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A maximal T-space of $\mathbb{F}_3[x]_0$

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Abstract. In earlier work, we have established that for any finite field k, the free associative k-algebra on one generator x, denoted by $k[x]_0$, has infinitely many maximal T-spaces, but exactly two maximal T-ideals (each of which is a maximal T-space). However, aside from these two T-ideals, no specific examples of maximal T-spaces of $k[x]_0$ were determined at that time. In a subsequent work, we proposed that for a finite field k of characteristic $p > 2$ and order q, for each positive integer n which is a power of 2, the T-space W_n, generated by $\{\, x + x^{q^n}, x^{q^n+1} \,\}$, is maximal, and we proved that W_1 is maximal. In this note, we prove that for $q = p = 3$, W_2 is maximal.

1. Introduction

Let k be a field, and let A be an associative k-algebra. A. V. Grishin introduced the concept of a T-space of A ([3], [4]); namely, a linear subspace of A that is invariant under the natural action of the transformation monoid of all k-algebra endomorphisms of A. A T-space of A that is also an ideal of A is called a T-ideal of A. For any $H \subseteq A$, the smallest T-space of A containing H shall be denoted by H^S. The set of all T-spaces of A forms a lattice under the inclusion ordering.

We shall let $k[x]_0$ denote the free associative k-algebra on the single generator x (so $k[x]_0 = xk[x]$). It was shown in [1] that there is a natural bijection between the set of maximal T-spaces of any free associative k-algebra and the set of maximal T-spaces of $k[x]_0$. It was also proven that if k is infinite, then any free associative k-algebra has a unique maximal T-ideal, and that maximum T-ideal is also the unique maximal T-space,

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while when k is finite, then any free associative k-algebra has two maximal T-ideals, each of which is a also a maximal T-space, but that now there are infinitely many maximal T-spaces. However, no explicit examples of maximal T-spaces, other than the two maximal T-ideals, were known. In a subsequent paper ([2]), we proposed that for any prime $p > 2$, and any finite field k of characteristic p and order q, the T-spaces W_n, where n is any power of 2 that is greater than 1, defined by

$$W_n = \{\, x + x^{q^n}, x^{q^n+1} \,\}$$

were each maximal in $k[x]_0$. We remark that we had also established in [2] that for $r = 2^s m$, with m odd, $W_r \subseteq W_{2^s}$, and that for $r > s \geq 0$, W_{2^s} is a proper subspace of $k[x]_0$ while $W_{2^r} + W_{2^s} = k[x]_0$. Further, we showed that when $p = 2$, $W_{2^n} = \{\, x + x^{q^{2^n}}, x^{q^{2^n}+1} \,\}^S \subseteq \{\, x + x^q, x^{q^{2^n}+1} \,\}^S$, and that $x + x^q \notin W_{2^n}$ for $n > 0$. For $n = 0$, the two T-spaces coincide, and we did prove that W_{2^0} is a maximal T-space of $k[x]_0$ for any prime p. Our conjecture then was that for $p > 2$, W_{2^n} is a maximal T-space of $k[x]_0$ for every $n \geq 0$, and for $p = 2$, $\{\, x + x^q, x^{q^{2^n}+1} \,\}^S$ is a maximal T-space of $k[x]_0$ for each $n \geq 0$. It does not appear that the methods that were used to prove that W_1 is maximal extend to W_n for $n \geq 2$. Our objective in this paper is to prove that for $q = p = 3$, W_2 is maximal in $\mathbb{F}_3[x]_0$, thereby lending some additional support for the conjecture. We remark that these values were chosen simply because the computation was feasible, and even then, it was only feasible because we were able to develop a reduction strategy that we were unable to duplicate even in the case $q = p = 5$ for example.

The following notion will be of fundamental importance in our work. Recall that for any finite field k of order q, monomials $u_i \in k[x]_0$ and $\alpha_i \in k$, $1 \leq i \leq t$, $f = \sum_{i=1}^t \alpha_i u_i$ are said to be q-homogeneous if for each i, j with $1 \leq i, j \leq t$, $\deg(u_i) \equiv \deg(u_j) \pmod{q-1}$. For any$T$-space T of $k[x]_0$, each $f \in T$ can be written as a sum of q-homogeneous elements, each of which belongs to T. Consequently, if for a positive integer i, we let T_i denote the linear space

$$T_i = \{\, 0 \,\} \cup \{\, f \in T - \{\, 0 \,\} \mid \deg(f) \equiv i \pmod{q-1} \,\},$$

then T is the direct sum of $T_1, T_2, \ldots, T_{q-1}$. In particular, if we let $H_i = (k[x]_0)_i$, then $k[x]_0$ is the direct sum of $H_1, \ldots, H_{q-1}$, and $T_i = T \cap H_i$. For each i, let $\pi_i \colon k[x]_0 \to H_i \subseteq k[x]_0$ denote the i^{th} projection mapping. Then $\pi_i(T) = T_i \subseteq T$; that is, each T-space of $k[x]_0$ is invariant under each of these projection mappings.

It was established in [2] that for $n \geq 1$ any power of 2, the T-ideal U_n that is generated by the set $\{\, x - x^{q^{2n}} \,\}$ is contained in W_n, and moreover, that the set $\{\, (x^{q^{2n}} - x)x^i \mid i \geq 0 \,\}$ is a linear basis for U_n. As well, it was proven that the set

$$\{\, x^{iq^n+j} + x^{i+jq^n} \mid q^n > i > j \geq 0 \,\} \cup \{\, (x^{q^n+1})^i \mid 1 \leq i \leq q^n - 1 \,\}$$

is linearly independent in $k[x]_0$, and so as a k-vector space, V_n, the linear span of this set in $k[x]_0$, has dimension $\binom{q^n}{2} + q^n - 1$. It was also proven that as linear spaces, $W_n = V_n \oplus U_n$, and that the set

$$B_n = \{\, x^{i+q^n j} \mid q^n > i > j \geq 0 \,\}$$

is linearly independent in $k[x]_0$, and Y_n, the linear span of B_n in $k[x]_0$, is complementary to W_n; that is, $k[x]_0 = Y_n \oplus W_n = Y_n \oplus V_n \oplus U_n$. Thus in order to establish that W_n is maximal in $k[x]_0$, it suffices to show that for any nonzero $f \in Y_n$, $W_n + \{\, f \,\}^S = k[x]_0$. Moreover, by the preceding discussion on q-homogeneity, it will suffice to prove that for any nonzero q-homogeneous polynomial $f \in Y_n$, $W_n + \{\, f \,\}^S = k[x]_0$.

Our general approach in this note will be to consider a T-space of the form $W_n + \{\, f \,\}^S$ for nonzero $f \in Y_n$. We shall take advantage of the following observation. For θ any k-linear operator on $k[x]_0 = Y_n \oplus W_n$ that preserves every T-space (for example, any algebra endomorphism, or any of the projection mappings $\pi_i : k[x]_0 \to H_i \subseteq k[x]_0$), then we have a linear operator on Y_n given by

$$Y_n \hookrightarrow Y_n \oplus W_n \xrightarrow{\theta} Y_n \oplus W_n \xrightarrow{\pi_Y} Y_n,$$

where π_Y is the projection mapping onto the subspace Y_n. The value to us of this observation is the following. Let V be any T-space of $k[x]_0$ that contains W_n. Then for $v \in V$, we have $v = y + w$ for unique $y \in Y_n$ and $w \in W_n$. Since $W_n \subseteq V$, we have $y = v - w \in V$, and so $y \in V \cap Y_n$. But then $y + W_n \subseteq V$, and so $V = (V \cap Y_n) \oplus W_n$. Since θ preserves V, we have $\theta(y) = y' + w' \in V$, where $y' \in Y_n$ and $w' \in W_n$. Thus $y' \in V$, and so $\pi_Y \circ \theta(y) = y' \in V \cap Y_n$; that is, $\pi_Y \circ \theta(V \cap Y_n) \subseteq V \cap Y_n$.

2. W_2 is maximal in the case $k = \mathbb{F}_3$

We specialize the results described in the preceding section to the case when $k = \mathbb{F}_3$ and $n = 2$. Since $q = p = 3$, $q - 1 = 2$ and thus q-homogeneity is simply parity; that is, $f \in \mathbb{F}_3[x]_0$ is 3-homogeneous if and only if all of its monomials have degrees of the same parity. We have $W_2 = V_2 \oplus U_2$, and $\mathbb{F}_3[x]_0 = Y_2 \oplus W_2$, where each of V_2 and Y_2 are finite dimensional (and thus finite) linear subspaces of $\mathbb{F}_3[x]_0$. Y_2 has dimension

$\binom{q^2}{2} = \binom{9}{2} = 36$, with basis B_2 the union of

$$O = \{\, x, x^3, x^5, x^7, x^{11}, x^{13}, x^{15}, x^{17}, x^{21}, x^{23}, \\ x^{25}, x^{31}, x^{33}, x^{35}, x^{41}, x^{43}, x^{51}, x^{53}, x^{61}, x^{71} \,\}$$

a set of size 20, and

$$E = \{\, x^2, x^4, x^6, x^8, x^{12}, x^{14}, x^{16}, x^{22}, x^{24}, x^{26}, x^{32}, x^{34}, x^{42}, x^{44}, x^{52}, x^{62} \,\},$$

a set of size 16. We may regard $\pi_Y \circ \pi_1$ as the projection from $\mathbb{F}_3[x]_0$ onto $\langle\, O \,\rangle$, and so we shall refer to $\pi_Y \circ \pi_1$ as π_O for convenience. Similarly, we shall refer to $\pi_Y \circ \pi_2$ as π_E.

There are two families of algebra endomorphisms that shall be of particular interest: those determined by mapping x to x^r or $x + x^r$ for r a positive integer. As we shall make frequent use of such endomorphisms, we introduce notation for them.

Definition 2.1. *For each positive integer r, let ϵ_r and ι_r be the algebra endomorphisms of $\mathbb{F}_3[x]_0$ that are determined by sending x to x^r, respectively, to $x + x^r$.*

It will be convenient to note that for any T-space W that contains W_2, $x^{80+i} \equiv x^i \pmod{W}$ and $x^{10i} \equiv 0 \pmod{W}$ for each positive integer i, and for $9 > i > j \geq 0$, $x^{9i+j} \equiv -x^{i+9j} \pmod{W}$.

Proposition 2.1. $W_2 + \{\, x^5 \,\}^S = \mathbb{F}_3[x]_0$.

Proof. Let $T = W_2 + \{\, x^5 \,\}^5$, so T is a T-space that contains W_2. We prove that $x \in T$. Since T is a T-space, $\iota_2(T) \subseteq T$. In particular, we have $\iota_2(x^5) = x^5 - x^6 + x^7 + x^8 - x^9 + x^{10} \in T$. As $x^5, x^{10} \in T$, we have $-x^6 + x^7 + x^8 - x^9 \in T$. The 3-homogeneous components of $-x^6 + x^7 + x^8 - x^9$ are $-x^6 + x^8$ and $x^7 - x^9$, so $-x^6 + x^8 \in T$ and $x^7 - x^9 \in T$. As well, $x + x^9 \in T$, so $x^7 - x^9 \equiv x^7 + x \pmod{T}$ and thus $x + x^7 \in T$. But then

$$\iota_2(x + x^7) = x + x^2 + x^7 + x^8 - x^{10} - x^{11} + x^{13} + x^{14} \in T.$$

Since $x + x^7 \in T$, $x^2 + (x^2)^7 = x^2 + x^{14} \in T$, and $x^{10} \in T$, we obtain that $x^8 - x^{11} + x^{13} \in T$. But then the 3-homogeneous component $x^8 \in T$. We now have that both x^8 and $-x^6 + x^8$ are elements of T, and so $x^6 \in T$ and thus $\iota_2(x^6) = x^6 - x^9 + x^{12} \in T$. Since x^6 and $(x^2)^6 = x^{12} \in T$, we finally obtain $x^9 \in T$. As $x \equiv x^9 \pmod{T}$, it follows that $x \in T$, as required. $\square$

Let $D_5 = \langle\, x^5, x^{15}, x^{25}, x^{35} \,\rangle$. Our next major objective is to prove that for any nonzero $f \in D_5$, $W_2 + \{\, f \,\}^S = \mathbb{F}_3[x]_0$. The fact that 5 is a prime factor of $q^{2n} - 1 = 80$ is at the heart of this observation.

The following result will prove to be useful.

Lemma 2.1. *For any $\alpha, \beta, \gamma, \delta \in \mathbb{F}_3$,*

$$\begin{aligned}\epsilon_5 \circ \pi_O \circ \iota_2(\alpha x^5 + \beta x^{15} + \gamma x^{25} + \delta x^{35}) \equiv (\alpha+\gamma+\delta)x^5 + (\beta+\gamma-\delta)x^{15} \\ + (\alpha+\beta-\gamma)x^{25} + (\alpha-\beta-\delta)x^{35} \ (mod\, W_2).\end{aligned}$$

Proof. We shall compute the effect of applying $\epsilon_5 \circ \pi_O \circ \iota_2$ to each of x^5, x^{15}, x^{25}, and x^{35}, with each computation carried out in stages. To begin with, we have

$$\begin{aligned}x^5 &\overset{\iota_2}{\mapsto} (x+x^2)^5 = (x+x^2)^3(x+x^2)^2 = (x^3+x^6)(x^2+2x^3+x^4) \\ &\overset{\pi_O}{\mapsto} x^5 + x^7 + 2x^9 \overset{\epsilon_5}{\mapsto} x^{25} + x^{35} + 2x^{45} \\ &\overset{W_2}{\equiv} x^5 + x^{25} + x^{35}\end{aligned}$$

and thus

$$\begin{aligned}x^{15} &\overset{\iota_2}{\mapsto} ((x+x^2)^5)^3 = (x^9+x^{18})(x^6+2x^9+x^{12}) \\ &\overset{\pi_O}{\mapsto} x^{15} + x^{21} + 2x^{27} \overset{\epsilon_5}{\mapsto} x^{75} + x^{105} + 2x^{135} \\ &\overset{W_2}{\equiv} x^{15} + x^{25} - x^{35}.\end{aligned}$$

Next, we have

$$\begin{aligned}x^{25} &\overset{\iota_2}{\mapsto} (x+x^2)^{18}(x+x^2)^6(x+x^2) = (x^9+x^{18})^2(x^3+x^6)^2(x+x^2) \\ &= (x^{18}+2x^{27}+x^{36})(x^6+2x^9+x^{12})(x+x^2) \\ &= (x^{18}+2x^{27}+x^{36})(x^7+2x^{10}+x^{13}+x^8+2x^{11}+x^{14}) \\ &\overset{\pi_O}{\mapsto} x^{25} + x^{31} + 2x^{29} + x^{37} + 2x^{35} + 2x^{41} + x^{43} + x^{49} + 2x^{47} \\ &\overset{\epsilon_5}{\mapsto} x^{125} + x^{155} + 2x^{145} + x^{185} + 2x^{175} + 2x^{205} + x^{215} + x^{245} + 2x^{235} \\ &\overset{W_2}{\equiv} -x^5 - x^{35} + x^{25} + x^{25} + 2x^{15} + x^5 - x^{15} + x^5 + x^{35} \\ &= x^5 + x^{15} - x^{25}.\end{aligned}$$

Finally, we have

$$\begin{aligned}x^{35} &\overset{\iota_2}{\mapsto} (x+x^2)^{27}(x+x^2)^6(x+x^2)^2 \\ &= (x^{27}+x^{54})(x^6+2x^9+x^{12})(x^2+2x^3+x^4) \\ &= (x^{33}+2x^{36}+x^{39}+x^{60}+2x^{63}+x^{66})(x^2+2x^3+x^4) \\ &\overset{\pi_O}{\mapsto} x^{35} + x^{41} + 2x^{65} + x^{39} + 2x^{63} + 2x^{69} + x^{37} + x^{43} + 2x^{67} \\ &\overset{\epsilon_5}{\mapsto} x^{175} + x^{205} + 2x^{325} + x^{195} + 2x^{315} + 2x^{345} + x^{185} + x^{215} + 2x^{335}\end{aligned}$$

$$\overset{W_2}{\equiv} x^{15} - x^5 + 2x^5 + x^{35} + x^{35} + 2x^{25} + x^{25} + 2x^{15} + 2x^{15}$$
$$= x^5 - x^{15} - x^{35}.$$

The result follows now by linearity. □

Corollary 2.1. $W_2 + \{\, x^{15} + x^{25} \,\}^S = \mathbb{F}_3[x]_0$.

Proof. Let $U = W_2 + \{\, x^{15} + x^{25} \,\}^S$. We have $\epsilon_3(x^{15} + x^{25}) = x^{45} + x^{75} \equiv -x^5 - x^{35} \pmod{W_2}$, so $x^5 + x^{35} \in U$. As well, we have $\epsilon_5 \circ \pi_O \circ \iota_2(x^{15} + x^{25}) \in T$, and by Lemma 2.1, $\epsilon_5 \circ \pi_O \circ \iota_2(x^{15} + x^{25}) = x^5 - x^{15} - x^{35}$. Since $x^5 + x^{35}$, $x^5 - x^{15} - x^{35} \in T$, we have $-(x^5 + x^{35} + x^5 - x^{15} - x^{35}) = x^5 + x^{15} \in T$. Thus $x^5 + x^{15} - (x^{15} + x^{25}) = x^5 - x^{25} \in T$, and so $\epsilon_3(x^5 - x^{25}) = x^{15} - x^{75} \in T$. Since $x^{75} \equiv -x^{35} \pmod{T}$, we then obtain that $x^{15} + x^{35} \in T$. Finally, as both $x^5 - x^{15} - x^{35}$ and $x^{15} + x^{35}$ belong to T, we have $x^5 \in T$. Now by Proposition 2.1, we obtain $W_2 + \{\, x^{15} + x^{25} \,\}^S = \mathbb{F}_3[x]_0$, as required. □

Proposition 2.2. *For any nonzero $f \in D_5$, $W_2 + \{\, f \,\}^S = \mathbb{F}_3[x]_0$.*

Proof. Let $f = \alpha x^5 + \beta x^{15} + \gamma x^{25} + \delta x^{35}$ and set $T = W_2 + \{\, f \,\}^S$, so that T is a T-space containing W_2. Note that modulo W_2 and thus modulo T, we have $x^{45} \equiv -x^5$, $x^{75} \equiv -x^{35}$, and $x^{105} \equiv x^{25}$, while $x^{r+80k} \equiv x^r$ for all positive integers k, r.

We have $\epsilon_7(f) = \alpha x^{35} + \beta x^{105} + \gamma x^{175} + \delta x^{245} \in T$, and since $\alpha x^{35} + \beta x^{105} + \gamma x^{175} + \delta x^{245} \equiv \alpha x^{35} + \beta x^{25} + \gamma x^{15} + \delta x^5$, we have $\delta x^5 + \gamma x^{15} + \beta x^{25} + \alpha x^{35} \in T$. Sum f and the latter element to obtain that

$$(\alpha + \delta)(x^5 + x^{35}) + (\beta + \gamma)(x^{15} + x^{25}) \in T. \tag{1}$$

Apply ϵ_3 to the expression in (1) to obtain that

$$(\alpha + \delta)(x^{15} + x^{105}) + (\beta + \gamma)(x^{45} + x^{75}) \in T \tag{1}$$

and thus

$$(\alpha + \delta)(x^{15} + x^{25}) - (\beta + \gamma)(x^5 + x^{35}) \in T. \tag{2}$$

Multiply (1) by $\beta + \gamma$ and (2) by $\alpha + \delta$ and sum to obtain that

$$[(\alpha + \delta)^2 + (\beta + \gamma)^2](x^{15} + x^{25}) \in T.$$

If $(\alpha + \delta)^2 + (\beta + \gamma)^2 \neq 0$, then $x^{15} + x^{25} \in T$, in which case it follows from Corollary 2.1 that $T = \mathbb{F}_3[x]_0$. Suppose that $(\alpha + \delta)^2 + (\beta + \gamma)^2 = 0$. Since $\alpha + \delta, \beta + \gamma \in \mathbb{F}_3$, this implies that $\alpha + \delta = 0$ and $\beta + \gamma = 0$, and so $f = \alpha x^5 + \beta x^{15} - \beta x^{25} - \alpha x^{35}$, with not both α and β equal to 0. Apply

Lemma 2.1 to f to obtain that

$$-\beta x^5 + \alpha x^{15} + (\alpha - \beta)x^{25} - (\alpha + \beta)x^{35} \in T. \tag{3}$$

Add f and the expression in (3) to obtain that

$$(\alpha - \beta)(x^5 + x^{35}) + (\alpha + \beta)(x^{15} + x^{25}) \in T. \tag{4}$$

As well, we have $\epsilon_5(f) = \alpha x^{25} + \beta x^{75} - \beta x^{125} - \alpha x^{175} \in T$, and so

$$\beta x^5 - \alpha x^{15} + \alpha x^{25} - \beta x^{35} \in T \tag{5}$$

Add f to the expression in (5) to obtain that

$$(\alpha + \beta)(x^5 - x^{35}) - (\alpha - \beta)(x^{15} - x^{25}) \in T. \tag{6}$$

After applying ϵ_3 to the expression in (6), we find that

$$(\alpha + \beta)(x^{15} - x^{25}) + (\alpha - \beta)(x^5 - x^{35}) \in T. \tag{7}$$

Now add the expressions in (4) and (7) to obtain that

$$(-\alpha + \beta)x^5 + (-\alpha - \beta)x^{15} \in T, \tag{8}$$

and so $\epsilon_3((-\alpha + \beta)x^5 + (-\alpha - \beta)x^{15}) \in T$. Thus

$$(-\alpha + \beta)x^{15} + (\alpha + \beta)x^5 \in T. \tag{9}$$

Add the expressions in (8) and (9) to find that $-\beta x^5 + \alpha x^{15} \in T$. Then (8) together with this fact results in

$$\alpha x^5 + \beta x^{15} \in T. \tag{10}$$

Apply ϵ_3 to the expression in (10) to obtain that

$$-\beta x^5 + \alpha x^{15} \in T. \tag{11}$$

Now multiply (10) by α, multiply (11) by β, and take the difference to obtain that $(\alpha^2 + \beta^2)x^5 \in T$. Since $\alpha, \beta \in \mathbb{F}_3$, not both zero, it follows that $\alpha^2 + \beta^2 \neq 0$ and thus $x^5 \in T$. Now by Proposition 2.1, we have $T = \mathbb{F}_3[x]_0$, as required. □

Proposition 2.3. *For any nonzero* $f \in \langle O \rangle$, $W_2 + \{ f \}^S = \mathbb{F}_3[x]_0$.

Proof. Our approach is to prove that for any nonzero $f \in \langle O \rangle$, there exists an algebra endomorphism θ of $\mathbb{F}_3[x]_0$ such that $\pi_O \circ \theta(f)$ is a nonzero element of D_5, in which case it follows from Proposition 2.2

that $W_2 + \{ f \}^S = \mathbb{F}_3[x]_0$. Since the image of the endomorphism ϵ_5 is D_5, we shall regard ϵ_5 as a linear map from $\mathbb{F}_3[x]_0$ into D_5. In particular, when we apply ϵ_5 to elements of $\langle O \rangle$, we find (using the programming language Python) that $\epsilon_5(\sum_{x_i\in O} \alpha_i x^i)) =$

$$\begin{aligned}&(\alpha_1+\alpha_{17}-\alpha_{25}+\alpha_{33}-\alpha_{41})x^5+(\alpha_3-\alpha_{11}+\alpha_{35}-\alpha_{43}+\alpha_{51})x^{15}\\&\quad+(\alpha_5-\alpha_{13}+\alpha_{21}+\alpha_{53}-\alpha_{61})x^{25}+(\alpha_7-\alpha_{15}+\alpha_{23}-\alpha_{31}+\alpha_{71})x^{35}.\end{aligned}$$

As well, for any algebra endomorphism τ of $\mathbb{F}_3[x]_0$, $\epsilon_5\circ\tau$ can be considered to be a linear map from $\mathbb{F}_3[x]_0$ to D_5. Let $\tau_1 = \epsilon_5\circ\pi_O\circ\iota_2$, and regard τ_1 as a linear map from $\mathbb{F}_3[x]_0$ to D_5. When we apply τ_1 to elements of $\langle O \rangle$, we find (again, using the programming language Python) that $\tau_1(\sum_{x_i\in O} \alpha_i x^i)) = a_1x^5 + a_2x^{15} + a_3x^{25} + a_4x^{35}$, where

$$\begin{aligned}a_1 &= \alpha_1+\alpha_5-\alpha_{17}-\alpha_{21}-\alpha_{23}+\alpha_{25}+\alpha_{33}+\alpha_{35}-\alpha_{43}+\alpha_{51}-\alpha_{53}-\alpha_{61}+\alpha_{71}\\a_2 &= \alpha_3+\alpha_7-\alpha_{11}+\alpha_{15}-\alpha_{17}-\alpha_{23}+\alpha_{25}-\alpha_{35}+\alpha_{41}-\alpha_{51}+\alpha_{53}+\alpha_{61}+\alpha_{71}\\a_3 &= \alpha_5-\alpha_7+\alpha_{11}-\alpha_{13}+\alpha_{15}-\alpha_{17}+\alpha_{21}-\alpha_{25}-\alpha_{31}\\&\qquad+\alpha_{43}-\alpha_{51}-\alpha_{53}-\alpha_{61}+\alpha_{71}\\a_4 &= \alpha_5+\alpha_7+\alpha_{13}-\alpha_{15}-\alpha_{17}+\alpha_{21}+\alpha_{23}-\alpha_{31}-\alpha_{33}\\&\qquad-\alpha_{35}+\alpha_{41}+\alpha_{51}-\alpha_{53}-\alpha_{71}.\end{aligned}$$

Next, let $\tau_2 = \tau_1\circ\pi_O\circ\iota_2$. Again using Python, we compute that

$$\tau_2(\sum_{x_i\in O}\alpha_i x^i) = b_1x^5 + b_2x^{15} + b_3x^{25} + b_4x^{35},$$

where

$$\begin{aligned}b_1 &= \alpha_1-\alpha_5+\alpha_{11}-\alpha_{13}-\alpha_{15}-\alpha_{17}+\alpha_{21}-\alpha_{31}+\alpha_{33}\\&\qquad-\alpha_{35}+\alpha_{41}-\alpha_{51}-\alpha_{53}+\alpha_{71}\\b_2 &= \alpha_3+\alpha_5-\alpha_7-\alpha_{11}-\alpha_{13}-\alpha_{15}+\alpha_{17}-\alpha_{25}+\alpha_{31}\\&\qquad+\alpha_{33}+\alpha_{43}-\alpha_{51}+\alpha_{53}+\alpha_{71}\\b_3 &= -\alpha_{11}-\alpha_{15}-\alpha_{21}+\alpha_{25}-\alpha_{31}+\alpha_{35}-\alpha_{41}+\alpha_{51}-\alpha_{61}+\alpha_{71}\\b_4 &= -\alpha_5-\alpha_7+\alpha_{13}+\alpha_{17}+\alpha_{23}-\alpha_{25}+\alpha_{33}+\alpha_{35}+\alpha_{43}-\alpha_{53}.\end{aligned}$$

Let $\tau_3 = \tau_2\circ\pi_O\circ\iota_2$, and compute

$$\tau_3(\sum_{x_i\in O}\alpha_i x^i) = c_1x^5 + c_2x^{15} + c_3x^{25} + c_4x^{35},$$

where

$$\begin{aligned}
c_1 &= \alpha_1 + \alpha_7 - \alpha_{11} + \alpha_{13} + \alpha_{21} + \alpha_{23} - \alpha_{31} - \alpha_{33} - \alpha_{35} - \alpha_{41} - \alpha_{61} - \alpha_{71} \\
c_2 &= \alpha_3 - \alpha_7 + \alpha_{11} - \alpha_{13} + \alpha_{21} - \alpha_{23} - \alpha_{25} - \alpha_{31} - \alpha_{33} - \alpha_{43} - \alpha_{53} - \alpha_{61} \\
c_3 &= \alpha_7 + \alpha_{13} + \alpha_{15} + \alpha_{17} - \alpha_{23} + \alpha_{31} + \alpha_{33} + \alpha_{35} + \alpha_{41} - \alpha_{43} - \alpha_{53} - \alpha_{61} - \alpha_{71} \\
c_4 &= \alpha_5 + \alpha_{11} - \alpha_{13} - \alpha_{21} + \alpha_{23} - \alpha_{25} + \alpha_{31} - \alpha_{41} - \alpha_{43} - \alpha_{51} + \alpha_{53} - \alpha_{61} - \alpha_{71}.
\end{aligned}$$

Let $\tau_4 = \tau_1 \circ \pi_O \circ \iota_3$. Then $\tau_4(\sum_{x_i \in O} \alpha_i x^i) = d_1 x^5 + d_2 x^{15} + d_3 x^{25} + d_4 x^{35}$, where

$$\begin{aligned}
d_1 &= \alpha_1 - \alpha_3 + \alpha_7 - \alpha_{11} + \alpha_{15} - \alpha_{17} - \alpha_{21} + \alpha_{25} - \alpha_{31} \\
&\qquad - \alpha_{35} - \alpha_{41} - \alpha_{43} - \alpha_{53} + \alpha_{61} - \alpha_{71} \\
d_2 &= \alpha_1 + \alpha_3 - \alpha_5 + \alpha_7 - \alpha_{13} + \alpha_{21} + \alpha_{23} - \alpha_{25} - \alpha_{33} \\
&\qquad - \alpha_{35} + \alpha_{41} - \alpha_{43} - \alpha_{51} - \alpha_{53} + \alpha_{71} \\
d_3 &= -\alpha_5 - \alpha_7 + \alpha_{13} + \alpha_{15} + \alpha_{17} - \alpha_{21} + \alpha_{23} + \alpha_{31} + \alpha_{35} + \alpha_{41} + \alpha_{43} - \alpha_{71} \\
d_4 &= \alpha_5 - \alpha_7 - \alpha_{13} + \alpha_{15} + \alpha_{21} - \alpha_{25} + \alpha_{31} + \alpha_{41} - \alpha_{43} - \alpha_{51} + \alpha_{53} + \alpha_{61}.
\end{aligned}$$

Finally, let $\tau_5 = \tau_4 \circ \epsilon_2$. We have $\tau_5(\sum_{x_i \in O} \alpha_i x^i) =$

$$\begin{aligned}
(\alpha_3 - \alpha_7 - \alpha_{17} + \alpha_{23} + \alpha_{33} + \alpha_{43})x^5 + (-\alpha_1 - \alpha_{11} - \alpha_{21} - \alpha_{41} - \alpha_{51} - \alpha_{61})x^{15} \\
+ (\alpha_{21} - \alpha_{31} + \alpha_{61} - \alpha_{71})x^{25} + (\alpha_7 + \alpha_{13} - \alpha_{23} + \alpha_{53})x^{35}
\end{aligned}$$

Each of ϵ_5, τ_1, τ_2, τ_3, τ_4, and τ_5 is a linear map from $\mathbb{F}_3[x]_0$ to D_5, and each maps any T-space T containing W_2 into itself. We consider the linear map $\theta \colon \langle O \rangle \to D_5^6$ given by

$$\theta(f) = (\epsilon_5(f), \tau_1(f), \tau_2(f), \tau_3(f), \tau_4(f), \tau_5(f))$$

for $f \in \langle O \rangle$. $\langle O \rangle$ has linear dimension 20, while D_5 has dimension 4 and so D_5^6 has dimension 24. We used the symbolic mathematics program SAGE ([5]) to determine that θ has rank 20, which means that θ is injective. Thus for each nonzero $f \in \langle O \rangle$, at least one of $\epsilon_5(f), \tau_i(f)$, $i = 1, 2, 3, 4, 5$, is nonzero; that is, $W_2 + \{ f \}^S$ contains a nonzero element of D_5, and thus by Proposition 2.2, $W_2 + \{ f \}^S = \mathbb{F}_3[x]_0$. □

Theorem 2.1. *W_2 is a maximal T-space of $\mathbb{F}_3[x]_0$.*

Proof. It remains only to prove that for nonzero $f \in \langle E \rangle$, $W_2 + \{ f \}^S = \mathbb{F}_3[x]_0$. We prove that if $f \in \langle E \rangle$ is nonzero, then $W_2 + \{ f \}^S$ contains a nonzero element of $\langle O \rangle$, at which point we may apply Proposition 2.3 to obtain that $W_2 + \{ f \}^S = \mathbb{F}_3[x]_0$.

Note that if we restrict each of $\pi_O \circ \iota_2$ and $\pi_O \circ \iota_4$ to $\langle E \rangle$, we have linear maps from $\langle E \rangle$ to $\langle O \rangle$, given by (again, computed with Python)

$$
\begin{aligned}
\pi_O \circ \iota_2\Big(\sum_{x_i\in E} \alpha_i x^i\Big) &= (\alpha_6+\alpha_8-\alpha_{42}-\alpha_{44}+\alpha_{52}+\alpha_{62})x \\
&+ (-\alpha_2+\alpha_{14}+\alpha_{24}+\alpha_{26}-\alpha_{44}-\alpha_{52})x^3 \\
&+ (\alpha_4+\alpha_{24}+\alpha_{26}+\alpha_{42}+\alpha_{52})x^5 + (\alpha_4+\alpha_{32}-\alpha_{44}+\alpha_{62})x^7 \\
&+ (-\alpha_8-\alpha_{14}+\alpha_{16}+\alpha_{52}-\alpha_{62})x^{11} + (-\alpha_8+\alpha_{26}-\alpha_{32}+\alpha_{34})x^{13} \\
&+ (-\alpha_8+\alpha_{12}-\alpha_{14}-\alpha_{44}-\alpha_{62})x^{15} \\
&\qquad + (\alpha_{14}+\alpha_{16}-\alpha_{44}-\alpha_{52}-\alpha_{62})x^{17} \\
&+ (\alpha_{12}+\alpha_{16}+\alpha_{26}-\alpha_{52})x^{21} \\
&\qquad + (\alpha_{14}+\alpha_{16}+\alpha_{22}+\alpha_{26}+\alpha_{44}+\alpha_{52})x^{23} \\
&+ (\alpha_{14}+\alpha_{16}+\alpha_{22}+\alpha_{34}-\alpha_{52}+\alpha_{62})x^{25} \\
&\qquad + (\alpha_{16}-\alpha_{22}+\alpha_{24}+\alpha_{62})x^{31} \\
&+ (-\alpha_{24}-\alpha_{26}-\alpha_{32}-\alpha_{42}-\alpha_{44})x^{33} \\
&+ (-\alpha_{22}-\alpha_{26}+\alpha_{32}+\alpha_{34}-\alpha_{42}-\alpha_{44})x^{35} \\
&+ (\alpha_{22}+\alpha_{34}+\alpha_{44}-\alpha_{62})x^{41} + (\alpha_{22}-\alpha_{26}-\alpha_{34}+\alpha_{52})x^{43} \\
&+ (-\alpha_{26}-\alpha_{32}+\alpha_{42}+\alpha_{44}-\alpha_{52})x^{51} \\
&+ (\alpha_{32}+\alpha_{34}-\alpha_{42}+\alpha_{44}-\alpha_{52}+\alpha_{62})x^{61},
\end{aligned}
$$

and $\pi_O \circ \iota_4(\sum_{x_i\in E} \alpha_i x^i) = \sum_{x_i\in O} e_i x^i$, where

$$
\begin{aligned}
e_1 &= \alpha_{26}+\alpha_{44}-\alpha_{52}-\alpha_{62} \\
e_3 &= -\alpha_{26}+\alpha_{44}+\alpha_{52}-\alpha_{62} \\
e_5 &= -\alpha_2+\alpha_{22}+\alpha_{32}-\alpha_{44}-\alpha_{52}-\alpha_{62} \\
e_7 &= \alpha_4-\alpha_{24}+\alpha_{42}+\alpha_{44} \\
e_{11} &= -\alpha_8-\alpha_{16}+\alpha_{52} \\
e_{13} &= \alpha_4-\alpha_{16}-\alpha_{34}-\alpha_{44} \\
e_{15} &= -\alpha_6+\alpha_{16}-\alpha_{26}-\alpha_{34}+\alpha_{44}-\alpha_{52} \\
e_{17} &= -\alpha_8-\alpha_{14}+\alpha_{52}+\alpha_{62} \\
e_{21} &= \alpha_8+\alpha_{12}-\alpha_{14}+\alpha_{52} \\
e_{23} &= -\alpha_8+\alpha_{26}-\alpha_{32}+\alpha_{34}+\alpha_{44}-\alpha_{52} \\
e_{25} &= -\alpha_{16}+\alpha_{22}+\alpha_{26}+\alpha_{44}+\alpha_{52}+\alpha_{62} \\
e_{31} &= -\alpha_{12}+\alpha_{22}-\alpha_{32}+\alpha_{52} \\
e_{33} &= -\alpha_{24}+\alpha_{32}-\alpha_{44}
\end{aligned}
$$

$$
\begin{aligned}
e_{35} &= -\alpha_{26} - \alpha_{32} + \alpha_{34} + \alpha_{44} - \alpha_{52} + \alpha_{62}\\
e_{41} &= \alpha_{14} + \alpha_{22} - \alpha_{26} + \alpha_{32}\\
e_{43} &= \alpha_{16} - \alpha_{34} + \alpha_{42} + \alpha_{62}\\
e_{51} &= -\alpha_{24} + \alpha_{26} - \alpha_{42} - \alpha_{44}\\
e_{53} &= -\alpha_{14} + \alpha_{34} + \alpha_{44} + \alpha_{52} + \alpha_{62}\\
e_{61} &= \alpha_{16} - \alpha_{22} + \alpha_{24} - \alpha_{44} - \alpha_{52} + \alpha_{62}\\
e_{71} &= -\alpha_{22} - \alpha_{26} + \alpha_{42} + \alpha_{44} - \alpha_{52}.
\end{aligned}
$$

Consider the linear map θ from $\langle E \rangle$ into $\langle O \rangle^2$ given by

$$\theta(f) = (\pi_O \circ \iota_2(f), \pi_O \circ \iota_4(f))$$

for $f \in \langle E \rangle$. $\langle E \rangle$ has dimension 16, and again using SAGE, we compute that θ has rank 16, and so conclude that θ is injective. Thus for each nonzero $f \in \langle E \rangle$, at least one of $\pi_O \circ \iota_2(f)$ or $\pi_O \circ \iota_4(f)$ is nonzero; that is, $W_2 + \{ f \}^S$ contains a nonzero element of $\langle O \rangle$, and thus by Proposition 2.3, $W_2 + \{ f \}^S = \mathbb{F}_3[x]_0$. $\square$

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