >$. We have

$$(yuV)^{gV} = (yV\ uV)^{gV} = (yV)^{gV}(uV)^{gV} =$$

$$= (y^gV)(u^gV) = (y^kV)(uV) = (yV)^k(uV).$$

Consider now a subgroup $< yuV > \times D/V$. This subgroup is not finitely generated, so it is normal in G/V. A subgroup $< yuV, QV/V >$ is not finitely generated, therefore it is normal in G/V. Furthermore, a subgroup $< yuV > \cap QV/V =< 1 >$, so that

$$(< yuV > \times D/V) \cap (< yuV, QV/V) =$$

$$= (< yuV > \times D/V) \cap (< yuV > \times QV/V) =< yuV > .$$

In other words, $< yuV >$ is normal in G/V. In particular, $(yuV)^{gV} = (yuV)^t$ for some positive integer t.

Then $(yV)^k(uV) = (yuV)^t = (yV\ uV)^t = (yV)^t(uV)^t$. It follows that $t \equiv k(mod\ q), t \equiv 1(mod\ p)$. It follows that $k \equiv 1(mod\ p)$, and we obtain a contradiction. This contradiction proves that G/Q is finitely generated, and result is proved. □

2. Groups whose locally nilpotent radical has finite periodic part

We will need the following property of abelian minimax group.

Lemma 4. *Let G be a torsion-free abelian minimax group and π an infinite set of primes. Then $\cap_{p\in\pi} G^p =< 1 >$.*

Proof. Since G is minimax, it includes a free abelian subgroup B such that G/B is a Chernikov group. Then $\Pi(G/B)$ is finite. Let $\sigma = \pi\ \Pi(G/B)$. Then a subset σ is infinite. Since B is free abelian,
$\cap_{p\in\pi} B^p =< 1 >$. Let $p \in \sigma$ and consider a factor-group G/B^p. The choice of shows that B/B^p is a Sylow p-subgroup of G/B^p, thus we obtain a direct decomposition $G/B^p = B/B^p \times A/B^p$ where $A/B^p \cong G/B$. It follows that $(G/B^p)^p = A/B^p$. On the other hand, $(G/B^p)^p = G^pB^p/B^p = G^p/B^p$. Then $G^p/B^p \cap B/Bp = A/Bp \cap B/Bp =< 1 >$, which implies that $G^p \cap B = B^p$. Put $K = \cap_{p\in\pi} G^p$. Then

$$K \cap B = (\cap_{p\in\pi} G^p) \cap B = \cap_{p\in\pi}(G^p \cap B) = \cap_{p\in\pi} B^p =< 1 > .$$

It follows that $K \cong K/(K \cap B) \cong KB/B$. We recall that G/B is a periodic group. Since G is torsion-free, $K =< 1 >$. □

Lemma 5. *Let G be a non-periodic radical group, whose not finitely generated subgroups are transitively normal, and L be a locally nilpotent radical of G. Suppose that L satisfies the following conditions:*

(i) L is minimax abelian and $\mathbf{Tor}(L)$ *is finite;*

(ii) L includes the G-invariant subgroups A, B such that A is torsion-free, $B \leq A$ and a factor A/B is infinite and periodic;

(iii) G/A is not periodic.

Then G is abelian.

Proof. Since A/B is abelian and minimax, $A/B = D/B \times K/B$ where D/B is a divisible Chernikov group and K/B is a finite group. Choose an infinite set π of primes such that $\pi \cap \Pi(A/B) = \emptyset$. Let $p \in \pi$ and consider a factor-group G/B^p. The choice of π yields that B/B^p is a Sylow p-subgroup of D/B^p. Then $D/B^p = D_p/B^p \times B/B^p$, where $D_p/B^p \cong D/B$ is a divisible Chernikov group. It follows that $(D/B^p)^p = D_p/B^p$. On the other hand, $(D/B^p)^p = D^p\ B^p/B^p = D^p/B^p$, that is $D_p = D^p$. Since D^p/B^p is a Chernikov divisible group and G/D is not periodic, Corollary 1 shows that $(G/B^p)/(D^p/B^p) \cong G/D^p$ is abelian. By Lemma 4 $\cap_{p\in\pi} D^p =< 1 >$. This equation together with Remak's theorem gives an embedding

$$G \hookrightarrow Cr_{p\in\pi} G/D^p,$$

which shows that G is abelian.

Let G be a group and A be a normal abelian subgroup of G. We say that A is rationally irreducible in G, if A/B is periodic for each non-identity G-invariant subgroup B of A. We say that A is almost irreducible in G, if A/B is finite for each non-identity G-invariant subgroup B of A. □

Proposition 2. *Let G be a radical group and L be a locally nilpotent radical of G. Suppose that L is minimax and $\mathbf{Tor}(L)$ is finite. Then G/L is abelian-by-finite and finitely generated.*

Proof. We recall that locally nilpotent minimax group has finite special rank. A factor-group $L/\mathbf{Tor}(L)$ is a locally nilpotent torsion-free group of finite special rank, therefore it is nilpotent [21]. Hence L has finite series of G-invariant subgroups

$$< 1 >= C_0 \leq C_1 \leq C_2 \leq ... \leq C_s = L,$$

where $C_1 = \mathbf{Tor}(L)$, $C_{j+1}/C_j = \xi(L/C_j)$, $1 \leq j \leq s-1$. The factors C_{j+1}/C_j of this series is torsion-free whenever $1 \leq j \leq s-1$ (see, for example, [22, §22]). We consider a refinement of this series

$$< 1 >= K_0 \leq K_1 \leq ... \leq K_t = L$$

such that $K_1 = \mathbf{Tor}(L) = C_1$, every subgroup K_j is G-invariant, the factors $K_j +_1 /K_j$ are torsion-free and rationally irreducible in G, $1 \leq j \leq$

$t-1$. Being minimax, $K_j +_1 /K_j$ has finite 0-rank r, $1 \leq j \leq t-1$. Then the factor-group $G/C_G(K_{j+1}/K_j)$ is isomorphic to some subgroup of $GL_r(Q)$. Being radical, $G/C_G(K_{j+1}/K_j)$ does not include the non-cyclic free subgroups. Then $G/C_G(K_{j+1}/K_j)$ is soluble (see, for example, [23, Corollary 10.17]). Furthermore, since K_{j+1}/K_j is rationally irreducible in $G, G/C_G(K_{j+1}/K_j)$ is abelian-by-finite (see, for example, [23, Lemma 3.5]). Let $V/C_G(K_{j+1}/K_j)$ be a normal abelian subgroup of $G/C_G(K_{j+1}/K_j)$, having finite index. Since $A_{j+1} = K_{j+1}/K_j$ has finite 0-rank, A_{j+1} includes a non-identity V-invariant subgroup W such that $r_0(W) = r_0(Y)$ for every non-identity V-invariant subgroup Y of W. In other words, W is rationally irreducible in V. Without loss of generality we can suppose that W is a pure subgroup of A_{j+1}. Since V has finite index in G, there are the elements $x_1, ..., x_m$ such that $W_1 = x_1^{-1}Wx_1...x_m^{-1}Wx_m$ is a G-invariant subgroup of A_{j+1}. Since A_{j+1} is rationally irreducible in $G, A_{j+1}/W_1$ is periodic. It follows that $C_G(A_{j+1}) = C_G(W_1)$. The fact that $V/C_G(A_{j+1})$ is abelian implies that $V/C_V(x_k^{-1}Wx_k)$ is finitely generated [24, Folgerung 3.2], $1 \leq k \leq m$. Obviously $C_V(W_1) = \cap_{1\leq k\leq m} C_V(x_k^{-1}Wx_k)$, therefore using Remak's theorem we obtain an embedding

$$V/C_V(W_1) \hookrightarrow Dr_{1\leq k\leq m} V/C_V(x_k^{-1}Wx_k),$$

which shows that $V/C_V(W_1)$ is finitely generated.

An equation $C_G(A_{j+1}) = C_G(W_1)$ shows that and $V/C_G(K_{j+1}/K_j)$ is finitely generated. In turn out, it follows that $G/C_G(K_{j+1}/K_j)$ is abelian-by-finite and finitely generated. We remark that it is true for each j, $1 \leq j \leq t-1$.

Put $B = \cap_{0\leq j\leq t-1} C_G(K_{j+1}/K_j)$, then Remak theorem shows that G/B is embedded in a group $Dr_{0\leq j\leq t-1} G/C_G(K_{j+1}/K_j)$.
Since $G/C_G(K_1)$ is finite and $G/C_G(K_{j+1}/K_j)$ is abelian-by-finite and finitely generated whenever $1 \leq j \leq t-1, G/B$ is likewise abelian-by-finite and finitely generated. Suppose that L does not include B. Then BL/L is non-identity, therefore it includes a non-identity G-invariant locally nilpotent subgroup R/L. We have $[B, K_{j+1}] \leq Kj$, $0 \leq j \leq t-1$. This shows that the series

$$< 1 >= K_0 \leq K_1 \leq ... \leq K_t = L$$

is central in B. Since R/L is locally nilpotent, R is locally nilpotent. Being normal, R must lie in L, and we obtain a contradiction. This contradiction shows that $B \leq L$ and hence G/L is abelian-by-finite and finitely generated.

For the proof of the next result we must use some module-theoretical concepts and results. Let G be a group, R a ring and A an RG-module. Put

$$\xi_{RG}(A) = \{a \in A/a(g-1) = 0 \ \text{ for each element } \ g \in G\} = C_A(G).$$

Clearly $\xi_{RG}(A)$ is an RG-submodule of A. This submodule is called the RG-center of A. Denote by ϖRG the augmentation ideal of a group ring RG, that is the two-sided ideal, generated by all elements $g-1,\ g \in G$. Then $\xi_{RG}(A)(\varpi RG) =< 0 >$.

Starting from the RG-center, we can construct the upper RG-central series of A :

$$< 0 >= \xi_{RG\,0}(A) \leq \xi_{RG\,1}(A) \leq \xi_{RG\,2}(A) \leq ... \leq$$

$$\leq \xi_{RG\,\alpha}(A) \leq \xi_{RG\,\alpha+1}(A) \leq ...\xi_{RG\,\gamma}(A),$$

defined the rule $\xi_{RG\,1}(A) = \xi_{RG}(A)$ is the center of G, and recursively $\xi_{RG\,\alpha+1}(A)/\xi_{RG\,\alpha}(A) = \xi_{RG}(A/\xi_{RG\,\alpha}(A))$ for all ordinals, $\alpha, \xi_{RG\,\alpha}(A) = \cup_{\mu<\lambda}\xi_{RG\,\mu}(A)$ for the limit ordinals λ and $\xi_{RG}(A/\xi_{RG\,\gamma}(A)) =< 0 >$. The last term $\xi_{RG\,\gamma}(A)$ of this series is called the *upper RG-hypercenter of A* and the ordinal γ is called the *RG-central length of a module A* and will denoted by $zl_{RG}(A)$. We observe that

$\xi_{RG\,\alpha+1}(A)(\varpi RG) \leq \xi_{RG\,\alpha}(A)$ for all $\alpha < \gamma$.

If the upper RG-hypercenter of A coincides with A, then A is called *RG-hypercentral.*

If A is an RG-hypercentral module and $zl_{RG}(A)$ is finite, then we will say that A is *RG-nilpotent.*

If B, C the RG-submodules of A and $B \leq C$, then a factor C/B is called *G-central* (respectively *G-eccentric*), if $G = C_G(C/B)$ (respectively $G \neq C_G(C/B)$).

We say that the RG-module A is *G-hypereccentric,* if A has an ascending series of RG-submodules

$$< 0 >= A_0 \leq A_1 \leq ...A_\alpha \leq A_{\alpha+1} \leq ...A_\gamma = A$$

whose factors $A_{\alpha+1}/A_\alpha$ are G-eccentric and simple FG-modules for all $\alpha < \gamma$.

We say that the RG-module A has *Z-RG-decomposition*, if $A = C \times E$ where C is the upper RG-hypercenter of A and E is an RG- submodule, which is G-hypereccentric (see, [17, Chapter 10]). We remark that this decomposition is unique (of course, if it exists) [17, Chapter 10]. $\square$

Lemma 6. *Let G be a non-periodic radical group, whose not finitely generated subgroups are transitively normal, and L be a locally nilpotent radical of G. Suppose that L satisfies the following conditions:*

(i) L is minimax abelian and $\mathbf{Tor}(L)$ *is finite;*

(ii) L includes the G-invariant subgroup A such that A is not finitely generated, torsion-free and almost irreducible in G;

(iii) L/A is not periodic and $\mathbf{Tor}(L/A)$ *is finite. Then G/A is finitely generated and nilpotent-by-finite.*

Proof. Let F/A be an arbitrary finitely generated subgroup of L/A. Since every finitely generated subgroup of locally nilpotent group is polycyclic, F is not finitely generated. It follows that F is transitively normal in G and F/A is transitively normal in G/A. If H/A is not finitely generated subgroup of L/A, then H/A is transitively normal in G/A. Hence every subgroup of L/A is transitively normal in G/A. In particular, L/A is a T-group. Being non-periodic, L/A is abelian [5, Theorem 6.1.1]. In turn out, it follows that every subgroup of L/A is G-invariant. Then $/(G/A)/C_{G/A}(L/A)/ \leq 2$ (see, for example, [16, Theorem 1.5.7]. By Proposition 2.3 G/L is finitely generated and abelian-by-finite. Let U/L be a normal abelian subgroup, having finite index in G/L. Put $K/A = C_{G/A}(L/A) \cap U/A$. Since $/(G/A)/C_{G/A}(L/A)/ \leq 2$, then index $/G : K/$ is finite and $L/A \leq \xi(K/A)$. Since K/L is abelian, K/A is nilpotent.

Suppose that L/A is not finitely generated. In usual way we can consider A as $Z(G/A)$-module. Let E be a Q-injective envelope of A. Since A has finite 0-rank r, then $dim_Q(E) = r$ is finite. We can extend the action of G on A to the action of G on E, so that E become a $Q(G/A)$-module. Being finite dimensional, E is artinian $Q(G/A)$-module. Then E has the Z - $Q(K/A)$-decomposition [17, Theorem 10.21], that is $E = C \times D$ where C is the upper $Q(K/A)$-hypercenter of E and D is an $Q(K/A)$-submodule, which has a finite series

$$< 0 >= D_0 \leq D_1 \leq ... \leq D_s = D$$

of $Q(K/A)$-submodules whose factors are $Q(K/A)$-simple and (K/A)-eccentric. The intersection $A \cap C$ obviously is an upper $Z(K/A)$-hypercenter of A. Since A is abelian and torsion - free, every term of an upper $Z(K/A)$-central series of A is pure in A. It follows that $A/(A \cap C)$ is torsion-free. On the other hand, since K is normal subgroup of G, C is a $Q(G/A)$-submodule. It follows that $A \cap C$ is G-invariant subgroup of A. Suppose now that $A \cap C \neq < 1 >$. Being G-invariant, $A \cap C$ has a finite index in A. By above remarked, this means that $A = A \cap C$. Moreover, every term of an upper $Z(K/A)$-central series is a pure G-invariant subgroup. It follows that $A = \xi_{Z(K/A)}(A)$, that is $A \leq \xi(K)$ and a subgroup K is nilpotent. Being normal in G, K lies in locally nilpotent radical of G, that is $K \leq L$. In particular, it implies that L has finite index in G. Choose in A a free abelian subgroup Y such that A/Y is periodic. Since A is not finitely generated, A/Y is infinite. Since L has finite index in G, there are the elements $g_1, ..., g_p$ such that $Y_1 = g_1^{-1} Y g_1 ... g_p^{-1} Y g_p$ is a G-invariant subgroup of A. Clearly this subgroup is finitely generated. Then Y_1/Y is finite and A/Y_1 is infinite periodic group, and we obtain a contradiction with the fact, that A is almost irreducible. This contradiction shows that $A \cap C = < 1 >$. Then A has a series

$$< 1 >= A_0 \leq A_1 \leq ... \leq A_s = A$$

of K-invariant subgroups modules whose factors are torsion-free, rationally irreducible in K and K-eccentric. Suppose first that A is rationally irreducible in K. Then K includes a nilpotent subgroup R such that $A \cap R = < 1 >$ and RA has finite index in K [25]. Consider a subgroup $R \cap L$. We have

$$R \cap L \cong (R \cap L)/(R \cap A) = (R \cap L)/(R \cap L \cap A) \cong (R \cap L)A/A =$$

$$(RA \cap L)/A = (RA/A) \cap (L/A).$$

Since RA/A has finite index in K/A, $(RA/A) \cap (L/A)$ has finite index in L/A. By above assumption L/A is not finitely generated, which follows that $(RA/A) \cap (L/A)$ is not finitely generated. This implies that $R\ L$ is not finitely generated. Then $R \cap L$ is transitively normal in G. On the other hand, $R \cap L$ is subnormal in G, thus $R \cap L$ is normal in G. An inclusion $[K, L] \leq A$ implies that $[K, R \cap L] \leq A$. On the other hand, since $R \cap L$ is normal in $G, [K, R \cap L] \leq R \cap L$, so that $[K, R \cap L] \leq A \cap R \cap L = < 1 >$. In other words, $R \cap L$ is central in K. It follows that $G/C_G(R \cap L)$ is finite. Since $R \cap L$ is not finitely generated, then using the above arguments, we can find in $R \cap L$ a G-invariant finitely generated subgroup Y_2 such that $(R \cap L)/Y_2$ is infinite periodic group. Clearly $L/(R \cap L)$ is not periodic, therefore an application of Lemma 2.2 shows that G must be abelian, and we obtain a contradiction. This contradiction proves that L/A must be finitely generated. As we have already seen, G/L is finitely generated, so that G/A is finitely generated.

Suppose now that A is not rationally irreducible. Then instead G we must consider K/A_{s-1} and repeat all above arguments. □

Lemma 7. *Let G be a non-periodic radical group, whose not finitely generated subgroups are transitively normal, and L be a locally nilpotent radical of G. Suppose that L satisfies the following conditions:*

(i) L is minimax abelian;

(ii) L includes the G - invariant subgroups A, B such that A is not finitely generated, torsion - free, B is free abelian, A/B is periodic;

(iii) G/A is finite.

Then every subgroup of A is G-invariant, in particular,

$/G : C_G(A)/ \leq 2$ and G/A is a $\overline{T}$-group.

Proof. Since A/B is periodic and minimax, it is Chernikov group, that is $A/B = D/B \times K/B$, where F/B is finite and D/B is divisible. Since G/A is finite, $K/B = (F/B)^{G/B}$ is finite. Then K is G-invariant, finitely generated and torsion-free. In other words, K is free abelian. Furthermore, A/K is a divisible Chernikov group. We have $K =< a_1 > \times ... \times < a_m >$. Factor-group G/K is Chernikov, in particular, $\Pi(G/K)$ is finite. Choose a prime p such that $2 \neq p \notin (G/K)$. Put $K_0 = K$, $K_1 = K_0^p$ and $K_{n+1} = K_n^p$, $n \in N$. By this choice K/K_n is a Sylow p-subgroup of A/K_n, so that A/K_n, $A/K_n = K/K_n \times C_n/K_n$, where $C_n/K_n \cong A/K_n$ is divisible. Let $t(n) = p^n$, then $(A/K_n)^{t(n)} = C_n/K_n$. On the other hand, $(A/K_n)^{t(n)} = A^{t(n)}K_n/K_n = A^{t(n)}/K_n$. This means that $A^{t(n)} = C_n$, or $A^{t(n)} \cap K = C_n \cap K = K_n$, $n \in N$. Since C_n/K_n is Chernikov subgroup, every subgroup of $(G/K_n)/(C_n/K_n) \cong G/C_n$ is transitively normal in G/C_n, that is G/C_n is a finite $\overline{T} - group$, in particular, G/A is a finite $\overline{T} - group$. It follows that $G/C_n = R_n/C_n \lambda V_n/C_n$ where R_n/C_n is a nilpotent residual of G/C_n, R_n/C_n is abelian, every its subgroup is normal in G/C_n, V_n/C_n is a Dedekind group. Moreover, $\emptyset = \Pi(R_n/C_n) \cap (V_n/C_n)$ and $2 \notin \Pi(R_n/C_n)$ (see, for example [26, Theorem 2.1.11]).

Suppose first that $p \in \Pi(V_1/C_1)$, then $A/C_1 \cap R_1/C_1 =< 1 >$. Since $p \neq 2$, Sylow p-subgroup of a Dedekind group G/R_1 is abelian and therefore lies in the center of G/R_1. In particular, $[G, A] \leq R_1$. On the other hand, A/C_1 is a normal subgroup of G/C_1, so that $[G/C_1, A/C_1] \leq A/C_1 \cap R_1/C_1 =< 1 >$. In other words, the center of G/C_1 includes A/C_1. Consider now a factor-group G/C_2 and suppose that $p \in \Pi(R_2/C_2)$. The mapping $\sigma : uC_2 \longrightarrow (uC_2)^p$, $u \in A$, is a G-endomorphism of A/C_2. Since A/C_2 is a direct product of cyclic subgroup of order p^2, $\mathbf{Ker}(\sigma) = \mathbf{Im}(\sigma) = \mathbf{\Omega}_1(A/C_2) = C_1/C_2$. It follows that $C_1/C_2 \cong_G A/C_1$. By above proved $G = C_G(A/C_1)$, so that $G = C_G(C_1/C_2)$. Hence the upper hypercenter of G/C_2 includes A/C_2. Since $p \notin \Pi(R_1/C_1)$, C_1/C_2 is a Sylow p-subgroup of R_1/C_2, so that $R_1/C_2 = C_1/C_2 \times X/C_2$. Clearly X/C_2 is G-invariant. The factor- group $(G/C_2)/(X/C_2)$ is an extension of A/C_2, which lies in the hypercenter, by the nilpotent group, therefore it is nilpotent. It follows that $R_2/C_2 \leq X/C2$. But in this case $p \notin \Pi(R_2/C_2)$, and we obtain a contradiction. This contradiction shows that $A/C_2 \cap R_2/C_2 =< 1 >$, which follows that $[G, A] \leq C_2$. Using the similar arguments, we obtain that $[G, A] \leq C_n$, for each $n \in N$. Let $C_\omega = \cap_{n\in N} C_n$, then

$$K \cap C_\omega = K \cap (\cap_{n\in N} C_n) = \cap_{n\in N}(K \cap C_n) = \cap_{n\in N} K_n =< 1 > .$$

Thus $C_\omega \cong C_\omega/(C_\omega \cap K) = C_\omega K/K \leq A/K$. Since A/K is periodic and A is torsion - free, $C_\omega =< 1 >$. In other words, $[G, A] \leq \cap_{n\in N} C_n =< 1 >$, that is $A \leq \xi(G)$.

Assume now that first that $p \in \Pi(R_1/C_1)$. Using the above arguments we can obtain that $A/C_n \leq R_n/C_n$ for each $n \in N$. Then every subgroup of A/C_n is G-invariant, in particular, $< a_j > C_n/C_n$ is G-invariant, $1 \leq j \leq m$. Let g be an arbitrary element of G, then $a_j^g = a_j^r j c_{j\,n}$ for some positive integer r_j and $c_{j\,n} \in C_n$. We have $c_{j\,n} = a_j^g a_j^{-r} j$. It follows that $c_{j\,n} \in A$, because A is normal in G. Moreover, $c_{j\,n} \in A \cap C_n = K_n$. Thus $a_j^g \in< a_j > K_n$ for every $n \in N$, that is $a_j^g \in \cap_{n\in N} < a_j > K_n$. Put $a_s = a_{s0},\ a_{s\,1} = a_{s\,0}^p, a_{s\,n+1} = a_{s\,n}^p$, $n \in N$. Then $K_n =< a_{1\,n} > \times ... \times < a_{m\,n} >$, and therefore

$$< a_j > K_n =< a_{1\,n} > \times ... \times < a_{j-1\,n} > \times$$

$$\times < a_j > \times < a_{j+1\,n} > \times ... \times < a_{m\,n} > .$$

Let x be an arbitrary element of $\cap_{n\in N} < a_j > K_n$, then $x = u_n y_n$ where $u_n \in< a_j >$, $y_n \in \mathbf{Dr}_{1\leq s\leq m,\ s\neq j} < a_{sn} >$. We have $u_1 y_1 = u_n y_n$, which implies $u_n^{-1} u_1 = y_n y_1^{-1}$. Since $u_n^{-1} u_1 \in< a_j > y_n y_1^{-1} \in \mathbf{Dr}_{1\leq s\leq m,\ s\neq j} < a_{s\,n} >$ and $< a_j > \cap \mathbf{Dr}_{1\leq s\leq m,\ s\neq j} < a_{s\,n} >=< 1 >$, $y_1 = y_n$ for each $n \in N$. It follows that $y_1 \in \cap_{n\in N}(\mathbf{Dr}_{1\leq s\leq m,\ s\neq j} < a_{s\,n} >) =< 1 >$. Thus $y_1 = 1$ and $x = u_1 \in< a_j >$. In other words $a_j^g \in< a_j >$ and a subgroup $< a_j >$ is G - invariant. Since $< a_j >$ is an infinite cyclic,
$|G : C_G(< a_j >)| \leq 2$. Suppose that there are j, s such that
$C_G(< a_j >) \neq C_G(< a_s >)$. Then we can find an element g such that $a_j^g = a_j^{-1}$ and $a_s^g = a_s$. It follows that $(a_j C_1)^{gC} 1 = (a_j C_1)^{-1} = (a_j C_1)^{p-1}$ and $(a_s C_1)^{gC} 1 = a_s C_1$. For a coset $a_j C_1 a_s C_1 = a_j a_s C_1$ we have $(a_j a_s C_1)^{gC} 1 = (a_j C_1)^{gC} 1 (a_s C_1)^{gC} 1 = (a_j C_1)^{-1}(a_s C_1)$. On the other hand, a subgroup $< a_j a_s C_1 >$ must be G-invariant, therefore

$$(a_j a_s C_1)^{gC} 1 = (a_j a_s C_1)^q = (a_j C_1 a_s C_1)^q = (a_j C_1)^q (a_s C_1)^q$$

for some positive integer q. Thus we have $q \equiv p-1(modp)$, $q \equiv 1(modp)$, and we obtain a contradiction, because $p \neq 2$. This contradiction shows that $C_G(< a_j >) \neq C_G(< a_s >)$ for every $j, s, 1 \leq j,\ s \leq m$. It follows that $C_G(< a_j >) = C_G(K)$. If we suppose that $C_G(K) = G$, then $A/C_n \leq \xi(G/C_n)$ for each $n \in N$. Using the above arguments, we obtain that $A/C_n \cap R_n/C_n =< 1 >$, so we have a contradiction with our assumption. This contradiction shows that $C_G(< a_j >) = C_G(K)$ has index $2, 1 \leq j \leq m$. Let $g \in G\ C_G(K)$, then $a_j^g = a_j^{-1}$ for all j, it follows that $a^g = a^{-1}$ for all element $a \in K$. Since A is torsion-free and A/K is periodic, $a^g = a^{-1}$ for each element $a \in A$. □

Theorem 2. *Let G be a non-periodic radical group, whose not finitely generated subgroups are transitively normal, and L be a locally nilpotent radical of G. Suppose that L is not finitely generated, minimax and* $\mathbf{Tor}(L) = T$ *is finite. Then G is a group of one from following types.*

(i) G is a group, whose not finitely generated subgroups are normal.

(ii) G includes a normal abelian torsion-free subgroup D such that D is almost irreducible in G and G/D is finitely generated and nilpotent-by-finite.

(iii) L includes the G-invariant subgroups A satisfying the following conditions:

A is not finitely generated, torsion-free and rationally irreducible in G; every proper pure subgroup of A is finitely generated; every subgroup of A is G-invariant, in particular, $|G : C_G(A) \leq 2$; G/A is a finite T-group.

Proof. As in Proposition 2 we obtain that L is nilpotent. Then every not finitely generated subgroup of L is G-invariant, in particular, if $L = G$, then every not finitely generated subgroup is normal in G.

Further we will assume that $G \neq L$. Suppose Suppose first that L is abelian. Then $L = T \times B$ where B is a torsion-free subgroup. Let $t = |T|$, then $A = L^t = B^t$, in particular, A is a torsion-free G-invariant subgroup.

Consider a case when G/L is not periodic. Proposition 2.3 shows that G/L is abelian-by-finite and finitely generated. Suppose that A includes proper G-invariant non-identity subgroup C such that $r_0(C) < r_0(A)$. Then C includes proper G-invariant non-identity subgroup E such that E is rationally irreducible in G. Put $D/E = \mathbf{Tor}(A/E)$, then D is G-invariant, A/D is non-identity and torsion-free and D is rationally irreducible in G. If D includes a non-identity G-invariant subgroup M such that D/M is infinite and periodic, then by Lemma 5 G is abelian. This shows that D is almost irreducible. Lemma 6 shows that in this case G/D is finitely generated and nilpotent - by - finite. Suppose now that

A is rationally irreducible in G. Then A must be almost irreducible. In fact, if we suppose that A includes a non-identity G-invariant subgroup D_1 such that A/D_1 is infinite, then by Lemma5 G is abelian.

Suppose now that G/L is periodic. Being abelian-by-finite and finitely generated, G/L is finite. Let K be a pure non-identity G-invariant subgroup of A. Suppose that K is not finitely generated. Let F be a finitely generated subgroup of K such that K/F is periodic. Since G/L is finite, F^G is finitely generated. Then F^G/F is finite, so that K/F^G is infinite. Lemma 5 shows again that in this case G must be abelian. Hence every pure non-identity G-invariant subgroup of A is finitely generated. It follows that A includes a G-invariant finitely generated subgroup V such that A/V is infinite and periodic. By Lemma 7 every subgroup of A is G-invariant, $|G : C_G(A)| \leq 2$ and G/A is a T-group.

Finally suppose that L is non-abelian. Since every not finitely generated subgroup of L is normal, $L = P \times U$ where P is a Dedekind group, $U \cong Q_2$ is a group 2-adic rational numbers (see, for example, survey [20, Theorem 2.4]). Since $G/C_G(U)$ is finite, $|G/C_G(U)| \leq 2$, and we come to the group of last type. □

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