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RESEARCH ARTICLE

# On the *le*-semigroups whose semigroup of bi-ideal elements is a normal band

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Abstract. It is well known that the semigroup $\mathcal{B}(S)$ of all bi-ideal elements of an $le$-semigroup $S$ is a band if and only if $S$ is both regular and intra-regular. Here we show that $\mathcal{B}(S)$ is a band if and only if it is a normal band and give a complete characterization of the $le$-semigroups $S$ for which the associated semigroup $\mathcal{B}(S)$ is in each of the seven nontrivial subvarieties of normal bands. We also show that the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of $S$ forms a rectangular band and that $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $\mathcal{B}(\mathcal{S})$.

## 1. Introduction

In the ideal theory of commutative rings, it was observed by W. Krull [15] that several results do not depend on the fact that the ideals are composed of elements. The same is true for the ideal theory of semigroups also. Consequently, these results can be formulated in a more general setting of lattice-ordered semigroups where an element represents an ideal of the ring or semigroup as an undivided entity. There are series of articles dealing with lattice-ordered semigroups, generalizing theorems from commutative ideal theory [1], [3], [5] and from the ideal theory of semigroups [12], [13], [21], [22]. Presently, lattice ordered semigroups are providing us a general setting not only for 'abstract ideal theory', but also

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for order-preserving transformations of a finite chain, power semigroups of an arbitrary semigroup, and for many other areas of algebra where the objects form similar kinds of lattice-ordered semigroups.

In the present paper we study $le$-semigroups globally; our aim here is to find out to what extent properties of the subsemigroup $\mathcal{B}(S)$ of all bi-ideal elements of an $le$-semigroup $S$ affect the structure of the $le$-semigroup as a whole. In 1952, R.A. Good and D.R. Hughes [6] introduced the notion of bi-ideals of a semigroup; these have been generalized again and again to rings, semirings, ternary semirings, $\Gamma$-semigroups, etc [4], [8]–[11], [14], [23]. It has also been proved that this notion is very useful for characterizing different types of regularity of rings, semirings, and semigroups [2], [16]– [19]. In [12], N. Kehayopulu defined bi-ideal elements of an $le$-semigroup as a generalization of bi-ideals. Here we introduce the notion of minimal bi-ideal elements and show that the product of any two bi-ideal elements is a bi-ideal element, and that the product of any two minimal bi-ideal elements is a minimal bi-ideal element. Thus the set $\mathcal{B}(S)$ of all bi-ideal elements and the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements are subsemigroups of $S$. It is well known that $S$ is both regular and intra-regular if and only if $b^2 = b$ for every bi-ideal element $b$ of $S$, equivalently $\mathcal{B}(S)$ is a band. Here we show that $\mathcal{B}(S)$ is a locally testable semigroup and hence a normal band (since a band is locally testable if and only if it is a normal band) if $S$ is both regular and intra-regular. The variety of normal bands has exactly eight subvarieties. Here we have characterized the $le$-semigroups $S$ such that $\mathcal{B}(S)$ is in each of these subvarieties of normal bands.

This introduction is followed by preliminaries. In Section 3, we characterize the $le$-semigroups $S$ such that $\mathcal{B}(S)$ is in each of the subvarieties of normal bands. In the last section, we show that the semigroup $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of $S$ is a bi-ideal of the semigroup $\mathcal{B}(S)$ whereas the set $\mathcal{L}_m(S)$ of all minimal left ideal elements is a left ideal of the semigroup $\mathcal{L}(S)$ of all left ideal elements of $S$.

## 2. Preliminaries and foundations

An *le-semigroup* $S$ is an algebra $(S, \cdot, \vee, \wedge, e)$ such that $(S, \cdot)$ is a semigroup, $(S, \vee, \wedge, e)$ is a lattice with a greatest element which is denoted by $e$, and for all $a, b, c \in S$,

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

For different examples and relevance, both classical and modern, of the $le$-semigroups we refer to [21]. Throughout the paper $S$ will stand for an $le$-semigroup $(S, \cdot, \vee, \wedge, e)$.

The usual order relation $\leqslant$ on the set $S$ is defined by: for $a, b \in S$

$$a \leqslant b \text{ if } a \vee b = b.$$

Since the multiplication is distributive over the lattice join, it follows that the order $\leqslant$ is compatible with the multiplication in $S$, that is, for all $a, b, c \in S$,

$$a \leqslant b \implies ac \leqslant bc \text{ and } ca \leqslant cb.$$

Let $A$ be a nonempty subset of $S$. We denote $(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}$. A nonempty subset $L$ is called a *left (right) ideal* of $S$ if $SL \subseteq L$ $(LS \subseteq L)$ and $(L] \subseteq L$. A subset $I$ is called an *ideal* if it is both a left and a right ideal of $S$. For $a \in S$, the left ideal generated by $a$ is given by

$$(a]_l = \{x \in S \mid x \leqslant sa \text{ for some } s \in S \cup \{1\}\}.$$

An element $a \in S$ is called *regular* if $a \leqslant aea$; and *intra-regular* if $a \leqslant ea^2e$. If every element of $S$ is regular (intra-regular) then the $le$-semigroup $S$ is defined to be *regular (intra-regular).* We also say that $a$ is

(i) a *subsemigroup element* if $a^2 \leqslant a$;

(ii) a *left ideal element* if $ea \leqslant a$;

(iii) a *right ideal element* if $ae \leqslant a$;

(iv) a *bi-ideal element* if it is a subsemigroup element and $aea \leqslant a$.

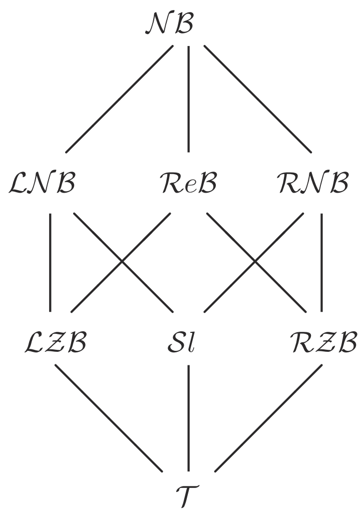
From the above definitions it is evident that every left and right ideal element is also a subsemigroup element. The definition of bi-ideal elements that we have given here is a little bit different from that of bi-ideal elements considered by Kehayopulu [12], Pasku and Petro [22]. According to these authors, a bi-ideal element $b$ needs not satisfy $b^2 \leqslant b$, i.e. needs not be a subsemigroup element, and is actually an abstraction of the generalized bi-ideals (of a semigroup) and not of the bi-ideals.

Let $a \in S$. Then $b = a \vee a^2 \vee aea$ is the least bi-ideal element in $S$ such that $a \leqslant b$. We call $a \vee a^2 \vee aea$ the *bi-ideal element generated by* $a$, and denote this by $\beta(a)$. Thus $a \in S$ is a bi-ideal element if and only if $\beta(a) = a$.

Now we recall some notions of semigroups (without order). A semigroup $F$ is called *regular* if for every $a \in F$ there is $x \in F$ such that $a = axa$. By a *band* we mean a semigroup $B$ such that $b^2 = b$ for all $b \in B$. A band $S$ is *normal* if for all $a, b, c \in S$, $abca = acba$. A subsemigroup $B$ of a semigroup $F$ is called a *bi-ideal* of $F$ if $BFB \subseteq B$.

In the diagram below, we use the following symbols to denote the different subvarieties of normal bands.

| | | |
|---|---|---|
| Normal band | $\mathcal{NB}$ | $abcd = acbd$, |
| Rectangular band | $\mathcal{R}e\mathcal{B}$ | $aba = a$, |
| Left normal band | $\mathcal{LNB}$ | $abc = acb$, |
| Right normal band | $\mathcal{RNB}$ | $abc = bac$, |
| Left zero band | $\mathcal{LZB}$ | $ab = a$, |
| Right zero band | $\mathcal{RZB}$ | $ab = b$, |
| Semilattice | $\mathcal{S}l$ | $ab = ba$, |
| Trivial semigroup | $\mathcal{T}$ | $a = b$. |




A semigroup is called *locally finite* if every finitely generated subsemigroup is finite. A *locally testable semigroup* [24] is a semigroup which is locally finite and in which $fSf$ is a semilattice for all idempotent $f \in S$. Nambooripad [20] proved that a regular semigroup $S$ is locally testable if and only if $fSf$ is a semilattice for all idempotent $f \in S$.

We refer the reader to [7] for the fundamentals of semigroup theory.

## 3. Subsemigroup of all bi-ideal elements

We denote the set of all left, right, and bi-ideal elements of $S$ by $\mathcal{L}(S), \mathcal{R}(S)$, and $\mathcal{B}(S)$, respectively. Then $\mathcal{L}(S), \mathcal{R}(S)$, and $\mathcal{B}(S)$ are all nonempty, since $e$ is a left ideal, a right ideal, and a bi-ideal element of $S$. Now for any two bi-ideal elements $a$ and $b$ of $S$, $(ab)^2 = (aba)b \leqslant ab$ and $abeab = (abea)b \leqslant ab$, since $a$ is a bi-ideal element of $S$, which shows that the product of any two bi-ideal elements is a bi-ideal element. Thus $\mathcal{B}(S)$ is a subsemigroup of $S$. Similarly both $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are subsemigroups of $S$.

Now we show that the regularity of an *le*-semigroup is equivalent to the regularity of the semigroup $\mathcal{B}(S)$. This, we think, is well known. But as we have seen the sufficient part nowhere, for the sake of completeness, we include a proof.

**Proposition 3.1.** *Let $S$ be an le-semigroup. Then $S$ is regular if and only if the semigroup $\mathcal{B}(S)$ of all bi-ideal elements is regular.*

*Proof.* First assume that $S$ is regular and that $b \in \mathcal{B}(S)$. Since $b$ is a bi-ideal element, $beb \leqslant b$. On the other hand, $b \leqslant beb$ by the regularity of $S$. Thus we have $b = beb$ which shows that $b$ is a regular element in $\mathcal{B}(S)$, since $e$ is also a bi-ideal element of $S$.

Conversely, suppose that $\mathcal{B}(S)$ is a regular semigroup. Consider $a \in S$. Then $\beta(a) = a \vee a^2 \vee aea \in \mathcal{B}(S)$ and so there is $b \in \mathcal{B}(S)$ such that $a \vee a^2 \vee aea = (a \vee a^2 \vee aea)b(a \vee a^2 \vee aea) \leqslant (a \vee a^2 \vee aea)e(a \vee a^2 \vee aea) \leqslant aea$. This implies that $a \leqslant aea$. Thus $S$ is a regular *le*-semigroup. □

If $S$ is a regular *le*-semigroup, then for every $a \in S$, $a \leqslant aea$ implies that $a^2 \leqslant aaea \leqslant aea$. Hence the bi-ideal element $\beta(a)$ generated by $a$ reduces to the form $\beta(a) = aea$. Thus in a regular *le*-semigroup the notions of bi-ideal elements as we have defined and that defined by N. Kehayopulu [12] are the same. Therefore in a regular *le*-semigroup $S$, an element $b \in S$ is a bi-ideal element if and only if $b = ca$ for some right ideal element $c$ and left ideal element $a$ [12, Lemma 2]. This can be reframed as:

**Theorem 3.2.** *Let $S$ be an le-semigroup. Then $\mathcal{R}(S)\mathcal{L}(S) \subseteq \mathcal{B}(S)$. If moreover, $S$ is a regular le-semigroup, then $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$.*

We also omit the proof of the following result, since this can be proved easily:

**Proposition 3.3.** *Let $S$ be a regular le-semigroup. Then $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are bands.*

The following important result can be proved similarly to that in [13] for the quasi-ideal elements.

**Theorem 3.4.** *An le-semigroup $S$ is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a band.*

Now we show that $\mathcal{B}(S)$ is in fact a normal band if $S$ is both a regular and intra-regular $le$-semigroup.

**Theorem 3.5.** *Let $S$ be an le-semigroup. Then $S$ is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a normal band.*

*Proof.* Let $a, b, c \in \mathcal{B}(S)$. Then $(bab)(bcb) = ba(bbcb) \leqslant ba(beb) \leqslant bab$. Similarly, $(bab)(bcb) \leqslant bcb$. Thus $(bab)(bcb) \leqslant (bab) \wedge (bcb)$. Now let $u = (bab) \wedge (bcb)$. Then $u \leqslant bab$ and $u \leqslant bcb$. Since $S$ is both regular and intra-regular, so $\mathcal{B}(S)$ is a band. Now $ueu = (bab \wedge bcb)e(bab \wedge bcb) = babebab \wedge babebcb \wedge bcbebab \wedge bcbebcb \leqslant bab \wedge babebcb \wedge bcbebab \wedge bcb \leqslant bab \wedge bcb = u$ shows that $u \in \mathcal{B}(S)$ which implies that $u = u^2 \leqslant (bab)(bcb)$. Thus $(bab) \wedge (bcb) \leqslant (bab)(bcb)$ and hence $(bab)(bcb) = (bab) \wedge (bcb)$.

Then $b\mathcal{B}(S)b = \{bab \mid a \in \mathcal{B}(S)\}$ is a semilattice for every $b \in \mathcal{B}(S)$. Thus $\mathcal{B}(S)$ is a locally testable semigroup. Since a locally testable semigroup is a band if and only if it is a normal band [24, Theorem 5], so $\mathcal{B}(S)$ is a normal band.

The converse follows from the Theorem 3.4. □

An ordered semigroup $S$ is said to be *left (right) duo* if every left (right) ideal of $S$ is a right (left) ideal of $S$; and $S$ is said to be *duo* if $S$ is both left and right duo.

**Lemma 3.6.** *An le-semigroup $S$ is left duo if and only if $ae \leqslant ea$ for all $a \in S$.*

*Proof.* First assume that $S$ is left duo and let $a \in S$. Then the left ideal $(a]_l = \{x \in S \mid x \leqslant sa$ for some $s \in S\}$ generated by $a$ is a right ideal also. Then $ae \in (a]_l$ implies that there is some $s \in S$ such that $ae \leqslant sa$ and this implies that $ae \leqslant ea$.

Conversely let $L$ be a left ideal of $S$ and $a \in L$. Then for every $s \in S$, $as \leqslant ae \leqslant ea \in L$ implies that $as \in L$. Thus $L$ is a right ideal of $S$ and hence $S$ is left duo. □

Immediately we have:

**Proposition 3.7.** *An le-semigroup $S$ is duo if and only if $ae = ea$ for all $a \in S$.*

Let $S$ be a regular left duo *le*-semigroup. Then for every $a \in S$, $a \leqslant aea \leqslant (ae)aea \leqslant ea^2ea$ shows that $S$ is intra-regular. Hence $\mathcal{B}(S)$ is a band. In fact we have:

**Theorem 3.8.** *An le-semigroup $S$ is regular left duo if and only if $\mathcal{B}(S)$ is a left normal band.*

*Proof.* First assume that $S$ is regular left duo. Then $\mathcal{B}(S)$ is a band. Let $a, b, c \in \mathcal{B}(S)$. Then $abc = (abc)(abc) = aabcabc \leqslant a(ae)cabc \leqslant a(ea)cabc \leqslant acabc = aca(bc)(bc) \leqslant acab(cb)e \leqslant acabecb \leqslant acb$. Similarly $acb \leqslant abc$. Thus $abc = acb$ and hence $\mathcal{B}(S)$ is a left normal band.

Conversely, assume that $\mathcal{B}(S)$ is a left normal band. Then $S$ is regular. Also for every $a \in S$, both $ea$ and $aea$ are bi-ideal elements of $S$, and hence $ae = (ae)(ae)(ae) = (aea)(ea)e = (aea)e(ea)$ [since $\mathcal{B}(S)$ is a normal band] $= (aeae^2)a \leqslant ea$ which shows that $S$ is left duo. □

The left-right dual of this theorem is as follows:

**Theorem 3.9.** *An le-semigroup $S$ is regular right duo if and only if $\mathcal{B}(S)$ is a right normal band.*

A band is a semilattice if and only if it is both a left and a right normal band. Hence it follows immediately that:

**Theorem 3.10.** *An le-semigroup $S$ is regular duo if and only if $\mathcal{B}(S)$ is a semilattice.*

**Theorem 3.11.** *Let $S$ be an le-semigroup. Then $\mathcal{B}(S)$ is a rectangular band if and only if $S$ is regular and $eae = ebe$ for all $a, b \in S$.*

*Proof.* First assume that $\mathcal{B}(S)$ is a rectangular band and that $a, b \in S$. Since $\mathcal{B}(S)$ is a band, so $S$ is regular and hence $\beta(a) = aea$ and $\beta(b) = beb$. Then $\beta(a) = \beta(a)\beta(b)\beta(a)$ implies that $a \leqslant aea = (aea)(beb)(aea) \leqslant ebe$. Then $eae \leqslant e^2be^2 \leqslant ebe$. Similarly $\beta(b) = \beta(b)\beta(a)\beta(b)$ implies that $ebe \leqslant eae$. Thus $eae = ebe$ for all $a, b \in S$.

Conversely let $a \in S$. Since $S$ is regular, so $a \leqslant aea \leqslant aeaea \leqslant aea^2ea$, by the given condition. Thus $a \leqslant ea^2e$, and hence $S$ is intra-regular. Therefore $\mathcal{B}(S)$ is a band, by Theorem 3.4. Now let $a, b$ be two bi-ideal elements of $S$. Since $a$ is a bi-ideal element and $S$ is already known to be regular, then $aea = a$, and so $a = aea = aeaea = aeabaea = aba$; and hence $\mathcal{B}(S)$ is a rectangular band. □

**Theorem 3.12.** *Let $S$ be an le-semigroup. Then $\mathcal{B}(S)$ is a left zero band if and only if $S$ is regular and $ae \leqslant eb$ for all $a, b \in S$.*

*Proof.* First assume that $\mathcal{B}(S)$ is a left zero band and $a, b \in S$. Since $\mathcal{B}(S)$ is band, so $S$ is regular and hence $\beta(ae) = ae^2ae$ and $\beta(b) = beb$. Then $\beta(ae) = \beta(ae)\beta(b)$ implies that $ae \leqslant ae^2ae = (ae^2ae)(beb) \leqslant eb$. Thus $ae \leqslant eb$ for all $a, b \in S$.

Conversely let $a \in S$. Since $S$ is regular, so $a \leqslant aea \leqslant aeaea \leqslant ae^2a^2a$, by the given condition. Thus $a \leqslant ea^2e$, and hence $S$ is intra-regular. Therefore $\mathcal{B}(S)$ is a band, by Theorem 3.4. Now let $a, b$ be two bi-ideal elements of $S$. Since $S$ is regular, $a = aea$, so that $ab \leqslant ae = ae(ae) \leqslant ae^2a \leqslant aea = a$ and $a = aeaea \leqslant ae(ae) \leqslant ae(eab) \leqslant (aea)b = ab$. Thus $a = ab$ and hence $\mathcal{B}(S)$ is a left zero band. □

The left-right dual of this theorem is as follows:

**Theorem 3.13.** *Let $S$ be an le-semigroup. Then $\mathcal{B}(S)$ is a right zero band if and only if $S$ is regular and $ea \leqslant be$ for all $a, b \in S$.*

## 4. Subsemigroup of all minimal bi-ideal elements

In this section we introduce minimal bi-ideal elements and minimal left ideal elements, and show that the set of all minimal bi-ideal elements of $S$ is a subsemigroup of $\mathcal{B}(S)$.

**Definition 4.1.** Let $S$ be an *le*-semigroup. A bi-ideal element $b$ is said to be minimal if for every bi-ideal element $a$ of $S$,

$$a \leqslant b \text{ implies that } a = b.$$

Minimal left (right) ideal elements are defined similarly.

We denote the set of all minimal bi-ideal, left ideal, and right ideal elements of $S$ by $\mathcal{B}_m(S), \mathcal{L}_m(S)$, and $\mathcal{R}_m(S)$, respectively.

Now we show that $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. For this consider $a, b \in \mathcal{B}_m(S)$. Then $ab$ is a bi-ideal element. To check the minimality, let $c$ be a bi-ideal element such that $c \leqslant ab$. Then $ca$ and $bc$ are bi-ideal elements such that $ca \leqslant aba \leqslant a$. Then by minimality of $a$ we have $ca = a$. Similarly, $bc = b$. Then $ab = cabc \leqslant cec \leqslant c$ and hence $c = ab$. Thus $ab \in \mathcal{B}_m(S)$.

Similarly, it can be proved that both $\mathcal{L}_m(S)$ and $\mathcal{R}_m(S)$ are subsemigroups of $\mathcal{B}(S)$.

We also have:

**Theorem 4.2.** *If $S$ is an le-semigroup then $\mathcal{B}_m(S) = \mathcal{R}_m(S)\mathcal{L}_m(S)$.*

*Proof.* First consider $a \in \mathcal{R}_m(S)$ and $c \in \mathcal{L}_m(S)$, and denote $b = ac$. Then $b$ is a bi-ideal element, by Theorem 3.2. To show the minimality of $b$, let $p \leqslant b$ be a bi-ideal element of $S$. Then $pe$ is a right ideal element of $S$ and $pe \leqslant be = ace \leqslant ae \leqslant a$ implies by the minimality of $a$ as a right ideal element that $pe = a$. Similarly we have $ep = c$, since $c$ is a minimal left ideal element. Then $p \leqslant b = ac = peep \leqslant p$ implies that $p = b$; and so $b$ becomes a minimal bi-ideal element. Thus $\mathcal{R}_m(S)\mathcal{L}_m(S) \subseteq \mathcal{B}_m(S)$.

Now consider $b \in \mathcal{B}_m(S)$. Then $be$ and $eb$ are a right ideal element and a left ideal element, respectively. Let $a \leqslant be$ be a right ideal element of $S$. Then $ab$ is a bi-ideal element of $S$ such that $ab \leqslant beb \leqslant b$, and so $ab = b$, since $b$ is a minimal bi-ideal element. Then $a \leqslant be = abe \leqslant ae \leqslant a$ implies that $a = be$. Thus $be$ is a minimal right ideal element of $S$. Similarly $eb$ is a minimal left ideal element of $S$. Then $beeb$ is a bi-ideal element, by Theorem 3.2. Now $beeb \leqslant b$ implies that $b = beeb$; and so $b \in \mathcal{R}_m(S)\mathcal{L}_m(S)$. Thus $\mathcal{B}_m(S) \subseteq \mathcal{R}_m(S)\mathcal{L}_m(S)$. Hence $\mathcal{B}_m(S) = \mathcal{R}_m(S)\mathcal{L}_m(S)$. □

**Theorem 4.3.** *a) Let $S$ be an le-semigroup such that the set $\mathcal{L}_m(S)$ of all minimal left ideal elements is non-empty. Then $\mathcal{L}_m(S)$ is a left ideal of the semigroup $\mathcal{L}(S)$. Moreover, $\mathcal{L}_m(S)$ is a right zero band.*
*b) Let $S$ be an le-semigroup such that the set $\mathcal{R}_m(S)$ of all minimal right ideal elements is non-empty. Then $\mathcal{R}_m(S)$ is a right ideal of the semigroup $\mathcal{R}(S)$. Moreover, $\mathcal{R}_m(S)$ is a left zero band.*

*Proof.* a) Let $l \in \mathcal{L}(S)$ and $a \in \mathcal{L}_m(S)$. Then $la$ is a left ideal element such that $la \leqslant ea \leqslant a$. This implies that $la = a$, since $a$ is a minimal left ideal element. Hence $la \in \mathcal{L}_m(S)$ and so $\mathcal{L}(S)\mathcal{L}_m(S) \subseteq \mathcal{L}_m(S)$. Thus $\mathcal{L}_m(S)$ is a left ideal of $\mathcal{L}(S)$.

Now $la = a$ for every $l \in \mathcal{L}(S)$ and $a \in \mathcal{L}_m(S)$ implies that $ab = b$ for every $a, b \in \mathcal{L}_m(S)$; and hence $\mathcal{L}_m(S)$ is a right zero band.
b) Follows as the left-right dual of a). □

Now we characterize the semigroup $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of $S$.

**Theorem 4.4.** *Let $S$ be an le-semigroup such that the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements is non-empty. Then $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $\mathcal{B}(S)$. Moreover, $\mathcal{B}_m(S)$ is a rectangular band.*

*Proof.* We have already shown that $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. Now consider $a, c \in \mathcal{B}_m(S)$ and $b \in \mathcal{B}(S)$. Then $abc$ is a bi-ideal element of $S$. To show the minimality of $abc$, let $d \leqslant abc$ be a bi-ideal element of $S$. Then $da$ is a bi-ideal element of $S$ and $da \leqslant abca \leqslant a$ implies that $a = da$, since $a$ is a minimal bi-ideal element. Similarly minimality of $c$ implies that $c = cd$. Then $abc = dabcd \leqslant d$ and so $d = abc$ which shows that $abc$ is a minimal bi-ideal element of $S$. Thus $\mathcal{B}_m(S)\mathcal{B}(S)\mathcal{B}_m(S) \subseteq \mathcal{B}_m(S)$ and hence $\mathcal{B}_m(S)$ is a bi-ideal of $\mathcal{B}(S)$.

If $b \in \mathcal{B}_m(S)$, then $b$ is a subsemigroup element of $S$ and so $b^2 \leqslant b$. Now minimality of $b$ implies that $b^2 = b$. Thus $\mathcal{B}_m(S)$ is a band. Let $a, b \in \mathcal{B}_m(S)$. Then $aba$ is a bi-ideal element such that $aba \leqslant aea \leqslant a$ which implies that $aba = a$, since $a$ is a minimal bi-ideal element of $S$. Thus $\mathcal{B}_m(S)$ is a rectangular band. □

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