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Closure operators in the categories of modules

Part II (Hereditary and cohereditary operators)

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Abstract. This work is a continuation of the paper [1] (Part I), in which the weakly hereditary and idempotent closure operators of the category R-Mod are described. Using the results of [1], in this part the other classes of closure operators C are characterized by the associated functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ which separate in every module $M \in R$-Mod the sets of C-dense submodules and C-closed submodules. This method is applied to the classes of hereditary, maximal, minimal and cohereditary closure operators.

1. Introduction. Preliminary definitions and results

The present work is devoted to the study of the closure operators of the module categories and it is a continuation of the paper [1] (Part I). On the basis of the results of Part I, where the weakly hereditary and idempotent closure operators of R-Mod are described, the characterizations of new types of closure operators C are shown by means of associated functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ which are defined by the sets of C-dense or C-closed submodules. With this purpose the following types of closure operators of R-Mod are studied: hereditary, weakly hereditary maximal, hereditary maximal, minimal, cohereditary (i.e. the most important subclasses of the classes of weakly hereditary or idempotent closure operators).

For every studied class of closure operators C the conditions to the associated functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ are indicated which are necessary and

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sufficient for C to belong to the investigated class. In the theory of radicals the corresponding results consist in the characterization of diverse kinds of preradicals by means of associated classes of torsion or torsion free modules ([5, 7, 8, 9]).

In what follows we use some notions and results of Part I ([1]). As it was mentioned in [1], our studies are based to the facts on closure operators of R-Mod [2, 3, 4, 5, 6] and on the radicals in modules ([5, 7, 8, 9]).

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. We denote by $\mathbb{L}({}_RM)$ the lattice of submodules of the module $M \in R$-Mod.

Definition 1.1. A **closure operator** of R-Mod is a function C which associates to every pair $N \subseteq M$, $N \in \mathbb{L}({}_RM)$, a submodule of M, denoted by $C_M(N)$ such that the following conditions are satisfied:

(c_1) $N \subseteq C_M(N)$;

(c_2) If $N \subseteq P$ for $N, P \in \mathbb{L}({}_RM)$, then $C_M(N) \subseteq C_M(P)$;

(c_3) If $f : M \to M'$ is an R-morphism and $N \subseteq M$, then $f(C_M(N)) \subseteq C_{M'}(f(N))$.

Denote by $\mathbb{CO}$ the class of all closure operators of R-Mod.

Definition 1.2. A closure operator $C \in \mathbb{CO}$ is called:

a) **weakly hereditary**, if $C_M(N) = C_{C_M(N)}(N)$ for every $N \subseteq M$;

b) **idempotent**, if $C_M(N) = C_M\big(C_M(N)\big)$ for every $N \subseteq M$.

Definition 1.3. Let $C \in \mathbb{CO}$. A submodule $N \in \mathbb{L}({}_R(M)$ is called:

a) C**-dense** in M, if $C_M(N) = M$;

b) C**-closed** in M, if $C_M(N) = N$.

We use the following notations:

$\mathcal{F}_1^C(M)$ is the set of all C-dense submodules of M;

$\mathcal{F}_2^C(M)$ is the set of all C-closed submodules of M.

Thus every closure operator $C \in \mathbb{CO}$ determines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ by which in some cases the operator C can be completely characterized. In order to formulate some conditions on these functions, we consider an abstract function $\mathcal{F}$ which for every module $M \in R$-Mod separates in $\mathbb{L}({}_RM)$ a (non empty) set of submodules $\mathcal{F}(M)$, where $\mathcal{F}$ is compatible with the isomorphisms and $M \in \mathcal{F}(M)$.

Definition 1.4. An abstract function $\mathcal{F}$ of a category R-Mod is called a **function of type $\mathcal{F}_1$** if it satisfies the following conditions:

1) If $N \in \mathcal{F}(M_\alpha)$, $M_\alpha \subseteq M$ $(\alpha \in \mathfrak{A})$, then $N \in \mathcal{F}(\sum_{\alpha\in\mathfrak{A}} M_\alpha)$;
2) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(P)$, then for every $K \subseteq M$ we have $N + K \in \mathcal{F}(P + K)$;
3) If $f : M \to M'$ is an R-morphism and $N \in \mathcal{F}(M)$, then $f(N) \in \mathcal{F}(f(M))$.

From the condition 2) it follows the property:

4) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $P \in \mathcal{F}(M)$.

For an abstract function $\mathcal{F}$ of R-Mod and for every $N \subseteq M$ we denote:

$$(C^{\mathcal{F}})_M(N) = \sum \{M_\alpha \subseteq M \mid N \subseteq M_\alpha, \;\; N \in \mathcal{F}(M_\alpha)\}. \tag{1.1}$$

The following statement describes the weakly hereditary closure operators of R-Mod by the abstract functions of type $\mathcal{F}_1$.

Theorem 1.1 ([1], Theorem 2.6)**.** *The mappings $C \longmapsto \mathcal{F}_1^C$ and $\mathcal{F} \longmapsto C^{\mathcal{F}}$ define a monotone bijection between the* ***weakly hereditary*** *closure operators C of R-Mod and the abstract functions $\mathcal{F}$ of type $\mathcal{F}_1$.* □

In a similar way the characterization of idempotent closure operators C of R-Mod is obtained, using the associated function $\mathcal{F}_2^C$.

Definition 1.5. An abstract function $\mathcal{F}$ of R-Mod is called a **function of type $\mathcal{F}_2$** if it satisfies the following conditions:

1*) If $N_\alpha \in \mathcal{F}(M)$, $N_\alpha \subseteq M$ $(\alpha \in \mathfrak{A})$, then $\bigcap_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}(M)$;
2*) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(P)$, then for every submodule $K \subseteq M$ we have $N \cap K \in \mathcal{F}(P \cap K)$;
3*) If $g : M \to M'$ is an R-morphism and $N' \in \mathcal{F}(g(M))$, then $g^{-1}(N') \in \mathcal{F}(M)$.

Remark that from the condition 2*) it follows the property:

4*) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $N \in \mathcal{F}(P)$.

For an abstract function $\mathcal{F}$ of R-Mod and $N \subseteq M$ we denote:

$$(C_{\mathcal{F}})_M(N) = \bigcap \{N_\alpha \subseteq M \mid N \subseteq N_\alpha, \quad N_\alpha \in \mathcal{F}(M)\}. \qquad (1.2)$$

The following statement shows the description of the idempotent closure operators of R-Mod by the abstract functions of type $\mathcal{F}_2$.

Theorem 1.2 ([1], Theorem 3.6). *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an antimonotone bijection between the* ***idempotent*** *closure operators of R-Mod and the abstract functions of type $\mathcal{F}_2$ of this category.* □

Using the previous results (Theorems 1.1 and 1.2) in a similar manner the weakly hereditary idempotent closure operators of R-Mod are described. For that the following *property of transitivity* is used:

5) = 5*) If $N \subseteq P \subseteq M$, $N \in \mathcal{F}(P)$ and $P \in \mathcal{F}(M)$, then $N \in \mathcal{F}(M)$.

Theorem 1.3 ([1], Corollaries 4.3, 4.6). a) *The mappings $C \longmapsto \mathcal{F}_1^C$ and $\mathcal{F} \longmapsto C^{\mathcal{F}}$ define a monotone bijection between the* ***weakly hereditary idempotent*** *closure operators of R-Mod and the transitive functions of type $\mathcal{F}_1$ of this category.*

b) *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an antimonotone bijection between the* ***weakly hereditary idempotent*** *closure operators of R-Mod and the transitive functions of type $\mathcal{F}_2$ of this category.* □

In continuation we will use the bijections of Theorems 1.1–1.3 with the intention to characterize the other important classes of closure operators by the abstract functions of types $\mathcal{F}_1$ or $\mathcal{F}_2$, associated to the studied closure operators.

2. Hereditary closure operators

In this section we consider a subclass of the class of weakly hereditary closure operators of R-Mod and give the description of such operators by the abstract functions of type $\mathcal{F}_1$, using the Theorem 1.1.

Definition 2.1. A closure operator $C \in \mathbb{CO}$ is called **hereditary** if for every submodules $L \subseteq N \subseteq M$ the following relation holds:

$$C_N(L) = C_M(L) \cap N. \qquad (2.1)$$

If $C \in \mathbb{CO}$ is hereditary and $N \subseteq M$, then by (2.1) in the situation $N \subseteq C_M(N) \subseteq M$ we have:

$$C_{C_M(N)}(N) = C_M(N) \cap C_M(N) = C_M(N),$$

thus C is weakly hereditary. So is true

Lemma 2.1. *Every hereditary closure operator of R-Mod is weakly hereditary.* □

Therefore, every hereditary closure operator $C \in \mathbb{CO}$ can be completely described by the associated function $\mathcal{F}_1^C$ (Theorem 1.1), which is an abstract function of type $\mathcal{F}_1$.

For an arbitrary weakly hereditary closure operator $C \in \mathbb{CO}$ we will find necessary and sufficient condition to $\mathcal{F}_1^C$ for the operator C to be hereditary. With this purpose we consider the condition $4^*)$ mentioned in the Section 1:

$4^*)$ If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $N \in \mathcal{F}(P)$.

Proposition 2.2. *Let C be an arbitrary weakly hereditary closure operator of R-Mod and $\mathcal{F}_1^C$ be the associated function of type $\mathcal{F}_1$ (Theorem 1.1). Then the operator C is hereditary if and only if the function $\mathcal{F}_1^C$ satisfies the condition $4^*)$.*

Proof. $(\Rightarrow)$ Let C be a hereditary closure operator of R-Mod, $N \subseteq P \subseteq M$ and $N \in \mathcal{F}_1^C(M)$. Then $C_M(N) = M$ and from (2.1) it follows that $C_P(N) = C_M(N) \cap P = M \cap P = P$, i.e. $N \in \mathcal{F}_1^C(P)$ and $\mathcal{F}_1^C$ satisfies the condition $4^*)$.

$(\Leftarrow)$ Let C be a weakly hereditary closure operator of R-Mod and the associated function $\mathcal{F}_1^C$ satisfies the condition $4^*)$. Then C can be re-established by $\mathcal{F}_1^C$ (see (1.1)) and for submodules $L \subseteq N \subseteq M$ we have:

$$C_M(L) = \sum \{K_\alpha \subseteq M \mid L \subseteq K_\alpha, \ \ L \in \mathcal{F}_1^C(K_\alpha)\},$$

$$C_N(L) = \sum \{K'_\alpha \subseteq N \mid L \subseteq K'_\alpha, \ \ L \in \mathcal{F}_1^C(K'_\alpha)\}.$$

Since $\mathcal{F}_1^C$ satisfies the condition 1) (Definition 1.4), from the relations $L \in \mathcal{F}_1^C(K_\alpha)$ $(\alpha \in \mathfrak{A})$ it follows that $L \in \mathcal{F}_1^C(\sum_{\alpha \in \mathfrak{A}} K_\alpha)$, where $\sum_{\alpha \in \mathfrak{A}} K_\alpha = C_M(L)$. Using the condition $4^*)$ in the situation $L \subseteq C_M(L) \cap N \subseteq C_M(L)$, from the relation $L \in \mathcal{F}_1^C(\sum_{\alpha \in \mathfrak{A}} K_\alpha)$ we conclude that $L \in \mathcal{F}_1^C(C_M(L) \cap N)$. Therefore the submodule $C_M(L) \cap N$, which is contained in N and contains L, is one of the submodules K'_α from the definition of $C_N(L)$. This implies $C_M(L) \cap N \subseteq C_M(L)$, the inverse inclusion being trivial. Thus the relation (2.1) holds, i.e. C is hereditary. □

From Theorem 1.1 and Proposition 2.2 it follows

Corollary 2.3. *The mappings* $C \longmapsto \mathcal{F}_1^C$ *and* $\mathcal{F} \longmapsto C^{\mathcal{F}}$ *define a monotone bijection between the* ***hereditary*** *closure operators of R-Mod and the abstract functions of type* $\mathcal{F}_1$*, which satisfy the condition* 4^*). □

3. Weakly hereditary maximal closure operators

In this section we consider the weakly hereditary maximal ([2]) closure operators C of R-Mod and show the condition, satisfied by the respective abstract functions $\mathcal{F}_1^C$.

Definition 3.1. A closure operator $C \in \mathbb{CO}$ is called **maximal** if for every $N \subseteq M$ the following relation holds:

$$C_M(N) \,/\, N = C_{M/N}(\bar{0}) \tag{3.1}$$

The role of such closure operators will be specified in other part of this work, studying the relation between the closure operators and preradicals of R-Mod. Now we remark that the maximal closure operators can be described by the following condition: if $K \subseteq N \subseteq M$, then

$$C_M(N) \,/\, K = C_{M/K}(N/\,K). \tag{3.2}$$

Lemma 3.1. *A closure operator* $C \in \mathbb{CO}$ *is maximal if and only if* C *satisfies the condition* (3.2).

Proof. (⇒) Let C be a maximal closure operator and $K \subseteq N \subseteq M$. From (3.1), substituting M by M/K and N by N/K, we obtain

$$[C_{M/K}(N/\,K)] \,/\, (N/\,K) = C_{(M/K)\,/\,(N/K)}(\bar{\bar{0}}). \tag{3.3}$$

On the other hand, from the isomorphism $(M/\,K)\,/\,(N/\,K) \cong M/\,N$ we have $[C_M(N)\,/\,K]\,/\,(N/\,K) \cong C_M(N)\,/\,N$. From (3.1) and the mentioned isomorphism it follows that

$$[C_M(N)\,/\,K]\,/\,(N/\,K) = C_{(M/K)\,/\,(N/K)}(\bar{\bar{0}}). \tag{3.4}$$

From (3.3) and (3.4) now we have

$$[C_M(N)\,/\,K]\,/\,(N/\,K) = [C_{M/K}(N/\,K)]\,/\,(N/\,K),$$

therefore $C_M(N)\,/\,K = C_{M/K}(N/\,K)$, i.e. (3.2) holds.

(⇐) If $K = N$, then (3.2) implies (3.1). □

In what follows we consider the weakly hereditary maximal closure operators C and using the Theorem 1.1 we give the characterization of such operators by the associated functions $\mathcal{F}_1^C$. For that we use the following modification of the condition 3^*) (Definition 1.5):

$\bar{3}$) If $K \subseteq N \subseteq M$ and $N/K \in \mathcal{F}(M/K)$, then $N \in \mathcal{F}(M)$.

Proposition 3.2. *Let $C \in \mathbb{CO}$ be a weakly hereditary closure operator. Then C is maximal if and only if the associated function $\mathcal{F}_1^C$ satisfies the condition $\bar{3}$).*

Proof. ($\Rightarrow$) Let C be a maximal closure operator and $K \subseteq N \subseteq M$. Then (3.2) holds (Lemma 3.1). If $N/K \in \mathcal{F}_1^C(M/K)$, then $C_{M/K}(N/K) = M/K$ and by (3.2) we have $C_M(N)/K = M/K$, thus $C_M(N) = M$, i.e. $N \in \mathcal{F}_1^C(M)$ and $\mathcal{F}_1^C$ satisfies the condition $\bar{3}$).

($\Leftarrow$) Let C be a weakly hereditary closure operator for which the function $\mathcal{F}_1^C$ satisfies $\bar{3}$). Then C can be re-established by $\mathcal{F}_1^C$ and in the situation $K \subseteq N \subseteq M$ we have:

$$C_M(N) = \sum \{M_\alpha \subseteq M \mid N \subseteq M_\alpha, \quad N \in \mathcal{F}_1^C(M_\alpha)\},$$

$$C_{M/K}(N/K) = \sum \{M'_\alpha/K \subseteq M/K \mid N/K \subseteq M'_\alpha/K, \quad N/K \in \mathcal{F}_1^C(M'_\alpha/K)\}.$$

To prove the maximality of C it is sufficient to verify in (3.2) the inclusion:

$$C_M(N)/K \supseteq C_{M/K}(N/K). \tag{3.5}$$

From the relations $N/K \in \mathcal{F}_1^C(M'_\alpha/K)$ $(\alpha \in \mathfrak{A})$ by the condition $\bar{3}$) it follows that $N \in \mathcal{F}_1^C(M'_\alpha)$ $(\alpha \in \mathfrak{A})$. Using the condition 1) of $\mathcal{F}_1^C$ (Definition 1.4), now we have $N \in \mathcal{F}_1^C(\sum_{\alpha \in \mathfrak{A}} M'_\alpha)$. Thus $\sum_{\alpha \in \mathfrak{A}} M'_\alpha$ is one of the submodules M_α from the definition of $C_M(N)$, therefore $C_M(N) \supseteq \sum_{\alpha \in \mathfrak{A}} M'_\alpha$ and $C_M(N)/K \supseteq (\sum_{\alpha \in \mathfrak{A}} M'_\alpha)/K$. But $(\sum_{\alpha \in \mathfrak{A}} M'_\alpha)/K = \sum_{\alpha \in \mathfrak{A}} (M'_\alpha/K) = C_{M/K}(N/K)$, so $C_M(N)/K \supseteq C_{M/K}(N/K)$, proving (3.5). Therefore C is maximal. □

Combining Theorem 1.1 and Proposition 3.2 we obtain

Corollary 3.3. *The mappings $C \longmapsto \mathcal{F}_1^C$ and $\mathcal{F} \longmapsto C^{\mathcal{F}}$ define a monotone bijection between the **weakly hereditary maximal** closure operators of R-Mod and the abstract functions of type $\mathcal{F}_1$ which satisfies the conditions $\bar{3}$).* □

4. Hereditary maximal closure operators

Restricting the bijection of Corollary 3.3 and using the characterization of hereditary closure operators (Corollary 2.3), we now obtain the following result on the hereditary maximal closure operators of R-Mod.

Proposition 4.1. *The mappings* $C \longmapsto \mathcal{F}_1^C$ *and* $\mathcal{F} \longmapsto C^{\mathcal{F}}$ *define a monotone bijection between the* ***hereditary maximal*** *closure operators of* R*-Mod and the functions of type* $\mathcal{F}_1$ *of* R*-Mod, which satisfy the conditions* $4^*)$ *and* $\overline{3})$*.* □

It is interesting the fact that the hereditary maximal closure operators of R-Mod can be described by well known sets of left ideals of R, namely by the preradical filters (or left linear topologies) of R ([5, 7, 8, 9]).

Definition 4.1. A set $\mathcal{E} \subseteq \mathbb{L}(_R R)$ of left ideals of the ring R is called **preradical filter** of R if it satisfies the following conditions:

(a_1) If $I \in \mathcal{E}$ and $a \in R$, then $(I:a) \in \mathcal{E}$ (where $(I:a) = \{r \in R \mid r\,a \in I\}$);
(a_2) If $I \in \mathcal{E}$ and $I \subseteq J$ $(J \in \mathbb{L}(_R R))$, then $J \in \mathcal{E}$;
(a_3) If $I, J \in \mathcal{E}$, then $I \cap J \in \mathcal{E}$.

Proposition 4.2. *Let* C *be a hereditary maximal closure operator of* R*-Mod. Then the set of left ideals*

$$\mathcal{E}^C = \mathcal{F}_1^C(_R R) = \{I \in \mathbb{L}(_R R) \mid C_R(I) = R\}$$

is a preradical filter of R*.*

Proof. (a_1) Let $I \in \mathcal{F}_1^C(_R R)$ and $a \in R$. Consider the R-morphism:

$$f : {}_R R \longrightarrow (R\,a + I)\,/\,I \subseteq R\,/\,I, \quad f(r) = ra + I \quad \forall r \in R.$$

Since $\operatorname{Ker} f = \{r \in R \mid ra \in I\} = (I : a)$, we have

$$R\,/\,(I : a) \cong (R\,a + I)\,/\,I \subseteq R\,/\,I.$$

Using the condition $4^*)$ for $\mathcal{F}_1^C$ in the situation $I \subseteq R\,a + I \subseteq R$, from the relation $I \in \mathcal{F}_1^C(_R R)$ we obtain $I \in \mathcal{F}_1^C(R\,a{+}I)$ and by the mentioned isomorphism we conclude that $(I : a) \in \mathcal{F}_1^C(_R R)$.

(a_2) Let $I \in \mathcal{F}_1^C(_R R)$ and $I \subseteq J$. Since $\mathcal{F}_1^C$ satisfies the condition 2) (Definition 1.4), it satisfies also the condition 4) which in the situation $I \subseteq J \subseteq R$ shows that the relation $I \in \mathcal{F}_1^C(_R R)$ implies $J \in \mathcal{F}_1^C(_R R)$.

(a_3) Let $I, J \in \mathcal{F}_1^C(_R R)$. Consider the module $M = (R\,/\,I) \oplus (R\,/\,J)$ and the R-morphism:

$$f:\ {}_RR \longrightarrow {}_RM, \quad f(r) = (r+I,\, r+J) \quad \forall\, r \in R.$$

Then $\operatorname{Ker} f = I \cap J$ and $R/(I\cap J) \cong \operatorname{Im} f \subseteq M$. From the assumption $I, J \in \mathcal{F}_1^C({}_RR)$ it follows that $\bar{0} \in \mathcal{F}_1^C(R/I)$ and $\bar{0} \in \mathcal{F}_1^C(R/J)$ (by condition 3)), therefore the condition 1) implies $\bar{0} \in \mathcal{F}_1^C(M)$. Now we apply the condition 4*) in the situation $\bar{0} \subseteq \operatorname{Im} f \subseteq M$ and from the relation $\bar{0} \in \mathcal{F}_1^C(M)$ we conclude that $\bar{0} \in \mathcal{F}_1^C(\operatorname{Im} f)$. From the indicated isomorphism it follows that $\bar{0} \in \mathcal{F}_1^C(R/(I\cap J))$. Since C is maximal, $\mathcal{F}_1^C$ satisfies the condition $\bar{3}$) (Proposition 3.2) which shows now that $I \cap J \in \mathcal{F}_1^C({}_RR)$. □

Proposition 4.3. *Let $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ be an arbitrary preradical filter of R and*

$$(C^{\mathcal{E}})_M(N) = \{m \in M \mid (N:m) \in \mathcal{E}\} \tag{4.1}$$

where $N \subseteq M$ and $(N:m) = \{r \in R \mid rm \in N\}$. Then $C^{\mathcal{E}}$ is a hereditary maximal closure operator of R-Mod.

Proof. From the conditions (a_1)–(a_3) of the Definition 4.1 it is obvious that the rule (4.1) defines a submodule of M, containing N. The monotony of $C^{\mathcal{E}}$ also follows from the definitions.

To verify the condition (c_3) (Definition 1.1) let $f: M \to M'$ be an R-morphism and $N \subseteq M$. If $m \in (C^{\mathcal{E}})_M(N)$, then $(N:m) \in \mathcal{E}$ and $(N:m) \subseteq (f(N):f(m))$, therefore from (a_2) we have $(f(N):f(m)) \in \mathcal{E}$. Thus $f(m) \in (C^{\mathcal{E}})_{f(M)}(f(N)) \subseteq C^{\mathcal{E}}_{M'}(f(N))$ and $C^{\mathcal{E}}$ is a closure operator of R-Mod.

Moreover, $C^{\mathcal{E}}$ is hereditary: if $L \subseteq N \subseteq M$, then $C_N(L) = C_M(L) \cap N$, since $n \in N$ and $n \in C_M(L)$ imply $(L:n) \in \mathcal{E}$ and $n \in C_N(L)$.

Finally, we verify the maximality of $C^{\mathcal{E}}$. Let $N \subseteq M$. From (4.1) we have:

$$(C^{\mathcal{E}})_{M/N}(\bar{0}) = \{m+N \in M/N \mid (\bar{0}:(m+N)) = (N:m) \in \mathcal{E}\},$$

$$[(C^{\mathcal{E}})_M(N)]/N = \{m+N \in M/N \mid m \in (C^{\mathcal{E}})_M(N)\} =$$

$$= \{m+N \in M/N \mid (N:m) \in \mathcal{E}\}.$$

Therefore $[(C^{\mathcal{E}})_M(N)]/N = (C^{\mathcal{E}})_{M/N}(\bar{0})$, so $C^{\mathcal{E}}$ is maximal by (3.1). □

Proposition 4.4.

a) *If C is a hereditary maximal closure operator of R-Mod, then $C = C^{\mathcal{E}^C}$.*

b) *If $\mathcal{E}$ is a preradical filter of R, then $\mathcal{E} = \mathcal{E}^{C^{\mathcal{E}}}$.*

Proof. a) To verify the inclusion $C_M(N) \subseteq \left(C^{\mathcal{E}^C}\right)_M(N)$ let $N \subseteq M$ and $m \in C_M(N)$. From the isomorphism:

$$R/(N : m) \overset{\varphi}{\cong} (Rm+N)/N \subseteq M/N, \;\; \varphi(r + (N : m)) = rm+N \;\; \forall r \in R$$

we have $(N : m) \in \mathcal{F}_1^C({}_RR)$ if and only if $N \in \mathcal{F}_1^C(Rm + N)$. Since by assumption C is hereditary, it is weakly hereditary $\left(C_{C_M(N)}(N) = C_M(N)\right)$, therefore $N \in \mathcal{F}_1^C\,(C_M(N))$. By the hereditary of C we have also the condition 4*) for $\mathcal{F}_1^C$ (Proposition 2.2) which in the situation $N \subseteq Rm + N \subseteq C_M(N)$, $N \in \mathcal{F}_1^C(C_M(N))$ implies $N \in \mathcal{F}_1^C(Rm + N)$. By the previous remark this means that $(N : m) \in \mathcal{F}_1^C({}_RR)$, i.e. $m \in \left(C^{\mathcal{E}^C}\right)_M(N)$, proving that $C_M(N) \subseteq \left(C^{\mathcal{E}^C}\right)_M(N)$.

For the inverse inclusion let $m \in (C^{\mathcal{E}^C})_M(N)$, i.e. $(N : m) \in \mathcal{F}_1^C({}_RR)$. From the mentioned isomorphism we have $N \in \mathcal{F}_1^C(Rm + N)$. Since C is weakly hereditary, it can be expressed by the function $\mathcal{F}_1^C$ as follows:

$$C_M(N) = \sum \{M_\alpha \subseteq M \mid N \subseteq M_\alpha, \;\; N \in \mathcal{F}_1^C(M_\alpha)\}.$$

From the relation $N \in \mathcal{F}_1^C(Rm + N)$ it is clear that $Rm + N$ is one of M_α from the definition of $C_M(N)$. Therefore $Rm + N \subseteq C_M(N)$, i.e. $m \in C_M(N)$, proving the needed inclusion. This means that $C = C^{\mathcal{E}^C}$.

b) By definitions we have:

$$\mathcal{E}^{C^{\mathcal{E}}} = \mathcal{F}_1^{C^{\mathcal{E}}}({}_RR) = \{I \in \mathbb{L}({}_RR) \mid (C^{\mathcal{E}})_R(I) = {}_RR\} =$$
$$= \{I \in \mathbb{L}({}_RR) \mid (I : r) \in \mathcal{E} \;\; \forall r \in R\}.$$

If $I \in \mathcal{E}$, then by the condition (a_1) we have $(I : r) \in \mathcal{E}$ for every $r \in R$, i.e. $I \in \mathcal{E}^{C^{\mathcal{E}}}$, proving that $\mathcal{E} \subseteq \mathcal{E}^{C^{\mathcal{E}}}$.

If $I \in \mathcal{E}^{C^{\mathcal{E}}}$, then $(I : r) \in \mathcal{E}$ for every $r \in R$, so $(I : 1_R) = I \in \mathcal{E}$. Thus $\mathcal{E}^{C^{\mathcal{E}}} \subseteq \mathcal{E}$. □

From the Propositions 4.2–4.4 we obtain

Corollary 4.5. *The mappings $C \longmapsto \mathcal{E}^C$ and $\mathcal{E} \longmapsto C^{\mathcal{E}}$ define a monotone bijection between the* ***hereditary maximal*** *closure operators of R-Mod and the preradical filters of the ring R.* □

It is a well known fact that every preradical filter defines an unique pretorsion (or: hereditary preradical) of R-Mod ([5, 7, 9]). Thus by Corollary 4.5 there exists a monotone bijection between the hereditary maximal closure operators of R-Mod and the pretorsions of this category. Other method of proving this result will be mentioned studying the relations of $\mathbb{CO}$ with the preradicals of R-Mod.

5. Minimal closure operators

In the previous studies the subclasses of the class of weakly hereditary closure operators were considered: hereditary, weakly hereditary maximal, hereditary maximal. Using the monotone bijection of Theorem 1.1, we obtained the characterizations of these kinds of closure operators C by means of the associated functions $\mathcal{F}_1^C$. In continuation we will operate in a similar manner, investigating some subclasses of the class of **idempotent** closure operators of R-Mod. Using the Theorem 1.2 we will show the characterizations of such types of closure operators C by means of the functions $\mathcal{F}_2^C$, adding some new conditions to the set of conditions 1*), 2*), 3*) (Definition 1.5).

Definition 5.1. A closure operator $C \in \mathbb{CO}$ is called **minimal** if

$$C_M(N) = C_M(0) + N \tag{5.1}$$

for every $N \subseteq M$ ([2]).

We indicate the other form of minimality of C, using the following condition: if $L \subseteq N \subseteq M$, then

$$C_M(N) = C_M(L) + N. \tag{5.2}$$

Lemma 5.1. *A closure operator $C \in \mathbb{CO}$ is minimal if and only if it satisfies the condition* (5.2).

Proof. (⇒) If $C \in \mathbb{CO}$ is minimal and $L \subseteq N \subseteq M$, then $C_M(N) = C_M(0) + N$ and $C_M(L) = C_M(0) + L$, therefore

$$C_M(L) + N = (C_M(0) + L) + N = C_M(0) + N = C_M(N),$$

i.e. (5.2) is true.

(⇐) From (5.2) for $L = 0$ we obtain (5.1). □

Lemma 5.2. *Every minimal closure operator of R-Mod is idempotent.*

Proof. If $C \in \mathbb{CO}$ is minimal, then for every $N \subseteq M$ we have:

$$C_M\left(C_M(N)\right) = C_M\left(C_M(0) + N\right) = C_M(0) + \left(C_M(0) + N\right) =$$
$$= C_M(0) + N = C_M(N). \qquad \square$$

Therefore every minimal closure operator C can be described by the corresponding function $\mathcal{F}_2^C$ (Theorem 1.2), which in this case satisfies the conditions 1*), 2*), 3*) (Definition 1.5).

Let C be an arbitrary idempotent closure operator of R-Mod. Then it is completely determined by the associated function $\mathcal{F}_2^C$. It is natural the question: what condition for $\mathcal{F}_2^C$ must be added to 1^*), 2^*), 3^*) for the operator C to be minimal? The answer can be obtained by the condition 4) of Section 1:

4) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $P \in \mathcal{F}(M)$.

Proposition 5.3. *Let C be an idempotent closure operator of R-Mod. Then the operator C is minimal if and only if the function $\mathcal{F}_2^C$ satisfies the conditions* 4).

Proof. ($\Rightarrow$) Let C be a minimal closure operator, $N \subseteq P \subseteq M$ and $N \in \mathcal{F}_2^C(M)$. Then $C_M(N) = N$, $C_M(0) + N = N$ and $C_M(0) \subseteq N$. Therefore $C_M(P) = C_M(0) + P \subseteq N + P = P$, so $C_M(P) = P$, i.e. $P \in \mathcal{F}_2^C(M)$ and $\mathcal{F}_2^C$ satisfies 4).

($\Leftarrow$) Suppose that C is idempotent and $\mathcal{F}_2^C$ satisfies condition 4). Then C can be re-established by $\mathcal{F}_2^C$ and for $L \subseteq N \subseteq M$ we have:

$$C_M(N) = \cap \, \{N_\alpha \subseteq M \mid N \subseteq N_\alpha, \;\; N_\alpha \in \mathcal{F}_2^C(M)\},$$

$$C_M(L) = \cap \, \{L_\alpha \subseteq M \mid L \subseteq L_\alpha, \;\; L_\alpha \in \mathcal{F}_2^C(M)\}.$$

Since $\mathcal{F}_2^C$ satisfies the condition 1^*) (Definition 1.5), from the relations $L_\alpha \in \mathcal{F}_2^C(M)$ $(\alpha \in \mathfrak{A})$ it follows that $\bigcap_{\alpha \in \mathfrak{A}} L_\alpha \in \mathcal{F}_2^C(M)$, i.e. $C_M(L) \in \mathcal{F}_2^C(M)$. Using the condition 4) in the situation $C_M(L) \subseteq C_M(L) + N \subseteq M$, from the relation $C_M(L) \in \mathcal{F}_2^C(M)$ we have $C_M(L) + N \in \mathcal{F}_2^C(M)$. Therefore $C_M(L) + N$ is one of the submodules N_α from the definition of $C_M(N)$, so $C_M(N) \subseteq C_M(L) + N$, the inverse inclusion being trivial. Thus (5.2) holds, i.e. C is minimal. □

From Theorem 1.2 and Proposition 5.3 it follows

Corollary 5.4. *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an antimonotone bijection between the **minimal** closure operators of R-Mod and the abstract functions of type $\mathcal{F}_2$, which satisfy the condition* 4). □

In the previous situation if we add for C the condition to be weakly hereditary, then for the function $\mathcal{F}_2^C$ we must join the condition of transitivity 5) = 5^*) (Theorem 1.3 b)). In such way by the restriction of the bijection of Corollary 5.4 we obtain

Corollary 5.5. *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an antimonotone bijection between the **weakly hereditary minimal** closure operators of R-Mod and the transitive functions of type $\mathcal{F}_2$ which satisfies the conditions* 4). □

6. Coherediary closure operators

In this section we will consider a new class of closure operators C which is a subclass of idempotent operators, therefore such operators possess the characterization by the associated functions $\mathcal{F}_2^C$ (Theorem 1.2).

Definition 6.1. A closure operator $C \in \mathbb{CO}$ will be called **cohereditary** if for every R-morphism $f : M \to M'$ of R-Mod and every $N \subseteq M$ the following relation holds:

$$f\big(C_M(N)\big) = C_{f(M)}\big(f(N)\big). \tag{6.1}$$

The other form of this condition is the following: for every submodules $K, N \in \mathbb{L}({}_R M)$ is true the relation

$$\big(C_M(N) + K\big) \,/\, K = C_{M/K}\big((N + K) \,/\, K\big). \tag{6.2}$$

Lemma 6.1. *Every cohereditary closure operator $C \in \mathbb{CO}$ is minimal, therefore it is also idempotent.*

Proof. Let $C \in \mathbb{CO}$ be cohereditary and $N \subseteq M$. Applying (6.1) to the natural morphism $\pi_N : M \to M/\,N$ and $N \subseteq C_M(N) \subseteq M$, we have $C_M(N) \,/\, N = C_{M/N}(\bar{0})$. By cohereditary for π_N and $0 \subseteq C_M(0) \subseteq M$ we obtain $\big(C_M(0) + N\big)/\,N = C_{M/N}(\bar{0})$. Therefore $C_M(N)/\,N = \big(C_M(0) + N\big)/\,N$ and $C_M(N) = C_M(0) + N$, i.e. C is minimal. By the Lemma 5.2 C is idempotent. $\square$

Lemma 6.2. *A closure operator C is cohereditary if and only if it is maximal and minimal.*

Proof. $(\Rightarrow)$ Let $C \in \mathbb{CO}$ be cohereditary and $K \subseteq N \subseteq M$. Then from (6.2) $C_M(N)/\,K = C_{M/K}(N/\,K)$, so (3.2) holds, i.e. C is maximal (Lemma 3.1). By the Lemma 6.1 C is also minimal.

$(\Leftarrow)$ Let $C \in \mathbb{CO}$ be maximal and minimal, and $K, N \in \mathbb{L}({}_R M)$. The minimality of C $\big($see (5.2)$\big)$ in the situation $N \subseteq N + K \subseteq M$ implies

$$C_M(N + K) = C_M(N) + (N + K) = C_M(N) + K,$$

therefore $\big(C_M(N) + K\big)/\,K = C_M(N + K) \,/\, K$.

On the other hand, since C is maximal by (3.2) in the situation $K \subseteq N + K \subseteq M$ we have:

$$\big(C_M(N + K)\big)/\,K = C_{M/K}\big((N + K) \,/\, K\big).$$

Comparing with the previous relation we obtain

$$\big(C_M(N) + K\big)/\,K = C_{M/K}\big((N + K) \,/\, K\big),$$

i.e. C is cohereditary. $\square$

Let $C \in \mathbb{CO}$ be an arbitrary cohereditary closure operator. Since it is idempotent (Lemma 6.1), it can be described by the associated function $\mathcal{F}_2^C$ (Theorem 1.2) which in this case satisfies the conditions 1^*), 2^*), 3^*) (Definition 1.5).

For an arbitrary *idempotent* closure operator $C \in \mathbb{CO}$ we consider the associated function $\mathcal{F}_2^C$ and find condition to $\mathcal{F}_2^C$ for C to be cohereditary. For that we use the condition 3) of Definition 1.4:

3) If $f : M \to M'$ is an R-morphism and $N \in \mathcal{F}(M)$, then $f(N) \in \mathcal{F}(f(M))$.

Proposition 6.3. *Let C be an idempotent closure operator of R-Mod and $\mathcal{F}_2^C$ be the associated function* (*of type $\mathcal{F}_2$*)*. Then C is cohereditary if and only if the function $\mathcal{F}_2^C$ satisfies the condition* 3)*.*

Proof. ($\Rightarrow$) Let $C \in \mathbb{CO}$ be cohereditary, $f : M \to M'$ be an R-morphism and $N \in \mathcal{F}_2^C(M)$. Then $C_M(N) = N$ and from (6.1) $f(C_M(N)) = C_{f(M)}(f(N))$, i.e. $f(N) = C_{f(M)}(f(N))$. Thus $f(N) \in \mathcal{F}_2^C(f(M))$ and $\mathcal{F}_2^C$ satisfies the condition 3).

($\Leftarrow$) Let $C \in \mathbb{CO}$ be idempotent and $\mathcal{F}_2^C$ satisfies the condition 3). Then C can be expressed by the function $\mathcal{F}_2^C$. So for $f : M \to M'$ and $N \subseteq M$ we have:

$$C_M(N) = \cap \, \{N_\alpha \subseteq M \mid N \subseteq N_\alpha, \;\; N_\alpha \in \mathcal{F}_2^C(M)\},$$
$$C_{f(M)}(f(N)) = \cap \, \{N'_\alpha \subseteq f(M) \mid f(N) \subseteq N'_\alpha, \;\; N'_\alpha \in \mathcal{F}_2^C(f(M))\}.$$

Since $\mathcal{F}_2^C$ satisfies the condition 1^*) (Definition 1.5), from the relations $N_\alpha \in \mathcal{F}_2^C(M)$ ($\alpha \in \mathfrak{A}$) it follows that $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}_2^C(M)$. Now from the condition 3) of $\mathcal{F}_2^C$ we obtain $f(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha) \in \mathcal{F}_2^C(f(M))$. Therefore the submodule $f(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha) \subseteq f(M)$ which contains $f(N)$ is one the submodules N'_α from the definition of $C_{f(M)}(f(N))$. So $C_{f(M)}(f(N)) \subseteq f(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha) = f(C_M(N))$ and the inverse inclusion is true by (c_3) (Definition 1.1). This proves that $f(C_M(N)) = C_{f(M)}(f(N))$, i.e. C is cohereditary. □

From Theorem 1.2 and Proposition 6.3 it follows

Corollary 6.4. *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an antimonotone bijection between the* ***cohereditary*** *closure operators C of R-Mod and the abstract functions $\mathcal{F}$ of type $\mathcal{F}_2$ which satisfy the condition* 3)*.* □

If we limit the previous bijection to the weakly hereditary operators C, then for the corresponding function $\mathcal{F}_2^C$ we must add the condition of transitivity 5) = 5*) (Theorem 1.3 b)), so we obtain

Corollary 6.5. *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an anti-monotone bijection between the* ***weakly hereditary and cohereditary*** *closure operators of R-Mod and the transitive functions of type $\mathcal{F}_2$ which satisfy the condition* 3)*.* □

References

[1] A.I. Kashu, *Closure operators in the categories of modules. Part* I (*Weakly hereditary and idempotent operators*), Algebra and Discrete Mathematics, v. **15**, №2, 2013, pp. 213–228.

[2] D. Dikranjan, E. Giuli, *Factorizations, injectivity and compactness in categories of modules*, Commun. in Algebra, v. **19**, №1, 1991, pp. 45–83.

[3] D. Dikranjan, E. Giuli, *Closure operators* I, Topology and its Applications, v. **27**, 1987, pp. 129–143.

[4] D. Dikranjan, E. Giuli, W. Tholen, *Closure operators* II, Proc. Intern. Conf. on Categorical Topology, Prague, 1988 (World Scientific Publ., Singapore, 1989).

[5] A.I. Kashu, *Radicals and torsions in modules*, Kishinev, Ştiinţa, 1983 (in Russian).

[6] A.I. Kashu, *Radical closures in categories of modules*, Matem. issled. (Kishinev), v. V, №4(18), 1970, pp. 91-104 (in Russian).

[7] L. Bican, T. Kepka, P. Nemec, *Rings, modules and preradicals*, Marcel Dekker, New York, 1982.

[8] J.S. Golan, *Torsion theories*, Longman Scientific and Technical, New York, 1976.

[9] B. Stenström, *Rings of quotients*, Springer Verlag, 1975.

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