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Colour class domination numbers of some classes of graphs

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Abstract. We compute the colour class domination number of fan graphs, double fan graphs, Helm graphs, flower graphs and sun flower graphs.

1. Preliminaries

Let $G = (V, E)$ be a graph, with the number of vertices $|V(G)| = n$. By the neighbourhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. We say that a vertex is isolated if it has no neighbour, while it is universal if it is adjacent to all other vertices. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighbourhood. Let $\delta(G)$ mean the minimum degree among all vertices of G.

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of G is dominated by at least one vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination of graphs, the reader is referred to [3, 4].

A proper colouring of a graph $G = (V, E)$ is an assignment of colours to the vertices of the graph, such that any two adjacent vertices have different colours. The chromatic number $\chi(G)$ is the minimum number

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of colours needed in a proper colouring of G. By a $\chi-$partition of G, we mean the partition $\{V_1, V_2, \cdots, V_\chi\}$ of $V(G)$ where each V_i is the colour class representing the colour i, $i = 1, 2, \cdots, \chi$.

Dominating and colouring concepts have nice interactions, one such combination is the dominator colouring introduced by Gera et al. A dominator colouring of a graph G is a proper colouring in which each vertex of graph dominates every vertex of some colour class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colour classes in a dominator colouring of a graph G. Swaminathan et al [6] relaxed the condition of 'every vertex dominating a colour class' with 'every colour class is dominated by a vertex of G' and defined a new parameter colour class domination number of a graph G. A colour class domination partition of a graph G is a proper colouring in which vertices in every colour class is dominated by a vertex in $V(G)$. The colour class domination number $\chi_{cd}(G)$ is the minimum number of colour classes in a colour class domination partition of a graph G. It is to be noted that dominator chromatic number and colour class domination number are not comparable. The domination colouring of some classes of graphs was found in [2,5] and the colour class domination number of middle graph and center graph of $K_{1,n}, C_n$ and P_n was calculated in [7]. In the present paper, we find the colour class domination number of fan graphs, double fan graphs, helm graphs, flower graphs and sun flower graphs.

2. Main results

We recall from [6] the following result.

Theorem 1. For a graph G, $max\{\chi(G), \gamma(G)\} \leqslant \chi_{cd}(G)$.

Given $n \geqslant 1$, the fan graph denoted by F_n, can be constructed by joining n copies of the cycle graph C_3 with a common vertex.

Theorem 2. For a fan graph F_n, with $n \geqslant 1$, we have $\chi_{cd}(F_n) = 3$.

Proof. By the definition of fan graph, n copies of C_3 is joined with a common vertex, $\chi(F_n) \geqslant 3$ and $\gamma(F_n) = 1$. Hence, $\chi_{cd}(F_n) \geqslant 3$.

Let $V(F_n) = \{v_1, v_2, \cdots, v_{2n+1}\}$ be the vertex set of F_n and let the vertex at the center be labeled by v_1. Let the vertex v_1 be coloured by colour 1 and the other two vertices of each copy of C_3 are coloured by colours 2 and 3. The vertex v_1 dominates itself namely vertex coloured 1 and the vertices coloured 2 and 3. Hence, $\chi_{cd}(F_n) \leqslant 3$. Therefore, $\chi_{cd}(F_n) = 3$. □

We recall from [1] that the double fan graph $F_{2,n}$ is isomorphic to $P_n + 2K_1$.

Theorem 3. For the double fan graph $F_{2,n}$, with $n \geqslant 2$, we have $\chi_{cd}(F_{2,n}) = 3$.

Proof. Let $V_1 = \{v_1, v_2, \cdots, v_n\}$ be the vertices of the path P_n and $V_2 = \{u_1, u_2\}$ be the vertices of $2K_1$. Then $V(F_{2,n}) = V_1 \cup V_2$. The vertices of V_2 are adjacent to the vertices of V_1. The set V_2 forms a minimum dominating set and $\gamma(G) = 2$. The vertices u_1, v_1, v_3 forms a cycle C_3 and therefore $\chi(F_{2,n}) \geqslant 3$. Hence, $\chi_{cd}(F_{2,n}) \geqslant 3$.

The vertices u_1 and u_2 are non adjacent, they are coloured by colour 1. Since, $v_i, 1 \leqslant i \leqslant n$ is adjacent to both u_1 and u_2, these vertices are alternatively coloured by colours 2 and 3. Any vertex v_i in the path dominates the vertices coloured 1 and the vertex u_1 dominates the vertices coloured 2 and 3. Hence, $\chi_{cd}(F_{2,n}) \leqslant 3$. Therefore, $\chi_{cd}(F_{2,n}) = 3$. □

The Helm graph H_n with $n \geqslant 1$, is defined to be the graph obtained from a wheel graph $W_{1,n}$ by attaching a pendant edge at each vertex of the $n-$cycle.

Theorem 4. For the Helm graph H_n, with $n \geqslant 4$, we have $\chi_{cd}(H_n) = n$.

Proof. Let $V(H_n) = \{v_1\} \cup V_1 \cup V_2$, where v_1 is the central vertex, $V_1 = \{v_i : 2 \leqslant i \leqslant n+1\}$ be the vertices on the $n-$cycle and $V_2 = \{v_i : n+2 \leqslant i \leqslant 2n+1\}$ be the pendant vertices incident with $n-$cycle such that v_{n+i} is adjacent with v_i, $2 \leqslant i \leqslant n+1$.

The vertices v_1 and $v_{n+i}, 2 \leqslant i \leqslant 2n+1$ are non adjacent, these vertices are coloured with colour 1. The remaining vertices v_i, $2 \leqslant i \leqslant n+1$ on the $n-$cycle can be coloured using a maximum of three colours. Therefore, $\chi(H_n) \leqslant 4 \leqslant n$. Since there are n pendant vertices, $\gamma(H_n) \geqslant n$. Hence, $\chi_{cd}(H_n) \geqslant n$.

Assume that the pendant vertices $v_{n+i+1}, 1 \leqslant i \leqslant n$ be coloured with i. The vertex v_1 is coloured with colour n. Let v_2 and v_4 be coloured with colour 2. The vertex adjacent to v_{2n+i}, $3 \leqslant i \leqslant n+1$; $i \neq 4$ namely v_i is coloured with colour $i-2$. The partition

$$\Pi = \{\{v_{n+2}, v_3\}, \{v_{n+3}, v_2, v_4\}, \{v_{n+4}, v_5\}, \cdots, \{v_{2n}, v_{n+1}\}, \{v_{2n+1}, v_1\}\}$$

is a minimum colour class domination partition. The first two colour classes are dominated by v_2 and v_3 and the last colour class is dominated by the vertex v_n. The colour classes $\{v_{n+i}, v_{i+1}\}$, $4 \leqslant i \leqslant n$ is dominated by the vertex v_i. Hence, $\chi_{cd}(H_n) \leqslant n$. Therefore, $\chi_{cd}(H_n) = n$. □

A flower graph Fl_n with $n \geqslant 1$, is defined to be the graph obtained from a Helm graph H_n by joining each pendant vertex to the central vertex of the Helm graph.

Theorem 5. For the flower graph Fl_n with $n \geqslant 3$, we have

$$\chi_{cd}(Fl_n) = \begin{cases} 3, & if \quad n \ is \ even \\ 4, & if \quad n \ is \ odd \end{cases}$$

Proof. The central vertex is a dominating set. Therefore, $\gamma(Fl_n) = 1$. Clearly, $\chi(Fl_n) = 3$ if n is even or $\chi(FL_n) = 4$ if n is odd. Therefore, $\chi(Fl_n) \geqslant 3$ if n is even and $\chi(Fl_n) \geqslant 4$ if n is odd.

Let v_1 be the central vertex of $W_{1,n}$. Let $v_2, v_3, \cdots, v_{n+1}$ be the vertices in the $n-$cycle. The vertices v_{n+i} is adjacent to v_i, $2 \leqslant i \leqslant n+1$ and v_1.

Case(i): n is odd. The vertex v_1 is coloured with colour 1. The sequence of vertices v_i, $2 \leqslant i \leqslant n$ are coloured with colours 2 and 3 alternatively and v_{n+1} is coloured with colour 4. The remaining vertices v_i, $n+2 \leqslant i \leqslant 2n+1$ are coloured with colours 3 and 2 alternatively. The vertex v_1 dominates all the colour classes. Hence, $\chi_{cd}(Fl_n) \leqslant 4$. Therefore, $\chi(Fl_n) = 4$.

Case(ii): n is even. The vertex v_1 is coloured with colour 1. The sequence of vertices v_i, $2 \leqslant i \leqslant n+1$ are coloured with colours 2 and 3 alternatively. The remaining vertices v_i, $n+2 \leqslant i \leqslant 2n+1$ are coloured with colours 3 and 2 alternatively. The vertex v_1 dominates all the colour classes. Hence, $\chi_{cd}(Fl_n) \leqslant 3$. Therefore, $\chi(Fl_n) = 3$. □

The sun flower graph Sf_n with $n \geqslant 1$, is defined to be the graph obtained by adding n pendant edges to the central vertex of the flower graph Fl_n.

Theorem 6. For the sun flower graph Sf_n with $n \geqslant 3$, we have

$$\chi_{cd}(Sf_n) = \begin{cases} 3, & if \quad n \ is \ even \\ 4, & if \quad n \ is \ odd \end{cases}$$

Proof. The central vertex is a dominating set. Therefore, $\gamma(Sf_n) = 1$. Clearly, $\chi(Sf_n) = 3$ if n is even or $\chi(Sf_n) = 4$ if n is odd. Therefore, $\chi(Sf_n) \geqslant 3$ if n is even and $\chi(Sf_n) \geqslant 4$ if n is odd.

Let $V(Sf_n) = \{v_i : 1 \leqslant i \leqslant 3n+1\}$.

Case(i): n is odd. The vertex v_1 is coloured with colour 1. The sequence of vertices v_i, $2 \leqslant i \leqslant n$ are coloured with colours 2 and 3 alternatively and v_{n+1} is coloured with colour 4. The remaining vertices v_i, $n+2 \leqslant i \leqslant 2n+1$ are coloured with colours 3 and 2 alternatively and the vertices v_i, $2n+2 \leqslant i \leqslant 3n+1$ are coloured with 2,3 or 4. The vertex v_1 dominates all the colour classes. Hence, $\chi_{cd}(Sf_n) \leqslant 4$. Therefore, $\chi(Sf_n) = 4$.

Case(ii): n is even. The vertex v_1 is coloured with colour 1. The sequence of vertices v_i, $2 \leqslant i \leqslant n+1$ are coloured with colours 2 and 3 alternatively. The remaining vertices v_i, $n+2 \leqslant i \leqslant 2n+1$ are coloured with colours 3 and 2 alternatively and the vertices $2n+2 \leqslant i \leqslant 3n+1$ are coloured with 2,3. The vertex v_1 dominates all the colour classes. Hence, $\chi_{cd}(Sf_n) \leqslant 3$. Therefore, $\chi(Sf_n) = 3$. □

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