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An amalgamation property for metric spaces

Aleksander Ivanov* **and Barbara Majcher-Iwanow****

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Abstract. In this paper we show that suficiently similar finite metric spaces can be amalgamated so that the distance between them is sufficiently small.

1. Introduction

The following is usually called *the free amalgamation property for finite metric spaces*, see Theorem 2.1 of [1].

Assume that (X, d_1) *and* (Y, d_2) *are finite metric spaces with* $Z = X \cap Y$ *and* $d_1 = d_2$ *for elements of* Z*. Then there is a metric* d *on* $X \cup Y$ *which agrees with* d_1 *on* X*, with* d_2 *on* Y *and is defined for* $x \in X \setminus Y$ *and* $y \in Y \setminus X$ *by*

$$d(x, y) = \min_{z \in Z}(d_1(x, z) + d_2(y, z)) \text{ when } Z \neq \varnothing;$$
$$d(x, y) = d_1(x, x^*) + d(x^*, y^*) + d_2(y^*, y) \text{ when } Z = \varnothing$$
$$\text{and } x^* \in X,\ y^* \in Y \text{ are distinguished together}$$
$$\text{with the distance between them.}$$

It has been applied in a number of places. We recommend [4] and [5] for some information concerning applications in the case of the *Urysohn space of diameter 1*, $\mathbb{U}$.

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The main result of our paper is an amalgamation property which roughly states that if A and B are finite metric spaces which are sufficiently similar then there is a metric on $A \cup B$ extending the metrics of A and B so that A and B are sufficiently close in $A \cup B$.

Some version of this theorem was presented in Section 3 of the first version of the preprint [3]. In that paper we apply it in order to evaluate the distance between types in $Th(\mathbb{U})$ with parameters. It is worth noting that the parameter-free case (i.e. when $A \cap B = \varnothing$) was also considered in [2], where the construction was based on Example 56 from [6], p. 295–296. Although the general case looks slightly more complicated than the case from [2], we have found that the idea of Example 56 of [6] works in the general case too. This improves the corresponding result from [3] and simplifies the proof. A model-theoretic version of this material can be found in Section 6 of the latest version of [3].

Since the essence of this material is elementary we have decided to present it in this paper in the simplest form. Below we do not use any mathematics beyond elementary metric topology.

2. Amalgamation

The following proposition states the property that if A and B are finite metric spaces which are sufficiently nice and sufficiently similar then there is a metric space C containing A and a copy of B under an isometry fixing $A \cap B$ so that A and the image of B are sufficiently close in C.

Proposition 1. *Let a finite metric space $A = \{a_1, \ldots, a_n\}$ and numbers $0 \leqslant q < n$ and $\varepsilon > 0$ satisfy all inequalities of the form*

$$4\varepsilon < d(a_i, a_j) \text{ for pairs } i < j \leqslant n \text{ with } q < j \text{ and}$$

$$4\varepsilon < d(a_i, a_j) + d(a_i, a_k) - d(a_j, a_k) \text{ for triples } a_i, a_j, a_k \text{ with } |\{i, j, k\}| = 3$$

$$\text{and } k \leqslant q < \min(i, j) < n.$$

Let B be an n-element metric space consisting of elements b_i so that for each pair $i < j \leqslant n$, $|d(b_i, b_j) - d(a_i, a_j)| \leqslant \varepsilon$. We assume that $a_1 = b_1, \ldots,$ $a_q = b_q$, $A \cap B = \{a_1, \ldots, a_q\}$, and the metric defined on $\{b_1, \ldots, b_q\}$ in the space B coincides with the metric defined on $\{a_1, \ldots, a_q\}$ in A.

Then there is a metric on $A \cup B$ extending metrics in A and B so that for each $q < i \leqslant n$, $d(a_i, b_i) = \varepsilon$.

Proof. The case $A \cap B = \varnothing$ is already considered in Example 56 [6], p. 295 - 296. We will assume that $A \cap B \neq \varnothing$. Let us build a metric space on $\{a_1, \ldots, a_n, b_{q+1}, \ldots, b_n\}$ so that $d(a_i, b_i) = \varepsilon$ for all $q < i \leqslant n$. We define $d(a_i, b_j)$ for $i \neq j$ with $i > q$ or $j > q$ as follows:

$$d(a_i, b_j) = \min(\{d(a_i, a_k) + d(a_k, b_j) : k \leqslant q\} \cup$$

$$\{d(a_i, a_k) + \varepsilon + d(b_k, b_j) : k > q\}).$$

Below we will use the observation that in the cases $i \leqslant q < j$ or $j \leqslant q < i$ the metric on B or A respectively which is given in the formulation of the proposition, satisfies the condition of this definition. To see this one should note that in these cases

$$d(a_i, b_j) \leqslant d(a_i, a_k) + \varepsilon + d(b_k, b_j) \text{ when } k > q.$$

Indeed, if for example $i \leqslant q < j$ then

$$d(a_i, b_j) \leqslant d(b_i, b_k) + d(b_k, b_j) \leqslant d(a_i, a_k) + \varepsilon + d(b_k, b_j).$$

Let us verify the triangle inequality. We may restrict ourselves by triangles which intersect both $A \setminus B$ and $B \setminus A$. We will use below the following consequence of the assumptions of the proposition:

$$3\varepsilon < d(b_i, b_j) \text{ for pairs } i < j \leqslant n \text{ with } q < j \text{ and}$$

$$\varepsilon < d(b_i, b_j) + d(b_i, b_k) - d(b_j, b_k) \text{ for triples } b_i, b_j, b_k \text{ with } |\{i, j, k\}| = 3$$

$$\text{and } k \leqslant q < \min(i, j) < n.$$

Case 1. $d(a_i, b_j) \leqslant d(a_i, a_l) + d(a_l, b_j)$. By the assumptions of the proposition we may assume that $i \neq j$. If $d(a_l, b_j) = d(a_l, a_k) + d(a_k, b_j)$ with $k \leqslant q$, then

$$d(a_i, a_l) + d(a_l, b_j) = d(a_i, a_l) + d(a_k, a_l) + d(a_k, b_j) \geqslant$$

$$d(a_i, a_k) + d(a_k, b_j) \geqslant d(a_i, b_j).$$

If $d(a_l, b_j) = d(a_l, a_k) + \varepsilon + d(b_k, b_j)$ for some $k > q$, then

$$d(a_i, a_l) + d(a_l, b_j) = d(a_i, a_l) + d(a_k, a_l) + \varepsilon + d(b_k, b_j) \geqslant$$

$$d(a_i, a_k) + d(b_k, b_j) + \varepsilon \geqslant d(a_i, b_j).$$

Case 2. $d(a_i, b_j) \leqslant d(a_i, b_l) + d(b_l, b_j)$. This case is similar to Case 1.

Case 3. $d(a_i, a_j) \leqslant d(a_i, b_l) + d(a_j, b_l)$. We may assume that $i > q$ or $j > q$. If $k, m \leqslant q$ and

$$d(a_i, b_l) = d(a_i, a_k) + d(a_k, b_l) \text{ and } d(a_j, b_l) = d(a_j, a_m) + d(a_m, b_l)$$

then

$$d(a_i, b_l) + d(a_j, b_l) \geqslant d(a_i, a_k) + d(a_k, b_l) + d(a_j, a_m) + d(a_m, b_l) \geqslant$$
$$\geqslant d(a_i, a_k) + d(a_k, a_m) + d(a_m, a_j) \geqslant d(a_i, a_j).$$

If

$$d(a_i, b_l) = d(a_i, a_k) + d(a_k, b_l) \text{ with } k \leqslant q \text{ and}$$
$$d(a_j, b_l) = d(a_j, a_m) + \varepsilon + d(b_m, b_l) \text{ with } m > q,$$

then $j > q < l$ (otherwise we are in the previous situation) and

$$d(a_i, b_l) + d(a_j, b_l) \geqslant d(a_i, a_k) + d(a_k, b_l) + d(a_j, a_m) + \varepsilon + d(b_m, b_l) \geqslant$$
$$\geqslant d(a_i, a_k) + d(a_k, a_l) - \varepsilon + d(a_m, a_j) + d(a_m, a_l).$$

By the assumptions of the proposition a_l is not between a_k and a_m and

$$\varepsilon < d(a_k, a_l) + d(a_m, a_l) - d(a_k, a_m).$$

Thus

$$d(a_i, a_k) + d(a_k, a_l) - \varepsilon + d(a_m, a_j) + d(a_m, a_l) \geqslant$$
$$d(a_i, a_k) + d(a_k, a_m) + d(a_m, a_j) \geqslant d(a_i, a_j).$$

Now assume that

$$d(a_i, b_l) = d(a_i, a_k) + \varepsilon + d(b_k, b_l) \text{ with } k > q \text{ and}$$
$$d(a_j, b_l) = d(a_j, a_m) + \varepsilon + d(b_m, b_l) \text{ with } m > q.$$

We may assume that $q < \min(i, j, l)$ (otherwise we are in one of the previous situations). Then

$$d(a_i, b_l) + d(a_j, b_l) \geqslant d(a_i, a_k) + \varepsilon + d(b_k, b_l) + d(a_j, a_m) + \varepsilon + d(b_m, b_l) \geqslant$$
$$\geqslant d(a_i, a_k) + d(a_k, a_l) + d(a_m, a_j) + d(a_m, a_l) \geqslant d(a_i, a_j).$$

Case 4. $d(b_i, b_j) \leqslant d(b_i, a_l) + d(a_l, b_j)$. This case is similar to Case 3. Note that we can use the inequality $\varepsilon < d(b_k, b_l) + d(b_m, b_l) - d(b_k, b_m)$ for $k \leqslant q < \min(l, m)$. □

We now formulate our main result.

Theorem 1. *Let $A_0 = \{a_1, \ldots, a_q\}$ be a finite metric space of size q. Let $n > q$. Consider spaces*

$$A = \{a_1, \ldots, a_q, \ldots, a_n\} \textit{ and } B = \{b_1, \ldots, b_q, \ldots, b_n\},$$

where $a_1 = b_1, \ldots, a_q = b_q$. Assume

$$max\{|d(b_i, b_j) - d(a_i, a_j)| : 1 \leqslant i < j \leqslant n\} \leqslant \varepsilon.$$

Then there is a metric on $A \cup B$ extending metrics in A and B so that for each $q < i \leqslant n$, $d(a_i, b_i) \leqslant 18\varepsilon$.

Proof. We start with a procedure which finds a minimal subset $A' \subset A$ containing A_0 so that the distances between pairs of elements of A' with at least one of them from $A' \setminus A_0$ are $> 4\varepsilon$ and A is contained in the neighbourhood of A' of radius 4ε. Let B' be the subset of B consisting of elements with the same numbers as elements of A' in A. Note that the distances between pairs of elements of B' are $> 3\varepsilon$ and B is contained in the neighbourhood of B' of radius 5ε.

Now consider all $a_i \in A' \setminus A_0$ which appear in triples a_i, a_j, a_k in A' with

$$d(a_i, a_j) + d(a_i, a_k) - d(a_j, a_k) \leqslant 2\varepsilon,$$
$$\text{where } |\{i, j, k\}| = 3 \text{ , } a_k \in A_0 \text{ , } a_i, a_j \notin A_0.$$

Firstly find indices of such elements in A and B. Then for each index i_j of this set apply free amalgamation of A with the two-element subspace $\{a_{i_j}, a'_{i_j}\}$ where the distance is rational and satisfies

$$2\varepsilon \leqslant d(a_{i_j}, a'_{i_j}) < 4\varepsilon.$$

We repeat this procedure for each element a_{i_j} of our list. We use Theorem 2.1 of [1] (see Introduction above) for every amalgamation. Let $\hat{A}$ be the resulting space. Removing $a_{i_0}, \ldots, a_{i_j}, \ldots$ from $(\hat{A} \setminus A) \cup A'$ we construct A'' which already satisfies the assumptions of Proposition 1. Indeed, it is now obvious that if a_i, a_j, a_k is a triple from A' as above, then a'_i, a_j, a_k fulfill the inequality

$$d(a'_i, a_j) + d(a'_i, a_k) - d(a_j, a_k) > 4\varepsilon.$$

Since $d(a_j, a_k) \geqslant max(d(a_i, a_j), d(a_i, a_k))$ the permutation of a'_i and a_j does not change this property. Moreover it is easy to see that no element of $A' \cap A''$ plays the role of a_i in any triple of A'' intersecting A_0 as above. Such an assumption whould imply the false statement that this element has the same property in A' (possibly replacing some a'_{i_j} in the triple by a_{i_j}). We use here the fact that the space $\hat{A}$ satisfies the condition that any non-trivial path from a'_{i_j} contains a_{i_j}.

We apply the same procedure to B' where we put

$$d(b_{i_j}, b'_{i_j}) = d(a_{i_j}, a'_{i_j}).$$

As a result we obtain the corresponding $\hat{B}$ and B''.

We enumerate A' and B' according the enumerations of A and B. Then we consider A'' under the enumeration induced by A' where the number of every a'_{i_j} is just i_j. Note that in $\hat{A}$ the distance between elements of A' and A'' with the same numbers (i.e for example $d(a_{i_j}, a'_{i_j})$) is not greater than 4ε. In particular A is contained in the neighbourhood of A'' of radius 8ε. Thus in $\hat{B}$ the space B is contained in the neighbourhood of B'' of radius 9ε. Moreover for each $a_i \in A$ there are a' and b' which have the same number in A'' and B'' respectively such that $d(a_i, a') \leqslant 8\varepsilon$ and $d(b_i, b') \leqslant 9\varepsilon$.

On the other hand it is easy to see that we still have

$$max\{|d(b_i, b_j) - d(a_i, a_j)| : 1 \leqslant i < j \leqslant n\} \leqslant \varepsilon,$$

for the corresponding elements of A'' and B''. As a result we have A'' and B'' which satisfy the assumptions of Proposition 1. Applying Proposition 1 to A'' and B'' over A_0 we obtain an amalgamated metric on $A'' \cup B''$, so that the distances between the corresponding elements with numbers $> q$ is ε. Using the free amalgamation property we amalgamate $\hat{A}$ with $A'' \cup B''$ and then amalgamate the result with $\hat{B}$. In this space A is distant from A'' by $\leqslant 8\varepsilon$ and B is distant from B'' by $\leqslant 9\varepsilon$. The rest is clear. □

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Contact information

A. Ivanov Institute of Mathematics, University of Wrocław, pl.Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-Mail(s): ivanov@math.uni.wroc.pl
Web-page(s): www.math.uni.wroc.pl/∼ivanov/

B. Majcher-Iwanow Institute of Mathematics, University of Opole, ul.Oleska 48, 45-052 Opole, Poland
E-Mail(s): bmajcher@math.uni.opole.pl

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