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On graphs with graphic imbalance sequences

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Abstract. The *imbalance* of the edge $e = uv$ in a graph G is the value $imb_G(e) = |d_G(u) - d_G(v)|$. We prove that the sequence M_G of all edge imbalances in G is graphic for several classes of graphs including trees, graphs in which all non-leaf vertices form a clique and the so-called complete extensions of paths, cycles and complete graphs. Also, we formulate two interesting conjectures related to graphicality of M_G.

Introduction

The notion of edge imbalance was introduced in 1997 by Albertson [2] as a measure of irregularity of a graph. By definition *irregularity* of G is the following value:

$$I(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)|.$$

There is quite extensive literature on graphs irregularity (see [1–3, 5–10] and references therein), which primarily consists of finding upper and lower bounds on $I(G)$. However, it seems like no author studied the sequence of edge imbalances in graphs as the main object. We hope that the present paper will open up things in this direction.

We consider only simple, finite and undirected graphs. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of a graph G respectively.

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Two vertices $u, v \in V(G)$ are *adjacent* if they are joined by an edge, i.e. if $uv \in E(G)$. Two graphs G_1 and G_2 are *isomorphic* if there exists adjacency preserving bijection $\phi : V(G_1) \to V(G_2)$. We write $G_1 \simeq G_2$ if G_1 and G_2 are isomorphic.

The *neighborhood* of a vertex $u \in V(G)$ in a graph G is the set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *degree* $d_G(u)$ of a vertex u in G is the number of its neighbors, i.e. $d_G(u) = |N_G(u)|$. A graph G is called *regular* if every vertex from G has the same degree.

The vertex $u \in V(G)$ is called *isolated* if $d_G(u) = 0$. Further, the vertex $u \in V(G)$ is called a *leaf* if $d_G(u) = 1$. The set of all leaves in G is denoted by $L(G)$.

The set of vertices $U \subset V(G)$ is called *clique* if every two vertices $u, v \in U$ are adjacent in G. A *matching* in a graph is a set of edges without common vertices.

A graph is *connected* if each pair of vertices is joined by a path. Otherwise graph is called *disconnected*. A maximal connected subgraph of a given graph G is called its *connected component*. The vertex $u \in V(G)$ is a *cut-vertex* if its deletion increases the number of connected components in G. A *tree* is a connected acyclic graph.

As usual, by K_n, $K_{1,n-1}$, P_n and C_n we denote the *complete graph*, *star*, *path* and *cycle* with n vertices respectively. Note that by definition $K_{1,0} = K_1$.

The *complement* of a graph G is a graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if $uv \notin E(G)$. The *empty graph* is a complement of complete graph.

Now let G_1 and G_2 be two graphs with disjoint vertex sets. The *union* of G_1 and G_2 is the graph $G = G_1 \cup G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. We write mH for the union $\cup_{i=1}^{m} G_i$ with $G_i \simeq H$, $1 \leqslant i \leqslant m$.

Similarly, the *join* of G_1 and G_2 is the graph $G = G_1 + G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

The *imbalance* of the edge $e = uv$ in a graph G is the value $imb_G(e) = |d_G(u) - d_G(v)|$. The multiset of all edge imbalances in G is denoted by M_G. Note that irregularity $I(G) = \sum M_G$ is even for every graph G.

Now suppose we have a finite sequence M (or more correctly, a multiset) of nonnegative integers. The sequence M is called *graphic* if it is the degree sequence of some graph G. Every such graph G is called a *realization* of M. A well-known Erdos-Gallai theorem [4] states that the sequence

$M = \{a_1 \geqslant \cdots \geqslant a_n\}$ is graphic if and only if the sum $\sum M$ is even and

$$\sum_{i=1}^{r} a_i \leqslant r(r-1) + \sum_{i=r+1}^{n} \min\{r, a_i\}.$$

for all $1 \leqslant r \leqslant n-1$.

We associate with every finite sequence of integers $M = \{a_1, \dots, a_n\}$ the function $f_M : \mathbb{Z} \to \mathbb{Z}$ in the following way:

$$f_M(m) = |\{1 \leqslant i \leqslant n : a_i = m\}|.$$

Also, by convention $m[n]$ denotes the sequence $a_1, \dots, a_n$ where $a_i = m$ for all $1 \leqslant i \leqslant n$. Thus, for example $\{1[2], 2[3]\} = \{1, 1, 2, 2, 2\}$. Note that the sequence $M = \{m[n]\}$ is graphic if and only if $m \leqslant n-1$ and mn is even.

1. Results

Firstly, we formulate some easy properties of graphs whose imbalance sequences are graphic.

Proposition 1. *Let G, G_1, G_2 be graphs. Then*

1) *if M_{G_1} and M_{G_2} are graphic, then so is $M_{G_1 \cup G_2}$;*
2) *if M_G is graphic, then M_{K_1+G} is also graphic;*
3) *if G is a graph with constant edge imbalance, then M_G is graphic;*

Proof. 1) Just observe that $M_{G_1 \cup G_2} = M_{G_1} \cup M_{G_2}$.

2) If H is a realization of M_G, then it is easy to see that $H \cup \overline{G}$ is a realization of M_{K_1+G}.
3) If $imb_G(e) = m \leqslant |E(G)| - 1$ for all edges $e \in E(G)$, then since $I(G) = \sum M_G = m|E(G)|$ is always even, the sequence M_G is graphic. □

Theorem 1. *If T is a tree, then M_T is graphic.*

Proof. Let T be a tree with $n \geqslant 1$ vertices. Firstly, suppose that $|V(T) - L(T)| = 0$. Then $T \simeq K_2$ and $M_T = \{0\}$ is obviously graphic. Similarly, if $|V(T) - L(T)| = 1$, then $T \simeq K_{1,n-1}$ and $M_T = \{n-2[n-1]\}$ is also graphic.

Further the proof goes by induction on n. If $n = 1$, then $T \simeq K_1$ and $M_T = \varnothing$ is trivially graphic. Now let T be a tree with $n \geqslant 2$ vertices. If $|V(T) - L(T)| \leqslant 1$, then we are done. If $|V(T) - L(T)| \geqslant 2$, then there exists a vertex $v \in V(T)$ that adjacent to $d_T(v) - 1$ leaves in T. Thus there exists a unique non-leaf vertex $u \in V(T)$ adjacent to v.

Now consider a tree $T' = T - (N_T(v) \cap L(T))$. It is easy to check that

$$M_T = (M_{T'} - \{d_T(u) - 1\}) \cup \{|d_T(u) - d_T(v)|, d_T(v) - 1[d_T(v) - 1]\}.$$

By induction hypothesis $M_{T'}$ is graphic. Let H be its realization. Fix a vertex $x \in V(H)$ with $d_H(x) = imb_{T'}(uv) = d_T(u) - 1$. We consider two cases.

Case 1: $d_T(u) \geqslant d_T(v)$.

Add to H $d_T(v) - 1$ new vertices $x_1, \ldots, x_{d_T(v)-1}$ with edges $x_i x_j$ for $1 \leqslant i, j \leqslant d_T(v) - 1$. Further, fix $d_T(v) - 1$ neighbors $y_1, \ldots, y_{d_T(v)-1} \in N_H(x)$ of x in H. Delete each edge xy_i, $1 \leqslant i \leqslant d_T(v) - 1$ and add new edges $x_i y_i$ for $1 \leqslant i \leqslant d_T(v) - 1$. Obtained graph is a realization of M_T.

Case 2: $d_T(u) \leqslant d_T(v) - 1$.

Let $N_H(x) = \{x_1, \ldots, x_{d_T(u)-1}\}$. Delete each edge xx_i, $1 \leqslant i \leqslant d_T(u) - 1$ from H. Now add $d_T(v) - 1$ new vertices $y_1, \ldots, y_{d_T(v)-1}$ with new edges $y_i y_j$ for $1 \leqslant i, j \leqslant d_T(v) - 1$ and xy_k for $d_T(u) \leqslant k \leqslant d_T(v) - 1$. Also, add new edges $x_k y_k$ for $1 \leqslant k \leqslant d_T(u) - 1$. Again, obtained graph is a realization of M_T. □

Definition 1. A graph G is called *cl-graph* if $V(G) - L(G)$ induces a clique in G.

It is easy to see that if G is disconnected cl-graph, then $G \simeq H \cup mK_2$, where H is a connected cl-graph. Therefore M_G is graphic if and only if M_H is graphic.

Theorem 2. *If G is a connected cl-graph graph, then M_G is graphic.*

Proof. Let $V(G) - L(G) = \{v_1, \ldots, v_n\}$. The proof goes by induction on n. If $n = 1$, then G is a star and M_G is graphic by Theorem 1.

Now for every $1 \leqslant i \leqslant n$ consider the following value:

$$l_i = |N_G(v_i) \cap L(G)|.$$

At first suppose that there exists $1 \leqslant i_0 \leqslant n$ such that $l_{i_0} = 0$. In this case the vertex v_{i_0} will be called *bald*.

From induction hypothesis it follows that $M_{G - v_{i_0}}$ is graphic. Suppose H its realization. Fix a bijection $f : E(G - v_{i_0}) \to V(H)$ with

$d_H(f(e)) = imb_{\,G-v_{i_0}}(e)$ for all edges $e \in E(G - v_{i_0})$. Add $n-1$ new vertices $x_1, \dots, x_{n-1}$ with new edges $x_i f(e)$ for all $e = v_i u$, where $u \in L(G - v_{i_0})$, $1 \leqslant i \leqslant n-1$. It is easy to see that obtained graph is a realization of M_G.

Now let $l_i \geqslant 1$ for all $1 \leqslant i \leqslant n$. It means that for every $1 \leqslant i \leqslant n$ one can choose a vertex $u_i \in N_G(v_i) \cap L(G)$. Consider the graph $G' = G - \{u_1, \dots, u_n\}$. We now show that if $M_{G'}$ is graphic, then M_G is graphic too.

Suppose that H' is a realization of $M_{G'}$. Fix a bijection $g : E(G') \to V(H')$ with $d_{H'}(g(e)) = imb_{\,G'}(e)$ for all edges $e \in E(G')$. Add n new vertices $x_1, \dots, x_n$ with new edges $x_i x_j$, $1 \leqslant i, j \leqslant n$ and $x_k f(e)$ for all $e = v_k u$, where $u \in L(G')$, $1 \leqslant k \leqslant n$. Again, the obtained graph is a realization of M_G.

Therefore one can always reduce a cl-graph G to a cl-graph G_0 with a bald vertex. But we have already proved that in this case M_{G_0} is graphic. Thus for every cl-graph G the sequence M_G is graphic. $\square$

Let G be a graph with n vertices and $a : V(G) \to \mathbb{Z}_+$ be a function with $a(v) \geqslant d_G(v)$ for all $v \in V(G)$.

Take n complete graphs $G_v \simeq K_{a(v)}$, $v \in V(G)$ with disjoint vertex sets. Since $a(v) \geqslant d_G(v)$, for every $v \in V(G)$ there exists injective map $\phi_v : N_G(v) \to V(G_v)$. A *complete extension* of G is a graph $G(a)$ which is obtained by taking the union of G_v, $v \in V(G)$ and adding new edges between $\phi_{v_1}(u_1)$ and $\phi_{v_2}(u_2)$ whenever $v_1 v_2 \in E(G)$ and $u_1 = v_2$, $u_2 = v_1$.

One can think of complete extension as of graph which is obtained from G by replacing each vertex $v \in V(G)$ with a complete graph of sufficiently large order.

Example 1. Fig. 1 shows a complete extension of the path $P_3 = \{v_1 - v_2 - v_3\}$ for $a(v_1) = 2$, $a(v_2) = 3$, $a(v_3) = 1$.

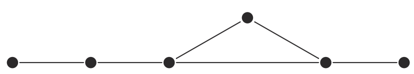

Figure 1. Complete extension of P_3.

Proposition 2. *Let G be a complete extension of a path P_n, $n \geqslant 1$. Then M_G is graphic.*

Proof. Let $V(P_n) = \{v_1, \ldots, v_n\}$, $E(P_n) = \{v_i v_{i+1} : 1 \leqslant i \leqslant n-1\}$ and $G = P_n(a)$, where $a : V(P_n) \to \mathbb{Z}_+$ with $a(v_i) \geqslant 2$ for all $2 \leqslant i \leqslant n-1$ and $a(v_1) \geqslant 1$, $a(v_n) \geqslant 1$.

Put $a_i = a(v_i)$ for each $1 \leqslant i \leqslant n$. We use induction on n.

If $n = 1$, then G is a complete graph and thus $M_G = \{0, \ldots, 0\}$ is graphic.

Now let $n \geqslant 2$. Consider a path $P_{n-1} = P_n - \{v_n\}$ with a' being restriction of a to $V(P_{n-1})$ and put $G' = P_{n-1}(a')$. One can check that

$$\begin{aligned} M_G = {} & (M_{G'} - \{0[a_{n-1} - 2], 1\}) \\ & \cup \left\{ 0[1 + \frac{(a_n - 1)(a_n - 2)}{2}], 1[a_n + a_{n-1} - 3], |a_{n-1} - a_n| \right\}. \end{aligned}$$

From induction hypothesis it follows that $M_{G'}$ is graphic. Suppose that H its realization. Fix a leaf $x' \in L(H)$ and delete the corresponding edge $x'x^0$ from H. Further, fix $a_{n-1} - 2$ isolated vertices $x_1, \ldots, x_{a_{n-1}-2} \in V(H)$. Now add a new vertex y and $a_n - 1$ new vertices $y_1, \ldots, y_{a_n - 1}$. Also, add $k = \frac{(a_n-1)(a_n-2)}{2}$ new isolated vertices $z_1, \ldots, z_k$.

We consider two cases.

Case 1: $a_n - 1 \geqslant a_{n-1} - 2$.

Add new edges $x_i y_i$ for all $1 \leqslant i \leqslant a_{n-1} - 2$. Also, add new edges yy_i for $a_{n-1} - 1 \leqslant i \leqslant a_n - 2$. Finally, add a new edge between x^0 and $y_{a_n - 1}$. Obtained graph is a realization of M_G.

Case 2: $a_{n-1} - 2 \geqslant a_n$.

Add new edges $y_i x_i$ for all $1 \leqslant i \leqslant a_n - 1$. Also, add new edges yx_i for $a_n \leqslant i \leqslant a_{n-1} - 2$. Finally, add a new edge between y and x^0. Again, the obtained graph is a realization of M_G. □

To prove analogous results for complete extensions of cycles and complete graphs we use a simple lemma which is a straightforward consequence of Erdos-Gallai criterion.

Lemma 1. *Let M be a sequence of nonnegative integers such that $\sum M$ is even and $2f_M(1) \geqslant \sum M$. Then M is graphic.*

Proposition 3. *Let G be a complete extension of a cycle C_n, $n \geqslant 3$. Then M_G is graphic.*

Proof. Let $V(C_n) = \{v_1, \ldots, v_n\}$, $E(C_n) = \{v_i v_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{v_1 v_n\}$ and $G = C_n(a)$, where $a : V(G) \to \mathbb{Z}_+$ with $a(v_i) \geqslant 2$ for all $1 \leqslant i \leqslant n$.

Now put $a_i = a(v_i)$ for each $1 \leqslant i \leqslant n$. Using induction on n we prove that the following inequality holds.

$$2\sum_{i=1}^{n} a_i - 4n \geqslant \sum_{i=1}^{n-1} |a_i - a_{i+1}| + |a_n - a_1|. \tag{1}$$

Firstly, suppose that $n = 3$. Without loss of generality we can assume that $a_1 \geqslant a_2 \geqslant a_3$. We have

$$\begin{aligned}(a_1 - a_2) + (a_2 - a_3) + (a_1 - a_3) &= 2a_1 - 2a_3\\ &= 2(a_1 + a_2 + a_3) - 2a_2 - 4a_3\\ &\leqslant 2(a_1 + a_2 + a_3) - 12.\end{aligned}$$

Now let $n \geqslant 4$. Further the proof splits into two casses.

Case 1: There exists $1 \leqslant k \leqslant n$ such that $a_{k-1} \leqslant a_k \leqslant a_{k+1}$, where $k-1$ and $k+1$ taken modulo n.

Without loss of generality we can assume that $k = n$. Therefore, let $a_{n-1} \leqslant a_n \leqslant a_1$. Using induction hypothesis and inequality $a_n \geqslant 2$ we obtain

$$\begin{aligned}2\sum_{i=1}^{n} a_i - 4n &\geqslant 2\sum_{i=1}^{n-1} a_i - 4(n-1)\\ &\geqslant \sum_{i=1}^{n-2} |a_i - a_{i+1}| + |a_{n-1} - a_1|\\ &= \sum_{i=1}^{n-2} |a_i - a_{i+1}| + |a_{n-1} - a_n| + |a_n - a_1|\\ &= \sum_{i=1}^{n-1} |a_i - a_{i+1}| + |a_n - a_1|.\end{aligned}$$

Case 2: For all $1 \leqslant k \leqslant n$ we have $a_{k-1} \leqslant a_k \geqslant a_{k+1}$ or $a_{k-1} \geqslant a_k \leqslant a_{k+1}$.

In this case n is even as C_n should be bipartite. Furthermore, it holds that $a_1 \leqslant a_2 \geqslant \dots \leqslant a_n \geqslant a_1$ or $a_1 \geqslant a_2 \leqslant \dots \geqslant a_n \leqslant a_1$. Assume that $a_1 \leqslant a_2$. Then

$$2\sum_{i=1}^{\frac{n}{2}} |a_i - a_{i+1}| + |a_n - a_1| = 2\sum_{i=1}^{\frac{n}{2}} (a_{2i} - a_{2i-1}).$$

Therefore

$$2\sum_{i=1}^{n} a_i - 4n - 2\sum_{i=1}^{\frac{n}{2}}(a_{2i} - a_{2i-1}) = 4\sum_{i=1}^{\frac{n}{2}} a_{2i} - 4n \geqslant 4 \cdot 2 \cdot \frac{n}{2} - 4n = 0.$$

Thus the inequality (1) holds. Now, it is easy to see that

$$f_{M_G}(1) \geqslant \sum_{i=1}^{n}(a_i - 2) = 2\sum_{i=1}^{n} a_i - 4n.$$

On the other hand

$$\sum M_G - f_{M_G}(1) \leqslant \sum_{i=1}^{n-1} |a_i - a_{i+1}| + |a_n - a_1|.$$

Thus $2f_{M_G}(1) \geqslant \sum M_G$ and the desired follows from Lemma 1. □

Proposition 4. *Let G be a complete extension of complete graph K_n, $n \geqslant 1$. Then M_G is graphic.*

Proof. Let $V(K_n) = \{v_1, \ldots, v_n\}$ and $G = K_n(a)$, where $a : V(G) \to \mathbb{Z}_+$ with $a(v_i) \geqslant n - 1$ for all $1 \leqslant i \leqslant n$.

Put $a_i = a(v_i)$ for each $1 \leqslant i \leqslant n$. Without loss of generality we can assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. Again, our goal is to show that $2f_{M_G}(1) \geqslant \sum M_G$.

Firstly, observe that

$$f_{M_G}(1) \geqslant \sum_{i=1}^{n}(n-1)(a_i - n + 1)$$

and

$$\sum M_G - f_{M_G}(1) \leqslant \sum_{i,j} |a_i - a_j| = \sum_{i=1}^{n}(n - 2i + 1)a_i.$$

Therefore it is sufficient to show that

$$\sum_{i=1}^{n}(n-1)(a_i - n + 1) \geqslant \sum_{i=1}^{n}(n - 2i + 1)a_i. \tag{2}$$

We have

$$\sum_{i=1}^{n}(n-1)(a_i - n + 1) = \sum_{i=1}^{n}(n - 2i + 1)a_i + \sum_{i=1}^{n}(2i - 2)a_i - n(n-1)^2.$$

Using $a_i \geqslant n-1$, $1 \leqslant i \leqslant n$ we obtain

$$\begin{aligned}\sum_{i=1}^{n}(2i-2)a_i &\geqslant 2(n-1)\sum_{i=1}^{n}(i-1)\\ &= 2(n-1)\cdot\frac{(n-1)n}{2}\\ &= n(n-1)^2.\end{aligned}$$

Thus the inequality (2) holds and the desired follows from Lemma 1. □

2. Conjectures

As it can be seen from Fig. 2 there exist graphs whose imbalance sequences aren't graphic.

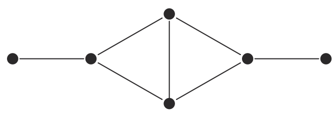

Figure 2. Graph whose imbalance sequence is not graphic.

In fact the following proposition is true.

Proposition 5. *Let M be a finite sequence of nonnegative integers such that $\sum M$ is even. Then there exists a graph G with $M_G - M = \{0, \ldots, 0\}$.*

Proof. For every even $m \in M$ construct a graph G_m in the following way. Take any $m+1$ - regular graph G such that G has a matching H of a size $\frac{m}{2}$ (for example $G = K_{m+1,m+1}$). Then delete all edges $E(H)$ from G and add a new vertex v with new edges from v to $V(H)$. Then again add a new vertex u and a new edge uv to obtain a graph G_m. Clearly, $M_{G_m} = \{m, 0, \ldots, 0\}$.

Now since $\sum M$ is even, the number of odd integers $m \in M$ is even. Divide the multiset of odd integers from M into pairs. For every such pair $\{m_1, m_2\}$ take any m_1+m_2+1 - regular graph G' with matching H' of a size $\frac{m_1+m_2}{2}$. Again, delete all edges $E(H')$ from G' and add two new vertices v_1, v_2 with new edges from v_1 to m_1 vertices from $V(H')$ and new edges from v_2 to remaining m_2 vertices from $V(H')$. Finally, add two new vertices u_1, u_2 with new edges u_1v_1, u_2v_2 to obtain a graph G_{m_1,m_2}. It is easy to see that $M_{G_{m_1,m_2}} = \{m_1, m_2, 0, \ldots, 0\}$.

Now take G as a union of G_m and G_{m_1,m_2} for all even $m \in M$ and all pairs $\{m_1, m_2\}$ of odd integers from M. Clearly $M_G = M \cup \{0, \dots, 0\}$. $\square$

The following conjecture originally appears on MathOverflow (question name "Graphs with graphic imbalance sequences").

> **Imbalance conjecture:** Suppose that for all edges $e \in E(G)$ we have $imb_G(e) > 0$. Then M_G is graphic.

Remark 1. This conjecture was verified for all such graphs with $\leqslant 9$ vertices.

Also, note that since for all graphs G the irregularity $I(G) = \sum M_G$ is always even, then M_G is always pseudographic, i.e. it is the degree sequence of some pseudograph (multiple edges and loops allowed). Furthermore, if G has no vertices with zero imbalance, then $I(G) = \sum M_G \geqslant \max M_G + |E(G)| - 1 \geqslant 2 \max M_G$. This means that M_G is multigraphic, i.e. it is the degree sequence of some multigraph (only multiple edges allowed). The Imbalance conjecture naturally generalizes these facts.

To formulate our second conjecture we need to consider the *mean imbalance* of every nonempty graph G. By definition it is the following value:

$$m(G) = \frac{I(G)}{|E(G)|}.$$

Here are some properties of mean imbalances.

Proposition 6. *Let G be a nonempty graph. Then*

1) $m(G) = 0$ *if and only if every component of G is regular;*
2) $m(G) \leqslant |E(G)| - 1$*;*
3) *for every $q \in \mathbb{Q}_+$ there exists a graph G with $m(G) = q$.*

Proof. 1) It is easy.

2) Follows from the inequality $I(G) \leqslant |E(G)|(|E(G)| - 1)$.
3) If $q \in \mathbb{Q}_+$, then $q = \frac{a}{b}$ for $a, b \in \mathbb{Z}_+$. Put $G = K_{1,a+1} \cup (a+1)(b-1)K_2$. We have
$$m(G) = \frac{a(a+1)}{a+1+(a+1)(b-1)} = \frac{a}{b} = q.$$ $\square$

Now observe that if the imbalance of every edge in G is nonzero, then $m(G) \geqslant 1$. Sadly, this inequality can't ensure the graphicality of M_G. However, we believe that there exists a "universal" constant $c > 0$ such that for every graph G with $m(G) \geqslant c$ the sequence M_G is graphic. Moreover, we think that $c = 2$ because of the following result.

Proposition 7. *The set of all mean imbalances of graphs with non-graphic imbalance sequences is dense in* $[0, 2]$.

Proof. For every $n \in \mathbb{N}$ and $1 \leqslant k \leqslant n$ construct a graph $G^{(n,k)}$ as follows. Take a complete graph $H \simeq K_{2n+2}$ with the vertex set $V(H) = \{v_1, \ldots, v_{2n+2}\}$. Delete from H every edge $v_{2i-1}v_{2i}$, $1 \leqslant i \leqslant k$ and add $2k$ new vertices $u_1, \ldots, u_{2k}$ with new edges $u_j v_j$ for $1 \leqslant j \leqslant 2k$ to obtain $G^{(n,k)}$.

Since $1 \leqslant k \leqslant n$, the sequence $M_{G^{(n,k)}} = \{2n[2k]\}$ is not graphic. We have

$$m(G^{(n,k)}) = \frac{4nk}{(n+1)(2n+1)+k}.$$

Now we show that the set $\{m(G^{(n,k)}) : n \in \mathbb{N}, 1 \leqslant k \leqslant n\}$ is dense in $[0, 2]$.

At first, observe that $m(G^{(n,1)}) \to 0,\ n \to \infty$. Now let $r \in (0, 2]$. Since $r > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ it holds that $\lfloor \frac{rn}{2} \rfloor \geqslant 1$. Furthermore, the inequality $r \leqslant 2$ implies $\lfloor \frac{rn}{2} \rfloor \leqslant n$ for every $n \in \mathbb{N}$. Now it is easy to see that

$$m(G^{(nn_0, \lfloor \frac{rnn_0}{2} \rfloor)}) \to r,\ n \to \infty. \qquad \square$$

Corollary 1. *For every* $\varepsilon > 0$ *there exists a graph* G *with non-graphic* M_G *such that* $m(G) \geqslant 2 - \varepsilon$.

Therefore we can formulate our final conjecture.

> **Conjecture:** Suppose that for G we have $m(G) \geqslant 2$. Then M_G is graphic.

References

[1] *H. Abdo, N. Cohen and D. Dimitrov*, Bounds and computation of irregularity of a graph // Preprint, arXiv:1207.4804 (2012).

[2] *M. O. Albertson*, The irregularity of a graph // Ars Comb. 46 (1997), 219-225.

[3] *F. K. Bell*, A note on the irregularity of graphs // Lin. Algebra Appl. 161 (1992), 45–54.

[4] *P. Erdos, T. Gallai*, Graphs with prescribed degrees of vertices // Mat. Lapok 11 (1960), 264–274.

[5] *F. Goldberg*, A spectral bound for graph irregularity // Preprint, arXiv:1308.3867 (2013).

[6] *P. Hansen, H. Melot*, Variable neighborhood search for extremal graphs 9. Bounding the irregularity of a graph // DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 69 (2005), 253-264.

[7] *M. A. Henning, D. Rautenbach*, On the irregularity of bipartite graphs // Discrete Math. 307 (2007), 1467-1472.

[8] *M. Tavakoli, F. Rahbarnia, M. Mirzavaziri, A. R. Ashrafi and I. Gutman*, Extremely irregular graphs // Kragujevac J. Mat. 37(1) (2013), 135-139.

[9] *W. Luo, B. Zhou*, On irregularity of graphs // Ars Comb. 88 (2008), 55-64.

[10] *W. Luo, B. Zhou*, On the irregularity of trees and unicyclic graphs with given matching number // Util. Math. 83 (2010), 141-147.

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