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Normal subdigroups and the isomorphism theorems for digroups

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Abstract. We discuss the notion of normality of a subobject in the category of digroups. This allows us to define quotient digroups, and then establish the corresponding analogues of the classical Isomorphism Theorems.

Introduction

Digroups are a generalization of groups that involves two operations. Their origins can be traced back to the work of J. L. Loday on Leibniz algebras and dialgebras, and for background on the subject we refer the reader to the detailed discussion in [5], and the references therein. Digroups are examples of dimonoids with bar-units considered recently in some other aspects in the paper of A.V. Zhuchok (see [7]).

As one would expect, there is a natural notion of homomorphism, so digroups constitute a category, and thus a rather evident question is: what kind of category do they constitute? Intuitively, one might suspect that the category of digroups should have more or less the same properties

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as that of groups; but since there are some obvious key differences, the generalization of the classical results for groups is not straightforward.

In this note we shall partially answer this question, by studying the construction of quotient digroups, and the corresponding analogues of the usual *Isomorphism Theorems* for groups (see for instance the classical text [3]).

A natural scenario for these isomorphism theorems is that of *Universal Algebra* (see *e.g.* [1] or [2]), where one studies algebraic structures with a finite number of operations, each one having finitely many arguments (n-ary operations). Digroups —as well as groups— fall of course into this context, since each has a nullary operation associated with a special element, and a unary operation of inversion, besides the binary products. The isomorphism theorems then describe the possibility of defining equivalence classes, or *congruences*, in such a way that the quotient inherits an algebraic structure of the same type, starting from a given morphism or sub-object.

Now, the presence of the nullary operation, either for groups or digroups, gives a so-called *zero object* in the corresponding category, and this makes them *pointed categories*. The point is then that while the isomorphism theorems for general universal algebras are not concerned with these “pointed properties”, in fact the existence of zero objects restricts the sub-structures in a very concrete manner, and as a rule this has a direct impact on the way the congruences can be constructed. For instance, it is well-known for groups and rings that the class of the zero sub-object, which is the same as the *kernel* of the morphism, is enough to determine completely the equivalence classes, and therefore the quotient; but in general this is not so simple. Thus, our intent here is to give a detailed answer to the question: what can be said for digroups in regard to the construction and properties of these congruences? This question is of interest because of the deep connections between the algebraic and geometrical properties of digroups.

The paper is organized as follows:

Section 1 is mostly a review of known results about digroups. We outline first a comparison between the category of digroups and the category of groups, and this leads us to a natural defintion of normal subdigroup. The key idea, however, is to recall a useful identification of a digroup with a G-set admitting a fixed point.

In Section 2 this identification is exploited to obtain a definition of quotient digroups. The main point is that the algebraic definition of normality is not as strong in this case as it is in the case of groups.

In Section 3, which is the core of the paper, we then present the analogues of the three classical Isomorphism Theorems. We give along the way several examples, illustrating the similarities as well as the differences between the two categories.

Finally, Section 4 contains a few concluding remarks.

We shall use the fairly standard notation $H < G$ to denote that H is a subgroup, and $H \lhd G$ to denote that H is a normal subgroup of G.

1. The category of digroups

1.1. Digroups and normal subdigroups

Let us begin with the usual definition of a digroup:

Definition 1. A *digroup* is a set D provided with two binary operations $\vdash, \dashv$, which are associative and satisfy the compatibility conditions:

$$\begin{aligned} x \dashv (y \dashv z) &= x \dashv (y \vdash z)\,; \\ (x \vdash y) \vdash z &= (x \dashv y) \vdash z; \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z)\,. \end{aligned}$$

It also possesses a neutral element and inverses in the sense that there exists a distinguished fixed element $\mathbf{e}$ such that for each x, $\mathbf{e} \vdash x = x \dashv \mathbf{e} = x$, and for each x there exists a unique element x^{-1} such that $x \vdash x^{-1} = x^{-1} \dashv x = \mathbf{e}$.

Remark 1. The conditions $\forall\, x$, $\mathbf{e} \vdash x = x \dashv \mathbf{e} = x$ define a so-called *bar unit*. In contrast to the neutral element in a group, in a non-trivial digroup (that is, a digroup that is not a group) bar units are not unique, and the set of these elements is sometimes called the *halo* of the digroup.

The corresponding definition of morphism is then:

Definition 2. Let D and D' be digroups, then:

A function $\Phi : D \to D'$ is a *homomorphism* if

$$\Phi\,(x \vdash y) = \Phi\,(x) \vdash \Phi\,(y)\ ,\ \ \Phi\,(x \dashv y) = \Phi\,(x) \dashv \Phi\,(y)\,,$$

and $\Phi(\mathbf{e}) = \mathbf{e}'$.

If Φ is a digroup homomorphism, then its *kernel* is $\Phi^{-1}(\mathbf{e}')$.

Finally, the sub-objects in the category are:

Definition 3. Let D be a digroup. Then, $S \subseteq D$ is a *subdigroup* of D if $\mathbf{e} \in S$ and S is closed under the digroup operations (including that of taking inverses).

It is easy to check that if $\Phi : D \to D'$ is a digroup homomorphism, then the image of a bar unit (inverse) is a bar unit (inverse) and the direct (inverse) image of a subdigroup is a subdigroup.

Remark 2. From the definitions above it is plain that, indeed, digroups form a pointed category: the trivial group, regarded as a digroup, is clearly a zero object. Obviously, zero objects can also be identified with the subdigroup consisting solely of the distinguished bar unit $\mathbf{e}$ of a given digroup D (see, *e.g.* [4]).

These are all direct adaptations of the corresponding notions in the classical theory of groups. Thus, based on the analogy with this theory, a natural choice for the definition of normality would be stability of the subdigroup under conjugation; but in principle we now have eight possibilities to define the conjugation $x * y * x^{-1}$, where $*$ denotes any of the two digroup operations. They are however not all independent, and from the digroup axioms we have, for instance, that $x \dashv (y \dashv x^{-1}) = x \dashv (y \vdash x^{-1})$, and hence $x \dashv (S \dashv x^{-1}) = x \dashv (S \vdash x^{-1})$; thus, we are left with the following three options: $\forall x \in D$,

$$x \vdash S \dashv x^{-1} \subseteq S\,;\;\; x \dashv S \dashv x^{-1} \subseteq S\,;\;\; \text{or } x \vdash S \vdash x^{-1} \subseteq S.$$

Moreover, no parentheses are needed if we write the conditions in this way, because $x \dashv (y \dashv x^{-1}) = (x \dashv y) \dashv x^{-1}$, etc.; and since in general $x \vdash y \dashv x^{-1}$, $x \vdash y \vdash x^{-1}$ and $x \dashv y \dashv x^{-1}$ are all different, the three possibilities listed are indeed not equivalent. We choose the following option, and in the next sections we shall give arguments explaining why this is the “correct” definition:

Definition 4. Let D be a digroup. A subdigroup S of D is normal, if $\forall$ $x \in D$, $x \vdash S \dashv x^{-1} \subseteq S$.

We notice here that this is in fact equivalent to the following condition: $\forall x$, $x \vdash S = S \dashv x$. The point of this remark (to be proved in Lemma 2, after some preliminary work) is that, just as in the case of groups, this equality gives well-defined equivalence classes on the digroup (that is, pairwise disjoint subsets of the digroup D), which of course is something necessary to define a quotient digroup. However, in marked contrast to the case of groups, in general these equivalence classes do not cover the whole group.

On the other hand, that this definition is adequate is further supported by the following result:

Proposition 1. *Let* $\Phi : D \to D'$ *be a digroup morphism. Then* $\ker(\Phi)$ *is a normal subdigroup of* D.

Proof. The proof is similar to the usual one: if we denote $S = \ker(\Phi)$, and let $x \in D$ be any element, then for $y \in S$

$$\Phi(x \vdash y \dashv x^{-1}) = \Phi(x) \vdash \Phi(e) \dashv \Phi(x^{-1}) = \Phi(x) \vdash e' \dashv (\Phi(x))^{-1} = e',$$

because in any digroup $e \dashv x^{-1} = x^{-1} \dashv x \dashv x^{-1} = x^{-1}$. □

We remark however that similar results hold (and the proofs are even more direct) for the other possible choices of conjugation in the digroup, so that the property of stability of the kernel of a morphism under conjugation alone does not suffice to determine uniquely the condition for normality, and therefore a choice has to be made.

1.2. Digroups and G-sets

To understand the properties of digroups better, it is convenient to use an alternative presentation, stemming from results due to M. Kinyon ([5]) that we can summarize as follows:

Theorem 1. *Given a digroup D, define its set of inverses,*

$$G = \left\{ y \mid \exists\ x \text{ such that } y = x^{-1} \right\},$$

and its set of bar units

$$E = \{ y \mid \forall\ x,\ y \vdash x = x \dashv y = x \}.$$

Then G is a group, and the projections

$$D \to G : x \mapsto a = \mathbf{e} \dashv x = (x^{-1})^{-1}\ ;\ D \to E : x \mapsto \alpha = x \dashv x^{-1},$$

yield a bijection $\rho : D \to G \times E$, with inverse $\rho^{-1} : G \times E \to D$ given by:

$$(a, \alpha) \mapsto \alpha \dashv a;\ \ a \in G, \alpha \in E.$$

The map ρ turns into an isomorphism of digroups if the operations on $G \times E$ are defined as

$$(a, \alpha) \dashv (b, \beta) = (a \dashv b, \alpha)\,; \quad (1)$$

$$(a, \alpha) \vdash (b, \beta) = \left(a \vdash b, a \vdash \beta \dashv a^{-1} \right). \quad (2)$$

We shall call G the group part (or factor) of D, and E the halo or bar unit part.

Actually, in a digroup D the group G is characterized by the fact that both operations restricted to G coincide, so that the symbol for the product might as well (and thus will) be omitted for products of two of these elements.

Moreover, the expression $a \vdash \beta \dashv a^{-1}$ determines a left action of G on E, which we shall write as $a \cdot \beta$. Thus, in any digroup D, E is a G-set, and the neutral element is a fixed point for this action, and the digroup is completely determined by the action of its group of inverses G on its bar units E.

Also, this identification between objects in two categories has a natural counterpart for the morphisms; namely, we can identify a digroup homomorphism with a pair composed by a group homomorphism and an equivariant map on the halo. The precise statement is given in the following lemma:

Lemma 1. *Let $\Phi : D \to D'$ be a digroup homomorphism; let $\rho : D \to G \times E$ and $\rho' : D' \to G' \times E'$ be the corresponding isomorphisms considered in Theorem 1. Then, there exists a unique homomorphism $\Phi' : G \times E \to G' \times E'$, that makes the following diagram commute:*

$$\begin{array}{ccc} D & \xrightarrow{\Phi} & D' \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ G \times E & \xrightarrow[\Phi']{} & G' \times E' \end{array}$$

In fact, $\Phi' \equiv (\phi, \mu)$, where $\phi : G \to G'$, $a \mapsto \phi(a) = \mathbf{e}' \dashv \Phi(a)$, is a group homomorphism, and $\mu : E \to E'$, $\alpha \mapsto \mu(\alpha) = \Phi(\alpha)$ is an equivariant map, i.e., $\mu(a \cdot \alpha) = \phi(a) \cdot \mu(\alpha)$, for all $(a, \alpha) \in G \times E$.

In particular, orbits in E under G are mapped into orbits in E' under $\phi(G) \leqslant G'$.

We can then summarize the above discussion in the following manner (see [6]):

Theorem 2. *A digroup is equivalent to the datum of a G-set E, or equivalently an action $G \times E \to E$, with the only condition for this action of the existence of a fixed point, ε (in particular, the fixed point for the action does not need to be unique). Under this equivalence, the neutral element of the digroup $D = G \times E$ is (e, ε), where e is the identity element of G.*

This relation between digroups and G-sets is in fact an equivalence of categories: each morphism $\Phi : G \times E \to G' \times E'$ corresponds to a pair

(ϕ, μ), *where* ϕ *is a group homomorphism and* μ *is an equivariant map that preserves the distinguished fixed points.*

Thus, in what follows we shall freely identify an element in a digroup, $x \in D$, with a pair $(a, \alpha) = \rho(x) \in G \times E$; the operations on the digroup then become

$$\begin{aligned} (a, \alpha) \dashv (b, \beta) &= (ab, \alpha)\,; \\ (a, \alpha) \vdash (b, \beta) &= (ab, a \cdot \beta)\,. \end{aligned}$$

Remark 3. When doing this identification we shall on occasion, as we already did, write $\mathbf{e} = (e, \varepsilon)$ for the neutral element of the digroup; here e refers to the neutral element of the group G, while ε denotes the preferred fixed point of the G-set of bar units E. This is mostly for bookkeeping convenience, because in fact the two elements might sometimes coincide; in particular, this is the case if we start with a digroup D, so that G and E are both subsets of D; indeed, in this case $G \cap E = \{\mathbf{e}\}$, and therefore $e = \varepsilon$.

Let us consider some examples to illustrate this correspondence:

Example 1. A basic class of digroups can be constructed as follows: Let G be a Lie group, and V a representation space of G; then $G \times V$ becomes a digroup, with the products defined as in (1) and (2). In particular, we can take a linear Lie group $G \subset GL(n, \mathbb{R})$, with the natural action of G on $\mathbb{R}^n$.

Example 2. Another simple class of digroups is the following: Let G be any group, and let it act on itself by conjugation. Then $D = G \times G$ becomes a digroup, with operations

$$\begin{aligned} (a, \alpha) \dashv (b, \beta) &= (ab, \alpha)\,; \\ (a, \alpha) \vdash (b, \beta) &= \left(ab, a\beta a^{-1}\right). \end{aligned}$$

Notice that here, as a subset of D, the group of inverses is $G \times \{e\}$, while $\{e\} \times G$ corresponds to the halo, so the two factors play indeed quite different roles, although here again $e = \varepsilon$.

2. Quotient digroups

2.1. Characterization of subdigroups and of normality

Theorem 2 gives the following very convenient characterization of subdigroups:

Proposition 2. *Let D be a digroup with fixed element (e, ε). A subset $S \subset D$ is a subdigroup if and only if it is of the form $S = R \times T$, where R is a subgroup of G, and $T \subset E$ is invariant under R and $\varepsilon \in T$.*

Proof. Indeed, S is a subdigroup if $\forall\ x = (a, \alpha), y = (b, \beta) \in S$, $x \vdash y = (ab, a \cdot \beta) \in S$, and $x \dashv y = (ab, \alpha) \in S$.

As any digroup, we know that S will have the form $S = R \times T$; now separate the above relations into coordinates: it follows that the group part, R, must be a subgroup of G; the condition given by the operation $\vdash$ then requires that the R-orbit of the bar unit part of any element in S be contained in T.

Finally, it is clear that we need to require $(e, \varepsilon) \in R \times T$, and this yields the desired result. □

Following the reasoning in this proof, let us next compare the three possible definitions of normality; a short and straightforward computation gives that they can be expressed as follows, condition (3) being the one we have chosen as definition of normality:

$$x \vdash S \dashv x^{-1} \subseteq S \iff \left(aba^{-1}, a \cdot \beta\right) \in R \times T, \tag{3}$$

$$x \dashv (S \vdash x^{-1}) \subseteq S \iff \left(aba^{-1}, \alpha\right) \in R \times T, \tag{4}$$

$$(x \dashv S) \vdash x^{-1} \subseteq S \iff \left(aba^{-1}, \varepsilon\right) \in R \times T, \tag{5}$$

$\forall\ x = (a, \alpha) \in D$, $y = (b, \beta) \in S$.

Again, splitting these conditions into coordinates, we see that in the factor G it makes no difference which one of the different possibilities we choose: the only option in the group factor for normality of the subdigroup is that $R \lhd G$, as expected. Thus, it is only on the bar units part that we have to make choices.

But now consider condition (4). Since α is arbitrary, this implies that $T \supseteq E$, that is $T = E$; hence this condition is too restrictive, because the only normal subdigroups allowed by it would correspond in a one-to-one fashion to the normal subgroups of G, and the quotients would be trivial as digroups. Condition (5), on the other hand, imposes no restriction on T (besides the R-invariance already implicit from the fact that S is a subdigroup); we will see in a moment that in a definite sense this is too flexible.

However, condition (3) imposes a non-trivial restriction, which we record as:

Proposition 3. *A subdigroup $N = H \times K$ of a digroup $D = G \times E$ is normal iff:*

1) *H is a normal subgroup of G*
2) *K is an invariant subset of E under the action of the whole group G.*

Proof. As said, it is immediate from the condition

$$\forall x \in D, \quad x \vdash N \dashv x^{-1} \subseteq N \iff$$

$$\forall x = (a, \alpha) \in D,\ y = (b, \beta) \in N,\ \left(aba^{-1}, a \cdot \beta\right) \in H \times K. \qquad \square$$

Let us give some examples of subdigroups and normal subdigroups:

Example 3. Let $D = G \times G$ be a digroup as in Example 2. Let us describe the subdigroups of D of the form $N = H \times K$, where K is also a subgroup of G:

For a general digroup of this form, since H acts on K by conjugation if and only if H is a subgroup of the normalizer of K, this is the only restriction on N.

On the other hand, for N to be a normal subdigroup, K also needs to be invariant under conjugation by the whole group, so both, H and K, must be normal subgroups of G.

Example 4. Let $G = S^1 = U(1)$ be the circle group, and make it act on the sphere S^2 by rotations around the z-axis; to get a digroup choose as fixed point one of the poles, say the south pole S.

Since G is abelian any subgroup is normal, and the G-orbits are the parallels of the sphere, together with the two poles. Therefore, the normal subdigroups are given by an arbitrary subgroup of G as the group factor, together with the set theoretic union of the singleton $\{S\}$ and an arbitrary set of orbits, as the bar-units factor.

Examples of non-normal subdigroups are obtained by taking as subdigroup $H \cong \mathbb{Z}_n$, the group of n-th roots of unity, and as K the union of $\{S\}$ and any orbit of H, such as a regular n-gon contained in a fixed parallel.

2.2. Normal subdigroups and quotient digroups

We now start the analysis of the construction of quotient digroups, by proving the assertion made after Definition 4:

Lemma 2. *A subdigroup N is normal iff $\forall x$, $x \vdash N = N \dashv x$.*

Any two of these sets either coincide or are disjoint.

Proof. For the direct implication, note that $x \vdash N \dashv x^{-1} \subseteq N$ implies

$$x \vdash N = x \vdash N \dashv \mathbf{e} = (x \vdash N) \dashv (x^{-1} \dashv x)$$
$$= (x \vdash N \dashv x^{-1}) \dashv x \subseteq N \dashv x \quad (6)$$

To get the converse inclusion, observe that since for any x, $\mathbf{e} \dashv x = (x^{-1})^{-1}$, we have that

$$N \dashv x = (x \vdash x^{-1}) \vdash N \dashv (\mathbf{e} \dashv x) = x \vdash (x^{-1} \vdash N \dashv (x^{-1})^{-1}) \subseteq x \vdash N$$

The converse implication is even simpler, since $x \vdash N = N \dashv x$ implies

$$x \vdash N \dashv x^{-1} = N \dashv (x \dashv x^{-1}) = N \dashv (x \vdash x^{-1}) = N \dashv \mathbf{e} = N.$$

For the last statement, if we write $x = (a, \alpha)$, then

$$x \vdash N = \{\ (ab, a \cdot \beta)\ ;\ b \in H\ ,\ \beta \in K\ \} = aH \times K.$$

Since $H \lhd G$, whenever two of these sets are distinct they are disjoint, and so the assertion follows. □

The previous lemma is the gist of the main difference between the group and digroup cases, for it shows that the algebraic condition for normality partitions in a non-trivial way the subset $G \times K \subset D$, but says *nothing* about its complement $G \times (E \setminus K)$ (but in fact neither do the other posibilities). As a consequence, this condition alone does not suffice to uniquely determine suitable equivalence classes in D, on which a digroup structure can be defined, and it turns out that such 'good' partitions of D are not unique. We now discuss this point, starting with the following definition:

Definition 5. Let $N = H \times K$ be a normal subdigroup of a digroup $D = G \times E$. We say that a partition of E is *admissible* for the subdigroup N, if it contains K and the group G/H acts on the partition; more precisely, if we denote the equivalence class containing $\alpha \in E$ by $[\alpha]$, then the partition satisfies $H \cdot [\alpha] = \{h \cdot \xi\, ;\ h \in H\, ,\ \xi \in [\alpha]\} = [\alpha]$, and for all $a \in G$ and $[\alpha]$ in the partition $[a \cdot \alpha] = a \cdot [\alpha]$, is also in the partition.

The first condition implies that the equivalence classes in such a partition are composed by H-orbits, while the second ensures that G/H acts on the equivalence classes. Examples of admissible partitions are K together with all remaining H-orbits in E, or K and a partition of $E \setminus K$ that is G-invariant, such as $E \setminus K$ itself.

It is also clear that an admissible partition of E also partitions D, namely, if we write $[a] = aH$, and let $x \in D$ be identified with $(a, \alpha) \in G \times E$, the equivalence class containing x, say Ξ, will be the 'rectangular box' in D $[a] \times [\alpha]$, and we shall also say that such a partition of D is admissible for N. Note that by hypothesis $[e] = H$ and $[\varepsilon] = K$. Figure 1 might be helpful to visualize what we are doing; it exhibits in particular the classes already discussed in Lemma 2.

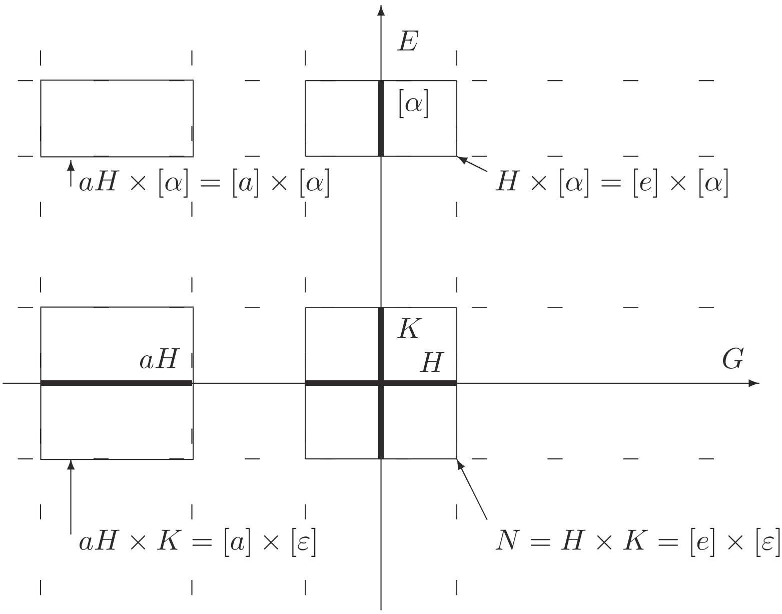

FIGURE 1. An admissible partition for a normal subdigroup $H \times K$.

The important point is that the set of these equivalence classes can be endowed with a structure of digroup:

Theorem 3. *Let $N = H \times K$ be a normal subdigroup of $D = G \times E$.*

Then, any partition of E admissible for N induces an admissible partition of D, so that the set of equivalence classes inherits a structure of digroup, with the operations defined as follows: Let $\Xi = [a] \times [\alpha]$ and $\Upsilon = [b] \times [\beta]$ be two equivalence classes, then set

$$\Xi \vdash \Upsilon = [a][b] \times [a] \cdot [\beta] = [ab] \times [a \cdot \beta]; \tag{7}$$

$$\Xi \dashv \Upsilon = [a][b] \times [\alpha] = [ab] \times [\alpha], . \tag{8}$$

The projection map

$$(a, \alpha) \mapsto [a] \times [\alpha]$$

is then a surjective morphism of digroups, whose kernel is N*.*

Conversely, any surjective morphism of digroups, whose kernel is N*,*

$$\Phi = \phi \times \mu : D = G \times E \to D' = G' \times E',$$

induces an admissible partition of D *by taking inverse images of elements of* E'*.*

Proof. Given a partition of D admissible for normal subdigroup N, denote the quotient $\hat{D} = \hat{G} \times \hat{E}$. It is then clear that the first equality in both (7) and (8) is exactly what is required for the set of these equivalence classes to be a digroup, according to Theorem 2, while the second equalities are the natural way to relate the operations on the classes to the original digroup operations.

We still need to check two things; first of all, that these products are well defined. So, let $a \sim a'$, $b \sim b'$, and $\alpha \sim \alpha'$, $\beta \sim \beta'$; then, for the group part, $[ab] = [a'b']$ is just the classical statement about normal subgroups, and in particular $\hat{G} = G/H$. As for the bar unit part, for the product $\dashv$ there is nothing to check, while for the $\vdash$ product, this holds true precisely because of the way admissible partitions were defined.

Second, that $\hat{G}$ acts on $\hat{E}$, so that (7) and (8) do define a quotient digroup structure on the set of equivalence classes. As already said, this is again by definition of admissible partition, for we have $aH \cdot [\alpha] = a \cdot (H \cdot [\alpha]) = a \cdot [\alpha]$, and because $H \lhd G$, this is indeed an action. Moreover, this also implies that the map induced on the bar units is equivariant, thus showing that the projection is a digroup homomorphism.

To prove the last assertion, if $\Phi : D \to D'$ is a surjective digroup homomorphism, with $\ker \Phi = N = H \times K$, then for a given $x' = (a', \alpha') \in D'$,

$$\Phi^{-1}(x') = \phi^{-1}(a') \times \mu^{-1}(\alpha'),$$

so the partition of D obtained this way is indeed by rectangular boxes, and we need to check that it is admissible.

Again, since $H = \ker \phi \lhd G$, for the group part the assertion follows immediately from the classical result, and hence, we only need to check that the partition of E induced by μ is admissible. Now, $K = \mu^{-1}(\varepsilon')$, is obviously in the partition, so consider any other bar-unit $\alpha' \in E'$, and let $\alpha \in \mu^{-1}(\alpha')$; we need to show that for $a \in G$, we have the equality of sets $a \cdot \mu^{-1}(\mu(\alpha)) = \mu^{-1}(\mu(a \cdot \alpha))$, and by equivariance we have that $\mu^{-1}(\mu(a \cdot \alpha)) = \mu^{-1}(\phi(a) \cdot \mu(\alpha))$.

Thus, if $\xi \in \mu^{-1}(\mu(\alpha))$, then $\mu(a \cdot \xi) = \phi(a) \cdot \mu(\xi) = \phi(a) \cdot \mu(\alpha)$, and we get one contention. For the other contention, if $\xi \in \mu^{-1}(\phi(a) \cdot \mu(\alpha))$, then $\mu(a^{-1} \cdot \xi) = \phi(a^{-1}) \cdot \mu(\xi) = \mu(\alpha)$, so that $a^{-1} \cdot \xi \in \mu^{-1}(\mu(\alpha))$, and hence $\xi \in a \cdot \mu^{-1}(\alpha)$; this shows that the partition is admissible, as desired. $\square$

Remark 4. Notice that because in (7) and (8) a is arbitrary, these constructions are consistent precisely because K is G-invariant, and not just H-invariant, since this ensures that K is a fixed point for the action on the quotient. This would not be the case had we chosen to require as normality condition (5), $(x \dashv N) \vdash x^{-1} \subseteq N$, and this is an additional justification to (3) as the proper definition of normality.

The previous theorem finally leads us to the following definition:

Definition 6. Let $N = H \times K$ be a normal subdigroup of $D = G \times E$. An admissible quotient for N is a digroup Q, together with a surjective digroup morphism $\Psi_Q : D \to Q$, such that $N = \ker(\Psi_Q)$.

Let us give some examples of these quotients:

Example 5. Consider the digroups of Example 2, $D = G \times G$; suppose that $H \lhd K \lhd G$, and consider the subdigroup $H \times K$. Then, if we choose as the admissible partition of the set of bar units G the classes induced by K, the quotient digroup constructed above can be identified to $G/H \times G/K$.

Moreover, although the fact that G/K also has a group structure is unimportant here, in this case the canonical projection $\pi_K : G \to G/K$ is equivariant, so that the map (π_H, π_K) is a digroup morphism, whose kernel is precisely $E = H \times K$.

But there are other admissible quotients: For instance, the partition of G, regarded as the set of bar units, $\{\, K\,,\ G \setminus K \,\}$, is also admissible, and the quotient digroup would then be $G/H \times \{\, 0, 1 \,\}$, with the trivial action of the group part on the bar units; but unless the order of K in G is 2, this is a different quotient.

Example 6. Consider the digroup of Example 4, $U(1) \times S^2$; for the normal subdigroup choose as $H = \mathbb{Z}^n$, the subgroup of n-th roots of unity, and as K the union $\{S\} \cup \{N\} \cup \mathcal{E}$, where N is the north pole and $\mathcal{E}$ is any parallel, other than the poles, such as the equator. Finally, as partition of the complement take the individual H-orbits.

The quotient is then isomorphic to a product of the form $S^1 \times \hat{E}$, where topologically $\hat{E}$ can be visualized as follows: first, contract the parallel $\mathcal{E}$

to its center; this leaves us with two smaller spheres touching at a point on the axis connecting the two original poles; then, pinch the resulting spheres until these three points get identified. The induced action of the new circle group $S^1 \cong U(1)/\mathbb{Z}_n$ on the quotient $\hat{E}$ can still be thought of as being by rotations, but performed n times 'faster' than for the original action, as the map $E \to \hat{E}$ is generically an n-fold covering (technically, it is a *ramified* covering).

3. The analogues of the Isomorphism Theorems

3.1. The Strong Kernel and the First Isomorphism Theorem

If we start with a fixed homomorphism $\Phi : D \to D'$, then we have associated the normal subdigroup $N = \ker(\Phi)$; but in contrast to the group case, knowledge of N alone does not allow us to recover the morphism, even assuming it to be onto. This is so because, when we identify $\Phi = (\phi, \mu)$, the subdigroup N does not keep track of the effect of the equivariant map μ, as N one only involves the subset of the bar units $K = \mu^{-1}(\varepsilon')$. Moreover, there is another admissible quotient naturally determined by Φ, given by $\Phi^{-1}(E') = \ker(\phi) \times E$ (which is also a normal subdigroup of D); but clearly this is still not enough to recover the morphism Φ, because $D/\Phi^{-1}(E') \cong G/\ker(\phi)$.

To address this situation we proceed as follows:

First, the equivariance of μ determines a natural partition of E, with classes $\mu^{-1}(\xi')$, $\xi' \in E'$. Thus, it defines another admissible quotient that can be associated to the normal subdigroup N, motivating the following:

Definition 7. If $\Phi = (\phi, \mu)$ is a morphism, we define its strong kernel $\mathrm{Sker}(\Phi)$, as the congruence induced by $\ker(\phi)$ together with the admissible partition induced by the equivariant map μ:

$$\alpha \sim \beta \iff \mu(\alpha) = \mu(\beta).$$

The partition given by the strong kernel is in fact the same as the one given by $x \sim y$ iff $\Phi(x) = \Phi(y)$. This is the standard way in universal algebra to obtain a congruence from a morphism, and quotients via this relation lead to isomorphism theorems. But in general with no regard to other properties, such as those associated with the presence of a zero object, in our case given by the special bar unit of the digroup. In a sense what we are trying to do is measure the obstructions to the isomorphism theorems when we attempt to define the quotient solely by the kernel defined by the zero object. This was the motivation for our analysis of the partitions of the space of bar units, leading to notions of admissible quotients and strong kernel.

In any case, Theorem 3 also suggests that for a fixed a normal subdigroup N, the admissible quotient associated to the *finest* possible partition plays a special role. The bar units of the corresponding quotient, obtained from the individual H-orbits, are $H \times H \cdot \alpha$, for $\alpha \in E \setminus K$, together with N, which has to be the neutral element of the quotient digroup, and this gives a sort of 'semi-universal' quotient. The whole idea is made precise in the following technical lemma:

Lemma 3. *Let $N = H \times K$ be a normal subdigroup of $D = G \times E$. Then, any partition of $E \setminus K$ as in Theorem 3, together with the corresponding natural projection, determines an admissible quotient.*

Conversely, any admissible quotient for N, $\Psi_Q : D \to Q$, determines a partition of E, so that the class of ε is K, and $E \setminus K$ is partitioned into H-invariant sets determined by $\mathrm{Sker}(\Psi_Q)$*.*

Also, the digroup D', defined on the partition determined by the bar units of the form $H \times H \cdot \alpha$, for each $\alpha \in E \setminus K$, together with the special bar unit K, is one of these admissible quotients for N; one that is moreover universal in the following sense:

For any admissible quotient Q associated to N there exists a unique surjective digroup morphism $\Phi_Q : D' \to Q$ such that the morphism Ψ_Q factorizes through $\Psi_{D'}$:

$$\Psi_Q = \Phi_Q \circ \Psi_{D'}.$$

Proof. The first assertions, and in particular that D' is an admissible quotient have essentially been established; it only remains to show that the mapping $(a, \alpha) \mapsto [a] \times [\alpha] = aH \times H \cdot \alpha$, $\alpha \in E \setminus K$, gives a digroup morphism, but this follows immediately from the definition of the products on the quotient, as given in Theorem 3.

Now, to prove the universal condition, assume $Q = M \times L$ is an admissible quotient. If $X \in Q$, let $x = (a, \alpha)$ be such that $X = \Psi_Q(x)$, and define $\Phi_Q(X) = \Psi_Q(x)$. We only need to check that Φ_Q is well defined.

But the digroup morphism Ψ_Q also splits into a group morphism and an equivariant map: $\Psi_Q(x) = (\psi_Q(a), \mu_Q(\alpha))$. Thus, if (b, β) is in the same class as (a, α), automatically $\psi_Q(b) = \psi_Q(a)$, as ψ_Q is an ordinary group projection, and hence we have just two possibilities left: either $\alpha, \beta \in K$ or $\beta = h \cdot \alpha$, $h \in H$.

In the first case trivially $\mu_Q(\beta) = \mu_Q(\alpha)$, because by definition of a morphism $\mu_Q(K) = \{ \varepsilon \} \subset N$; in the second

$$\mu_Q(\beta) = \mu_Q(h \cdot \alpha) = \psi_Q(h) \cdot \mu_Q(\alpha) = e \cdot \mu_Q(\alpha) = \mu_Q(\alpha),$$

and thus we reach the desired conclusion. □

For convenience, let us introduce the following terminology:

Definition 8. Given a digroup morphism $\Phi = (\phi, \mu)$, the quotient digroup $D/\mathrm{Sker}(\Phi)$, is the admissible quotient associated to the normal subdigroup $N = \ker(\Phi)$ defined by the partition given by $\mathrm{Sker}(\Phi)$, as in Definition 7.

Given a normal subdigroup N, we denote the quotient digroup constructed by the finest partition, as in Lemma 3, as D/N, and call it the *universal* quotient of $D \mod N$.

The two quotients might coincide, as shown by the following example; but it is clear that in general they do not:

Example 7. Let $H \lhd G$, and consider the digroup $D = G \times G$ as before. The universal quotient of D determined by the normal subdigroup $N = H \times H$ is $D/N = G/H \times G/H$. This can also be described as the admissible quotient $D/\mathrm{Sker}(\Phi)$, associated to the digroup morphism $\Phi = \pi_H \times \pi_H$, where $\pi_H : G \to G/H$ is the natural projection.

We now state the analogue for digroups to the standard First Isomorphism Theorem; as consequence of the above discussion, the conclusion has to be split into two parts:

Theorem 4. *Each morphism $\Phi : D \to D'$ determines a normal subdigroup $N = \ker(\Phi)$ and two admissible quotients for N, D/N and $D/\mathrm{Sker}(\Phi)$, so that:*

1) *There is a 1 to 1 correspondence between normal subdigroups N of a digroup D and universal quotients D/N.*
2) *The digroup $\Phi(D)$ is a subdigroup of D' isomorphic to $D/\mathrm{Sker}(\Phi)$.*

The proof is now immediate, but to conclude this part let us make the following useful observation:

Remark 5. Given the normal subdigroup $N \lhd D$, because the set $(E \backslash K)$ is an H-set which we partition into its H-orbits, the universal quotient D' can be expressed in the following form:

$$D/N = (G/H) \times (\{K\} \cup (E \setminus K)/H)\,.$$

Example 8. If D is a digroup, and $H \lhd G$, there is clearly a minimal normal subdigroup having H as its group factor, namely $H \times \{\varepsilon\}$; correspondingly, the maximal universal quotient having group part G/H is $(G/H) \times (E/H)$.

3.2. The lattice of subdigroups and the Second and Third Isomorphism Theorems

To obtain the remaining isomorphism theorems we will regard them as assertions concerning the universal quotients, as these are the non-trivial admissible quotients independent of the additional datum of a specific morphism.

First we define the join of two subdigroups S_1, S_2, denoted S_1S_2, as the minimal subdigroup containing both subdigroups; this, together with the intersection of subdigroups defines the lattice structure in the subdigroups of a given digroup D. But in order to state the version for digroups of the Second Isomorphism Theorem, a more explicit description of the join is convenient; this is given in the next lemma:

Lemma 4. *Let $S_1 = R_1 \times T_1$, $S_2 = R_2 \times T_2$, be two subdigroups of $D = G \times E$, and let S_1S_2 denote the minimal subdigroup containing both S_1 and S_2.*

Then $S_1S_2 = R_1R_2 \times (R_1R_2) \cdot (T_1 \cup T_2)$, where R_1R_2 is the subgroup generated by R_1 and R_2 (that is, the standard join of the subgroups), and $(R_1R_2) \cdot (T_1 \cup T_2)$ denotes the union of R_1R_2-orbits of the elements of $T_1 \cup T_2$.

Proof. Again, we know that the join S_1S_2 will have a product structure $R \times T$. Since it is minimal, the group factor R is necessarily the standard join R_1R_2 of the corresponding group parts.

But then, the bar units factor T must contain all the R_i-orbits of elements of T_i, $i = 1, 2$, so it must include $T_1 \cup T_2$, and must be R_1R_2-invariant, and so must include the orbits under the action of R_1R_2, that is $(R_1R_2) \cdot (T_1 \cup T_2) \subset T$.

Finally, since $R_1R_2 \times (R_1R_2) \cdot (T_1 \cup T_2)$ has the structure of a subdigroup, and it has to be included in the join S_1S_2, by minimality this must be the join of the subdigroups. □

Now we can state the analogue for digroups of the Second Isomorphism Theorem; notice that for the bar units factors the statement is weaker than for the group parts:

Theorem 5. *Let S be a subdigroup and N a normal subdigroup of the digroup D. Then $S \cap N$ is a normal subdigroup of S and we have, for the corresponding universal quotients, a surjective digroup morphism:*

$$S/(S \cap N) \twoheadrightarrow SN/N,$$

that is an isomorphism on the group factors.

In other words, SN/N is isomorphic to an admissible quotient for the normal subdigroup $S \cap N$ of S, but in general not to the universal quotient $S/(S \cap N)$.

Proof. That $S \cap N$ is normal in S is clear, the proof being identical to the case of groups.

To prove the existence of the morphism, write the digroups as $N = H \times K$, $S = R \times T$, and observe first that for the group factors this is just the classical theorem; therefore, we can express the quotients as digroups of the form

$$SN/N = (RH/H) \times \tilde{E}\ ; \quad S/(S \cap N) = R/(R \cap H) \times \hat{E},$$

and it now suffices to consider the bar units part. We want to construct an equivariant projection $\mu : \hat{E} \to \tilde{E}$, and to this end, we further decompose these sets (according to the characterization of the universal quotients in Theorem 4) as:

$$\tilde{E} = \{\, K \,\} \cup \{\, H \cdot \alpha\ ;\ \alpha \in RH \cdot (T \setminus K) \,\} = \{\, K \,\} \cup \{\, H \cdot \alpha\ ;\ \alpha \in H \cdot (T \setminus K) \,\},$$

where the last equality holds because H is normal and $T \setminus K$ is R-invariant, and

$$\hat{E} = \{\, K \cap T \,\} \cup \{\, (H \cap R) \cdot \alpha\ ;\ \alpha \in T \setminus K \,\}.$$

Now, consider the map

$$\mu :\ (H \cap R) \cdot \alpha \mapsto (H \cdot \alpha)$$

for $\alpha \in T \setminus K$, and $\mu(\{K \cap T\}) = \{K\}$. It is well defined, for if $\beta \in (H \cap R)$ is another representative, then there exists $h \in H$ such that $\beta = h \cdot \alpha$; but this means that $H \cdot \alpha = H \cdot \beta$. Further, it is trivially a surjection.

Finally, the equivariance of the map also follows easily from its definition, since for $r \in R$ the actions are, on the one side,

$$r(H \cap R) \cdot ((H \cap R) \cdot \alpha) = (H \cap R) \cdot (r \cdot \alpha),$$

while on the other,

$$rH \cdot (H \cdot \alpha) = H \cdot (r \cdot \alpha),$$

both equalities holding true by normality of H. □

But as mentioned, the result is weaker than for groups: the map is not in general an isomorphism, since it is not necessarily injective. This is due to the fact that the orbits of the larger group H might contain more than one orbit of the smaller group $R \cap H$, and these will get identified in the quotient. In other words, to have an isomorphism $S/(S \cap N) \approx SN/N$ we would need the additional requirement that no two different $H \cap R$-orbits in $T \setminus K$ are contained in the same H-orbit in $H \cdot (T \setminus K)$. This can be seen for instance in the following simple example:

Example 9. Consider the digroup $D = \mathbb{Z}_6 \times E$, where $E = P_1 \cup P_6$, and where by P_n we denote the regular n-sided polygon (P_1 being a single point and P_2 two points). Thus, here E can be viewed as a regular hexagon together with its center, and the action is the natural one, by rotations by 60° angles; the point P_1 is needed to have a fixed point of the action. Now, let $N = \mathbb{Z}_2 \times P_1$, and $S = \mathbb{Z}_3 \times E$, so that $H \cap R = \{e\}$.

Then, it is easy to see that $SN = D$ and $S \cap N = \{e\} \times P_1 = \{\mathbf{e}\}$ is the trivial subdigroup. Therefore,

$$SN/N = D/N \cong \mathbb{Z}_3 \times \hat{E},$$

where $\hat{E} = P_1 \cup P_3$ is a triangle, corresponding to the antipodal identification of the vertices of the hexagon given by the action of $\mathbb{Z}_2$, together with a point (corresponding to the projection of the center P_1), while obviously

$$S/(S \cap N) = S/\{\mathbf{e}\} \cong \mathbb{Z}_3 \times E,$$

and so the quotient digroups are different.

Finally, the analogue of the Third Isomorphism Theorem is:

Theorem 6. *Let N, S be normal subdigroups of D, such that $N \subset S$. Then, for the corresponding universal quotients we have:*

$$S/N \quad \textit{is a normal subdigroup of} \quad D/N$$

and there is a digroup isomorphism

$$(D/N)/(S/N) \to D/S.$$

Proof. Write N, S and D as in the previous theorem. To see that S/N is normal in D/N we need to show that $(T \setminus K)/H$ is invariant under the action of the group G/H. But this is clear, since the action is

$$(gH, H \cdot \alpha) \mapsto gH \cdot (H \cdot \alpha) = H \cdot (g \cdot \alpha) \subset T \setminus K,$$

because H is normal and both T and K are G-invariant. Thus, the quotient $(D/N)/(S/N)$ is well-defined.

Moreover, according to Remark 5, this quotient can be written as follows:

$$(D/N)/(S/N) = (G/H)/(R/H)\times \Big(\{\{K\} \cup (T \setminus K)/H\} \cup (((E \setminus K)/H) \setminus ((T \setminus K)/H))/(R/H)\Big) = (G/H)/(R/H)\times (\{\{K\} \cup (T \setminus K)/H\} \cup ((E \setminus T)/H)\,/(R/H)),$$

because $K \subset T$, so that

$$(((E \setminus K)/H) \setminus ((T \setminus K)/H))/(R/H) = ((E \setminus T)/H)\,/(R/H)).$$

Now, for the group parts the claim is again just the classical result, and so it suffices to see what happens to the bar units. For these, obviously we need to map $\{\{K\} \cup (T \setminus K)/H\} \mapsto \{T\}$, so it remains to define the map on $((E \setminus T)/H)/(R/H)$. We do this by sending a class represented by $H \cdot \alpha$, $\alpha \in E \setminus T$, into the class $R \cdot \alpha$.

The map is well-defined, since if $H \cdot \alpha$ and $H \cdot \beta$ are equivalent modulo R/H, then $rH \cdot \alpha = H \cdot (r \cdot \alpha) = H \cdot \beta$, for some $r \in R$; but this is the same as $\beta = hr \cdot \alpha$ for some $h \in H$, and because $H \subset R$, this means that $R \cdot \beta = R \cdot \alpha$. Moreover, this is clearly a surjective map.

But now we can define an inverse map, by sending $R \cdot \alpha \mapsto (R/H) \cdot (H \cdot \alpha)$, and this is also well-defined, because if $R \cdot \alpha = R \cdot \beta$, then $\exists\, r \in R$ such that, $\beta = r \cdot \alpha$; therefore,

$$(R/H) \cdot (H \cdot \beta) = (R/H) \cdot (H \cdot r \cdot \alpha) = r \cdot (R/H) \cdot (H \cdot \alpha) = (R/H) \cdot (H \cdot \alpha).$$

Finally, by construction, this map is clearly equivariant, and so we have a digroup isomorphism, as claimed. □

For the sake of completeness, let us conclude by giving an explicit example illustrating the last theorem:

Example 10. With notations as in Example 9, let $D = \mathbb{Z}_{12} \times (P_1 \cup P_{12} \cup \hat{P}_{24})$, $N = \mathbb{Z}_3 \times P_1$, and $S = \mathbb{Z}_6 \times (P_1 \cup P_{12})$ (where we put the hat to stress that the two $\mathbb{Z}_{12}$-orbits contained in $\hat{P}_{24}$ are not those in S). It is clear that all the hypotheses of the theorem are satisfied.

Then it is easy to see that, for instance,

$$D/S \cong \mathbb{Z}_2 \times (P_1 \cup \hat{P}_4),$$

where P_1 now corresponds to the identification of the original $P_1 \cup P_{12}$ to a point, while $\hat{P}_4$ corresponds to the quotient of $\hat{P}_{24}$ under the action of $\mathbb{Z}_6$. Similarly, one has

$$D/N \cong \mathbb{Z}_4 \times (P_1 \cup P_4 \cup \hat{P}_8) \ ; \quad S/N \cong \mathbb{Z}_2 \times (P_1 \cup P_4).$$

Hence, for $(D/N)/(S/N)$, the bar unit space of D/N is partitioned into $P_1 \cup P_4$, which will be identified to a point, while $\hat{P}_8$ will be partitioned into four $\mathbb{Z}_2$-orbits; thus we get again a point and a square, so that

$$(D/N)/(S/N) \cong \mathbb{Z}_2 \times (P_1 \cup \hat{P}_4),$$

and therefore both quotients are indeed isomorphic.

4. Some final remarks

There are several directions in which this work can be continued and refined:

First, having a good control over the notion of normality and the associated quotients, one can attempt a finer analysis of the structure of digroups. To name just one possibility, it would be an interesting question to decide if there is an analogue of the concept of a normal series and the Jordan-Hölder Theorem for digroups.

Secondly, the interplay between the algebraic questions about digroups and the more geometric questions about G-sets must give more insight in both directions: In this work we saw, particularly through the examples, such as Example 6, how some natural constructions coming from the algebraic side translate into geometric properties of the quotients. Obviously, we can expect that much more could be obtained in this direction, and it would be interesting to establish some general results; but the point we want to stress now is that this is most certainly a two-way road, and so there should be also some natural geometrical questions implying some interesting algebraic properties. Suffice it to recall that, in fact, the very notion of digroup arose from the geometrical problem of attempting to define a structure furnishing integral manifolds for Leibniz algebras (see *e.g.* [5] or [6]).

As an example, we mention finally the problem of extending the analysis to the so-called *generalized digroups*, which are defined by dropping the requirement that inverses be bilateral, and which, for this reason, do not require actions with fixed points. A straightforward generalization of the constructions done here is probably not possible, but it seems certain that at least some of the results can be recovered in this ampler context.

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