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# Profinite closures of the iterated monodromy groups associated with quadratic polynomials

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Abstract. In this paper we describe the profinite closure of the iterated monodromy groups arising from the arbitrary post-critically finite quadratic polynomial.

## Introduction

In this article we consider the iterated monodromy groups arising from the post-critically finite (i.e. orbit of the critical point is finite) quadratic polynomial $z^2 - c$ and we denote such group as $\mathrm{IMG}(z^2 - c)$. Group $\mathrm{IMG}(z^2 - c)$ could be represented as the automorphism group of the rooted binary tree (see [1] or [6]). These groups were studied by Bartholdi and Nekrashevych [1]. They show that properties of such group highly depend on whether the orbit of the critical point of $z^2 - c$ is periodic (the orbit of the critical point contain critical point itself) or strictly pre-periodic (the orbit of the critical point do not contain critical point). Bartholdi and Nekrashevych [1] have proved that group $\mathrm{IMG}(z^2 - c)$ is weakly branch on its commutator for the periodic case and also they have proved that group $\mathrm{IMG}(z^2 - c)$ is regular branch for the pre-periodic case (except for the infinite dihedral group).

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One interesting example of the iterated monodromy group is Basilica group [5] arising from the complex polynomial $z^2 - 1$. This group is generated by a three-state automaton. It is the first example of an amenable group not belonging to the class of sub-exponentially amenable groups [3].

Closure of some iterated monodromy groups of post-critically finite quadratic polynomials $z^2 - c$ was studied by Pink [7]. He studied the Hausdorff dimension, the maximal abelian factor groups, and the normalizers. Also he showed that the closure of $\mathrm{IMG}(z^2 - c)$ don't depend up to conjugacy in $\mathrm{Aut}\, T$ on words that define the automorphism group which is associated with $\mathrm{IMG}(z^2 - c)$ (we improve this result and we show that in many cases for different words we get the same closure of $\mathrm{IMG}(z^2 - c)$, see Corollary 1 and Theorem 4).

In this article, we provide the description of the closure for the group $\mathrm{IMG}(z^2 - c)$. We prove that for the pre-periodic case closure of the group $\mathrm{IMG}(z^2 - c)$ is a self-similar group of finite type except for the infinite dihedral group. We use another approach compare to Pink [7] that allow us to improve some of his results.

## 1. Preliminaries

Denote by $T$ a rooted binary tree. We can denote vertices of the tree $T$ as words on the alphabet $\{0, 1\}$. The empty word $\varnothing$ corresponds to the root of the tree $T$; edges are $(v, vx)$ for all $v \in T$ and $x \in \{0, 1\}$. Words $\underbrace{1 \ldots 1}_{n}$ and $\underbrace{0 \ldots 0}_{n}$ we denote by $1^n$ and $0^n$ respectively. The set of all words of length $n$ we call as *$n$-th level* of the tree $T$ and we denote this set by $T^n$, $n \geqslant 0$. Let $0T^n$ denote the set of all words of length $n+1$ that starts with 0 and let $1T^n$ denote the set of all words of length $n+1$ that starts with 1.

The subtree of $T$ of all vertices of the first $n$ levels is denoted by $T^{[n]}$. The group of all automorphisms of the tree $T$ we denote by $\mathrm{Aut}\, T$. We denote trivial element in $\mathrm{Aut}\, T$ by the symbol 1. For an automorphism $g \in \mathrm{Aut}\, T$ and a vertex $v \in T$ we denote by $g_{(v)}$ such action on the tree $T$ that

$$g(vw) = g(v)g_{(v)}(w)$$

for all $w \in T$.

*A stabilizer of the $n$-th level* of $G < \mathrm{Aut}\, T$, $\mathrm{St}_G(n)$ (or just $\mathrm{St}(n)$) is defined to be a set of all elements in $G$ which fix all vertices of the level $n$. Group $G|_{T^{[n]}}$ is a restriction of the group $G$ to the subtree $T^{[n]}$ (i.e. if $g \in G|_{T^{[n]}}$ then $g \in \mathrm{Aut}\, T^{[n]}$ and there exists such $h \in G$ that for all

$v \in T^{[n]}$ equality $g(v) = h(v)$ holds). For the element $g \in G$ we denote by $g|_{T^{[n]}} \in G|_{T^{[n]}}$ the restriction of the element $g$ to the subtree $T^{[n]}$. The group

$$G = \{g \in \operatorname{Aut} T \mid g_{(v)}|_{T^{[n]}} \in F \text{ for every } v \in T\}$$

called as *self-similar group of finite type* with depth $n$ given by the pattern group $F < \operatorname{Aut} T^{[n]}$. If self-similar group of finite type with depth $n$ can't be given by any pattern group $F < \operatorname{Aut} T^{[n-1]}$ then $n$ is *minimal pattern depth.*

We know [2] that

$$\operatorname{Aut} T \cong \wr_{i=1}^{\infty} \operatorname{Sym}(\{0,1\})$$

where $\operatorname{Sym}(\{0,1\}) = \{e, \sigma\}$ is a symmetric group on $\{0,1\}$. Thus $\operatorname{Aut} T$ is a profinite group

$$\operatorname{Aut} T = \varprojlim \wr_{i=1}^{n} \operatorname{Sym}(\{0,1\}).$$

Also we know [2] that

$$\operatorname{Aut} T \cong \operatorname{Aut} T \wr \operatorname{Sym}(\{0,1\})$$

therefore every element $g \in \operatorname{Aut} T$ we can write by the following *wreath recursion*

$$g = (g_{(0)}, g_{(1)})\sigma^i$$

where $i \in \{0,1\}$. Product of elements $g = (g_1, g_2)\sigma^i$ and $h = (h_1, h_2)\sigma^j$ is defined as follows

$$gh = \begin{cases} (g_1h_1, g_2h_2)\sigma^j, & \text{if } i = 0, \\ (g_1h_2, g_2h_1)\sigma^{1-j}, & \text{otherwise.} \end{cases}$$

To describe profinite closure of the group $G$ it is enough to describe all $G|_{T^{[n]}}$ for $n \geqslant 1$. In this article we describe $G|_{T^{[n]}}$ for $n \geqslant 2$ recursively i.e. for $g_0, g_1 \in G|_{T^{[n-1]}}$ and $i \in \{0,1\}$ we determine necessary and sufficient condition for $g = (g_0, g_1)\sigma^i \in G|_{T^{[n]}}$.

Let us fix some integer $n \geqslant 1$ and automorphism $g = (g_0, g_1)\sigma^{i_\varnothing} \in \operatorname{Aut} T$, where $i_\varnothing \in \{0,1\}$. We define following functions

$$\begin{aligned}
L_n(g) &= g_{(00\ldots00)} \cdot g_{(00\ldots01)} \cdot \cdots \cdot g_{(0j_2\ldots j_{n-1}j_n)} \cdot \cdots \cdot g_{(01\ldots11)}, \\
R_n(g) &= g_{(10\ldots00)} \cdot g_{(10\ldots01)} \cdot \cdots \cdot g_{(1j_2\ldots j_{n-1}j_n)} \cdot \cdots \cdot g_{(11\ldots11)}, \\
l_{n,0}(g) &= r_{n,0}(g) = l_0(g) = r_0(g) = i_\varnothing, \\
l_{n,i}(g) &= l_{n,i-1}(L_n(g)), \text{for } i > 0, \\
r_{n,i}(g) &= r_{n,i-1}(R_n(g)), \text{for } i > 0.
\end{aligned}$$

We can define functions $l_{n,i}$ and $r_{n,i}$ for the element $g \in \operatorname{Aut} T^{[k]}$ where $k \geqslant ni+1$ in the same way as above.

We use notation $g^h$ for $h^{-1}gh$ where $g$ and $h$ are elements of a group. For real number $x$ denote by $\lfloor x \rfloor$ the largest integer not greater than $x$ and denote by $\lceil x \rceil$ the smallest integer not less than $x$.

## 2. Periodic case

We consider profinite closures of iterated monodromy groups associated with quadratic polynomials with periodic critical points in this section.

Let us fix an integer $t \geqslant 1$ and a word $x = x_1 \dots x_t$ over the alphabet $\{0,1\}$. We define elements $a_1, ..., a_t \in \operatorname{Aut} T$ by recursions

$$a_1 = \begin{cases} (a_t, 1)\sigma, & \text{if } x_1 = 0, \\ (1, a_t)\sigma, & \text{otherwise.} \end{cases}$$

and for $2 \leqslant k \leqslant t$

$$a_k = \begin{cases} (a_{k-1}, 1), & \text{if } x_k = 0, \\ (1, a_{k-1}), & \text{otherwise.} \end{cases}$$

We will study the group $G_{t,x} = \langle a_1, \dots, a_t \rangle$ and its profinite closure $\overline{G_{t,x}}$. We consider case $x_1 = 0$ further without limiting the generality because we can get $G_{t,1x}$ from $G_{t,0y}$ for some $y$ by switch of the letters of the alphabet $\{0,1\}$. Then

$$a_1 = (a_t, 1).$$

Note that

$$a_1^{-1} = (1, a_t^{-1})\sigma$$

and for $2 \leqslant k \leqslant t$

$$a_k^{-1} = \begin{cases} (a_{k-1}^{-1}, 1), & \text{if } x_k = 0, \\ (1, a_{k-1}^{-1}), & \text{otherwise.} \end{cases}$$

Let $u$ be a word in $\{a_1, a_2, \dots, a_t, a_1^{-1}, a_2^{-1}, \dots, a_t^{-1}\}$ that represents element $g \in G_{t,x}$. We denote by $|u|_k$ the exponent sum of $a_k$ in $u$. By the proposition 3.3 from [1] it follows that this numbers does not depend on word $u$ representing element $g \in G_{t,x}$. Thus we define $|g|_k$ as the exponent sum of $a_k$ for the word $u$.

**Lemma 1.** *For every integer $n \geqslant 0$ we have $a_k^{2^n} \in \mathrm{St}(nt+k-1)$, $1 \leqslant k \leqslant t$.*

*Proof.* We will prove this by induction on $n$. Statement for $n=0$ follow from the definition of elements $a_k$, $1 \leqslant k \leqslant t$. We assume that the statement of the Lemma holds for $n-1 \geqslant 0$. By the definition of $a_1$ we have

$$a_1^{2^n} = (a_1^2)^{2^{n-1}} = (a_t, a_t)^{2^{n-1}} = (a_t^{2^{n-1}}, a_t^{2^{n-1}}),$$

and we see that $a_1^{2^n} \in \mathrm{St}(nt)$ by the inductive hypothesis. For $2 \leqslant k \leqslant t$ the inclusion $a_k^{2^n} \in \mathrm{St}(nt+k-1)$ follows from the equalities $a_k^{2^n} = (a_{k-1}^{2^n}, 1)$ for $x_k = 0$, $a_k^{2^n} = (1, a_{k-1}^{2^n})$ for $x_k = 1$ and the inclusion $a_{k-1}^{2^n} \in \mathrm{St}(nt+k-2)$. □

**Lemma 2.** *If $g \in G_{t,x}$ then*

$$\sum_{v \in 0T^{t-1}} |g_{(v)}|_1 = \left\lceil \frac{|g|_1}{2} \right\rceil \quad \textit{and} \quad \sum_{v \in 1T^{t-1}} |g_{(v)}|_1 = \left\lfloor \frac{|g|_1}{2} \right\rfloor .$$

*Proof.* We could represent element from $G_{t,x}$ as product over the set

$$\{a_1, \ldots, a_t, a_1^{-1}, \ldots, a_t^{-1}\}.$$

We prove the statement by induction on the length of the product. The base is true as we have the trivial element in that case.

We suppose that the statement is true for $h \in G_{t,x}$

$$h = (h_0, h_1)\sigma^{i_\varnothing}.$$

We will multiply element from the set $\{a_1, \ldots, a_t, a_1^{-1}, \ldots, a_t^{-1}\}$ on $h$. We denote the result of multiplying by $g$. There are $2t$ possibilities for the element $g$ and we consider all that possibilities. Note that for any $v \in T^t$ we have $(a_i)_{(v)} = 1$ or $(a_i)_{(v)} = a_i$ and for any $v \in T^t$ we have $(a_i^{-1})_{(v)} = 1$ or $(a_i^{-1})_{(v)} = a_i^{-1}$.

i) Case $g = a_1 h$. Then $|g|_1 = |h|_1 + 1$ and

$$a_1 h = (a_t h_1, h_0)\sigma^{1-i_\varnothing}.$$

For the element $(a_1 h)_{(0)}$ we have the following sum

$$\sum_{v \in 0T^{t-1}} |(a_1 h)_{(v)}|_1 = |a_1|_1 + \sum_{v \in 1T^{t-1}} \left| h_{(v)} \right|_1 .$$

Since the last sum is equal to $1+\left\lfloor\frac{|h|_1}{2}\right\rfloor$ by the inductive hypothesis then we have

$$1+\left\lfloor\frac{|h|_1}{2}\right\rfloor=\left\lceil\frac{|h|_1+1}{2}\right\rceil=\left\lceil\frac{|g|_1}{2}\right\rceil.$$

For the element $(a_1h)_{(1)}$ we have

$$\sum_{v\in 1T^{t-1}}|(a_1h)_{(v)}|_1=\sum_{v\in 0T^{t-1}}|h_{(v)}|_1=\left\lceil\frac{|h|_1}{2}\right\rceil=\left\lfloor\frac{|h|_1+1}{2}\right\rfloor=\left\lfloor\frac{|g|_1}{2}\right\rfloor.$$

ii) Case $g=a_1^{-1}h$. Then $|g|_1=|h|_1-1$ and

$$a_1^{-1}h=(h_1,a_t^{-1}h_0)\sigma^{1-i_\varnothing}.$$

We have the following sum for the element $(a_1^{-1}h)_{(0)}$

$$\sum_{v\in 0T^{t-1}}|(a_1^{-1}h)_v|_1=\sum_{v\in 1T^{t-1}}\left|h_{(v)}\right|_1.$$

This number is equal to $\left\lfloor\frac{|h|_1}{2}\right\rfloor$ by the inductive hypothesis. So we have

$$\sum_{v\in 1T^{t-1}}\left|h_{(v)}\right|_1=\left\lfloor\frac{|h|_1}{2}\right\rfloor=\left\lceil\frac{|h|_1-1}{2}\right\rceil=\left\lceil\frac{|g|_1}{2}\right\rceil.$$

For the element $(a_1^{-1}h)_{(1)}$ we have

$$|a_1^{-1}|_1+\sum_{v\in 0T^{t-1}}|h_{(v)}|_1=\left\lceil\frac{|h|_1}{2}\right\rceil-1=\left\lfloor\frac{|h|_1-1}{2}\right\rfloor=\left\lfloor\frac{|g|_1}{2}\right\rfloor.$$

iii) We consider all cases $a_ih$ and $a_i^{-1}h$, $2\leqslant i\leqslant t$ together as for them we get the same exponent sum of $a_1$: $|a_ih|_1=|a_i^{-1}h|_1=|h|_1$ and

$$a_ih=\begin{cases}(a_{i-1}h_0,h_1)\sigma^{i_\varnothing}, & \text{if } x_i=0,\\ (h_0,a_{i-1}h_1)\sigma^{i_\varnothing}, & \text{otherwise.}\end{cases}$$

$$a_i^{-1}h=\begin{cases}(a_{i-1}^{-1}h_0,h_1)\sigma^{i_\varnothing}, & \text{if } x_i=0,\\ (h_0,a_{i-1}^{-1}h_1)\sigma^{i_\varnothing}, & \text{otherwise.}\end{cases}$$

Therefore

$$\sum_{v\in 0T^{t-1}}|(a_ih)_{(v)}|_1=\sum_{v\in 0T^{t-1}}|(a_i^{-1}h)_{(v)}|_1=\sum_{v\in 0T^{t-1}}|h_{(v)}|_1=\left\lceil\frac{|h|_1}{2}\right\rceil,$$

$$\sum_{v\in 1T^{t-1}}|(a_ih)_{(v)}|_1=\sum_{v\in 1T^{t-1}}|(a_i^{-1}h)_{(v)}|_1=\sum_{v\in 1T^{t-1}}|h_{(v)}|_1=\left\lfloor\frac{|h|_1}{2}\right\rfloor. \square$$

We prove that functions $l_{t,i}$ and $r_{t,i}$ depend only on the exponent sum of $a_1$.

**Lemma 3.** *For every element $g \in G_{t,x}$ and for every not negative integer $i$ the following equalities hold*

$$l_{t,i}(g) = \left\lceil \frac{|g|_1}{2^i} \right\rceil \mod 2 \quad \textit{and} \quad r_{t,i}(g) = \left\lfloor \frac{|g|_1}{2^i} \right\rfloor \mod 2.$$

*Proof.* We prove this Lemma by induction on $i$. In the case $i = 0$ we have $l_{t,0}(g) = r_{t,0}(g) = |g|_1 \mod 2$. Now we suppose that the lemma holds for $i-1$ where $i \geqslant 1$. By the definition of $l_{t,i}$ we have

$$l_{t,i}(g) = l_{t,i-1}(L_t(g)) = \left\lceil \frac{|L_t(g)|_1}{2^{i-1}} \right\rceil \mod 2 = \left\lceil \frac{\sum_{v \in 0T^{t-1}} |g_{(v)}|_1}{2^{i-1}} \right\rceil \mod 2.$$

We conclude by the Lemma 2 that

$$l_{t,i}(g) = \left\lceil \frac{\lceil |g|_1/2 \rceil}{2^{i-1}} \right\rceil \mod 2 = \left\lceil \frac{|g|_1}{2^i} \right\rceil \mod 2.$$

Analogously, for the function $r_{t,i}$ we have

$$r_{t,i}(g) = r_{t,i-1}(R_t(g)) = \left\lfloor \frac{|R_t(g)|_1}{2^{i-1}} \right\rfloor \mod 2 = \left\lfloor \frac{\lfloor |g|_1/2 \rfloor}{2^{i-1}} \right\rfloor \mod 2$$
$$= \left\lfloor \frac{|g|_1}{2^i} \right\rfloor \mod 2. \qquad \square$$

**Theorem 1.** *For every $g \in G_{t,x}$ and every not negative integer $k$ the following holds*

$$\sum_{i=0}^{k} l_{t,i}(g)2^i + \sum_{i=0}^{k} r_{t,i}(g)2^i \equiv 0 \mod 2^{k+1}. \tag{1}$$

*Proof.* It follows from the Lemma 3 that

$$\sum_{i=0}^{k} l_{t,i}(g)2^i = \sum_{i=0}^{k} \left( \left\lceil \frac{|g|_1}{2^i} \right\rceil \mod 2 \right) 2^i,$$
$$\sum_{i=0}^{k} r_{t,i}(g)2^i = \sum_{i=0}^{k} \left( \left\lfloor \frac{|g|_1}{2^i} \right\rfloor \mod 2 \right) 2^i.$$

This equalities are the 2−adic representations for $-|g|_1 \mod 2^{k+1}$ and $|g|_1 \mod 2^{k+1}$ respectively. $\square$

**Remark 1.** Equality 1 also holds for every automorphism $g \in G_{t,x}|_{T^{[n]}}$ where $n \geqslant kt+1$. We can choose such element $g' \in G_{t,x}$ that $g'|_{T^{[n]}} = g$. Equality 1 holds for $g'$ and so it holds for $g$.

**Lemma 4.** *If for the automorphism $g = (g_0, g_1)\sigma^{i_\varnothing} \in \operatorname{Aut} T$ where $g_0, g_1 \in G_{t,x}$, $i_\varnothing \in \{0,1\}$ and for the integer $k \geqslant 0$ following congruence relation holds*

$$\sum_{i=0}^{k} l_{t,i}(g)2^i + \sum_{i=0}^{k} r_{t,i}(g)2^i \equiv 0 \mod 2^{k+1},$$

*then $|g_1|_t + i_\varnothing \equiv |g_0|_t \mod 2^k$.*

*Proof.* At first we make the following transformation

$$\begin{aligned}\sum_{i=0}^{k} l_{t,i}(g)2^i + \sum_{i=0}^{k} r_{t,i}(g)2^i &= 2\sum_{i=1}^{k} l_{t,i}(g)2^{i-1} + 2\sum_{i=1}^{k} r_{t,i}(g)2^{i-1} + 2i_\varnothing \\ &\equiv 0 \mod 2^{k+1}.\end{aligned}$$

Then we divide last congruence by 2 and we get

$$\sum_{i=0}^{k-1} l_{t,i+1}(g)2^i + \sum_{i=0}^{k-1} r_{t,i+1}(g)2^i + i_\varnothing \equiv 0 \mod 2^k.$$

By the definition of $l_{t,i+1}$ and $r_{t,i+1}$ we have

$$\sum_{i=0}^{k-1} l_{t,i+1}(g)2^i + \sum_{i=0}^{k-1} r_{t,i+1}(g)2^i = \sum_{i=0}^{k-1} l_{t,i}(L_t(g))2^i + \sum_{i=0}^{k-1} r_{t,i}(R_t(g))2^i.$$

As $g_0, g_1 \in G_{t,x}$ then $g_{(v)} \in G_{t,x}$ for all non-empty $v$ and we can use Lemma 3 for the previous expression and we get

$$\begin{aligned}&\sum_{i=0}^{k-1} \left( \left\lceil \frac{|L_t(g)|_1}{2^i} \right\rceil \bmod 2 \right) 2^i + \sum_{i=0}^{k-1} \left( \left\lfloor \frac{|R_t(g)|_1}{2^i} \right\rfloor \bmod 2 \right) 2^i \\ &\quad = \sum_{i=0}^{k-1} \left( \left\lceil \frac{\sum_{v \in 0T^{t-1}} |g_{(v)}|_1}{2^i} \right\rceil \bmod 2 \right) 2^i + \sum_{i=0}^{k-1} \left( \left\lfloor \frac{\sum_{v \in 1T^{t-1}} |g_{(v)}|_1}{2^i} \right\rfloor \bmod 2 \right) 2^i.\end{aligned}$$

From the definition of generators of $G_{t,x}$ we have

$$\sum_{v \in T^{t-1}} |(g_0)_{(v)}|_1 = |g_0|_t \quad \text{and} \quad \sum_{v \in T^{t-1}} |(g_1)_{(v)}|_1 = |g_1|_t.$$

Now we have the condition

$$\sum_{i=0}^{k-1}\left(\left\lceil\frac{|g_0|_t}{2^i}\right\rceil \bmod 2\right)2^i+\sum_{i=0}^{k-1}\left(\left\lfloor\frac{|g_1|_t}{2^i}\right\rfloor \bmod 2\right)2^i+i_\varnothing\equiv 0 \mod 2^k.$$

Congruence above contains 2-adic representations for $-|g_0|_t \bmod 2^k$ and $|g_1|_t \bmod 2^k$ respectively which finally lead us to

$$-|g_0|_t+|g_1|_t+i_\varnothing\equiv 0 \mod 2^k. \qquad \square$$

**Theorem 2.** *We assume that* $g=(g_0,g_1)\sigma^{i_\varnothing}\in G_{t,x}|_{T[n]}$ *and* $g'=(g_0',g_1')\sigma^{i_\varnothing}$ *where* $g_0',g_1'\in G_{t,x}|_{T[n]}$, $g_0'|_{T[n-1]}=g_0$ *and* $g_1'|_{T[n-1]}=g_1$ *for* $n\geqslant 1$*. Then* $g'\in G_{t,x}|_{T[n+1]}$ *if and only if the following congruence holds for* $k=\lfloor\frac{n}{t}\rfloor$

$$\sum_{i=0}^{k} l_{t,i}(g')2^i+\sum_{i=0}^{k} r_{t,i}(g')2^i\equiv 0 \mod 2^{k+1}. \tag{2}$$

*Proof.* Necessary condition follow from the Theorem 1.

Here we prove the sufficient condition. We can choose such elements $w_0,w_1\in G_{t,x}$ that $w_0|_{T[n]}=g_0'$ and $w_1|_{T[n]}=g_1'$. By the Lemma 4 there exists such integer $j$ that $|w_1|_t+i_\varnothing=|w_0|_t+j2^k$. By the Lemma 1 we have $a_t^{2^k}\in \mathrm{St}(kt+t-1)$. As $n\leqslant kt+t-1$ then $a_t^{2^k}|_{T[n]}$ is trivial and $w_0a_t^{j2^k}|_{T[n]}=g_0'$. If $i_\varnothing=0$ then $|w_1|_t=|w_0|_t+j2^k$. Stabilizer St(1) is generated by $a_1^2=(a_t,a_t)$, $a_1^{-x_i}a_ia_1^{x_i}=(a_{i-1},1)$ and $a_1^{x_i-1}a_ia_1^{1-x_i}=(1,a_{i-1})$ where $2\leqslant i\leqslant t$. So we have

$$(w_0a_t^{j2^k},w_1)=(a_t^{|w_1|_t},w_1)(a_t^{-|w_1|_t},a_t^{-|w_1|_t})(w_0a_t^{j2^k},a_t^{|w_1|_t})\in \mathrm{St}(1).$$

If $i_\varnothing=1$ then we can generate element $a_1\cdot(w_0a_t^{j2^k},w_1)\sigma=(a_tw_1,w_0a_t^{j2^k})$ because of $1+|w_1|_t=|w_0|_t+j2^k$. $\square$

**Remark 2.** The congruence 2 from Theorem 2 gives us a restriction only in the case $n=kt$ for some positive integer $k$. In other cases congruence 2 holds because $kt<n$ therefore action of $g'$ on the level $n+1$ not used for the congruence 2.

**Remark 3.** We do not use order in products from definitions of $L_n(g)$ and $R_n(g)$.

**Remark 4.** We prove our formula for the case $x_1 = 0$ only. But if we switch letters in the alphabet $\{0,1\}$ then by the previous Remark we conclude that it is enough to switch functions $l_{t,i}$ and $r_{t,i}$ in the congruence 2 of the Theorem 2 to get the criteria for $G_{t,x}|_{T^{[n]}}$ for the case $x_1 = 1$. But then we get the same criteria as for the case $x_1 = 0$.

Pink [7] already showed that group $\overline{G_{t,x}}$ depend only on $t$ up to conjugacy in $\operatorname{Aut} T$. The following corollary improves that result for the periodic case.

**Corollary 1.** *For all words $x^1, x^2 \in \{0,1\}^t$ equality $\overline{G_{t,x^1}} = \overline{G_{t,x^2}}$ holds.*

*Proof.* Any group $G_{t,x}$ is $\operatorname{Sym}\{0,1\}$ on the first level. Congruence 2 do not depend on $x$ therefore $\overline{G_{t,x}}$ do not depend on the word $x$. □

**Remark 5.** In general group $G_{t,x}$ depend on a word $x$. For example when $t = 3$ groups corresponding to “Douady Rabbit” and “Airplane” [6] are not isomorphic.

## 3. Strictly pre-periodic case

In this section, we consider profinite closures of iterated monodromy groups associated with quadratic polynomials with pre-periodic critical points.

Let us fix integers $t > s \geqslant 1$ and a word $x = x_1 x_2 \dots x_{t-1}$ over the alphabet $\{0,1\}$. We define elements $a_1, ..., a_t \in \operatorname{Aut} T$ by recursions

$$a_1 = (1,1)\sigma,$$

$$a_{s+1} = \begin{cases} (a_s, a_t), & \text{if } x_s = 0, \\ (a_t, a_s), & \text{otherwise,} \end{cases}$$

$$a_i = \begin{cases} (a_{i-1}, 1), & \text{if } x_{i-1} = 0, \\ (1, a_{i-1}), & \text{otherwise.} \end{cases}$$

We will study group $G_{s,t,x} = \langle a_1, \dots, a_t \rangle$ and its profinite closure $\overline{G_{s,t,x}}$. Almost all groups $\overline{G_{s,t,x}}$ are self-similar groups of finite type, only one exception is the case $s = 1$ and $t = 2$. We consider this exception later.

**Lemma 5.** *(Proposition 4.2 from[1]) For all $1 \leqslant i \leqslant t$ equality $a_i^2 = 1$ holds. Abealinization $G_{s,t,x}/G'_{s,t,x}$ is $(\mathbb{Z}/(2\mathbb{Z}))^t$ generated by the images of the $a_i$.*

Let $u$ be a word in $\{a_1, a_2, \dots, a_t\}$ that represents element $g \in G_{s,t,x}$. We denote by $|u|_k$ the exponent sum of $a_k$ in $u$. By the Lemma 5 it follows that number $|u|_k \bmod 2$ does not depend on the word $u$ representing element $g \in G_{s,t,x}$. Thus we define $|g|_k$ as $|u|_k \bmod 2$.

**Lemma 6.** *Element $g \in G_{s,t,x}$ belong to the commutator $G'_{s,t,x}$ iff $|g|_i = 0$ for all $1 \leqslant i \leqslant t$.*

*Proof.* Statement follow from the Lemma 5. □

**Lemma 7.** *If $g \in \mathrm{St}(i)$ then $|g|_i = 0$ for all $1 \leqslant i \leqslant t$.*

*Proof.* We prove this by induction on $i$. If $g \in \mathrm{St}(1)$ then $|g|_1 = 0$. Let $g = (g_1, g_2)$. If $g \in \mathrm{St}(i)$ then $g_1, g_2 \in \mathrm{St}(i-1)$ and by induction suppose for $g_1$ and $g_2$ we have that numbers $|g_1|_{i-1}$ and $|g_2|_{i-1}$ are zeroes. Note that equality $|g|_i = (|g_1|_{i-1} + |g_2|_{i-1}) \bmod 2$ holds therefore $|g|_i = 0$. □

**Lemma 8.** *If $g \in \mathrm{St}(t)$ then $g \in G'_{s,t,x}$.*

*Proof.* Statement follow from the Lemma 6 and the Lemma 7. □

**Theorem 3.** *Suppose that $s \neq 1$ and $t \neq 2$ simultaneously. Then group $\overline{G_{s,t,x}}$ is a self-similar group of finite type with the following minimal depth*

$$d = \begin{cases} 5, & \text{if } s = 2 \text{ and } t = 3, \\ t+1, & \text{otherwise.} \end{cases}$$

*Proof.* We use Sunic's [8] criteria from the Theorem 3 for a self-similar group of finite type. It is already proven that the group $G_{s,t,x}$ is regular branch over some group $H_{s,t,x}$ (Theorem 4.10 from [1]). It is only left to show that $H_{s,t,x}$ contain a stabilizer of the level $d$. Let us to consider following cases.

• If $s \geqslant 2$ and $t \geqslant s+2$ or if $s \geqslant 3$ and $t = s+1$ than $H_{s,t,x} = G'_{s,t,x}$ and statement follow from the Lemma 8.

• If $s = 1$ and $t \geqslant 3$ then $H_{s,t,x} = \langle [a_i, a_j]_{2 \leqslant i < j \leqslant t}, [a_1, a_j]_{1 \leqslant j < t} \rangle^{G_{s,t,x}}$. Group $G'_{s,t,x}/H_{s,t,x} \cong \mathbb{Z}_2$ generated by $[a_1, a_t]$. Therefore

$$G'_{s,t,x} = H_{s,t,x} \cup ([a_1, a_t] H_{s,t,x}).$$

By the Lemma 8 we have $\mathrm{St}_{G_{s,t,x}}(t) < G'_{s,t,x}$. We prove that $H_{s,t,x}$ contain a stabilizer of the level $t$ in the following way: for any element $g \in G'_{s,t,x}$ we prove that if $g \notin H_{s,t,x}$ then $g \notin \mathrm{St}_{G_{s,t,x}}(t)$.

It is easy to see that for all $h=(h_1,h_2)\in H_{s,t,x}$ we have $|h_1|_{t-1}=|h_2|_{t-1}=0$. As $[a_1,a_t]=(a_{t-1},a_{t-1})$ then for all $g=(g_1,g_2)\in[a_1,a_t]H_{s,t,x}$ we conclude that $|g_1|_{t-1}$ and $|g_2|_{t-1}$ are ones; but by the Lemma 7 it follow that $g_1,g_2\notin \mathrm{St}_{G_{s,t,x}}(t-1)$. Therefore $g\notin \mathrm{St}_{G_{s,t,x}}(t)$.

• Case $s=2,t=3$. We consider case $x_1=0$ only without limiting the generality. In this case we have $H_{2,3,x}=\langle[a_1,a_2a_3]\rangle^{G_{2,3,x}}$. Also

$$G'_{2,3,x}=H_{2,3,x}\cup([a_1,a_2]H_{2,3,x})\cup([a_2,a_3]H_{2,3,x})\cup([a_1,a_2][a_2,a_3]H_{2,3,x}).$$

Note that $\mathrm{St}(4)<G'_{2,3,x}$. We prove for any $g\in G'_{2,3,x}$ that if $g\notin H_{2,3,x}$ then $g\notin \mathrm{St}(4)$ using GAP in the following way: we compute group $H_{2,3,x}|_{T^{[4]}}$ and we check that $e\notin([a_1,a_2]H_{2,3,x})|_{T^{[4]}}\cup([a_2,a_3]H_{2,3,x})|_{T^{[4]}}\cup([a_1,a_2][a_2,a_3]H_{2,3,x})|_{T^{[4]}}$.

Minimality for all cases except $s=2$ and $t=3$ follow from the fact that $G_{s,t,x}|_{T^{[t]}}=\mathrm{Aut}(T^t)$. For the case $s=2$ and $t=3$ it can be seen using GAP. □

Group $\overline{G_{s,t,x}}$ barely depend on the word $x$. Pink [7] proved that for all words $z,y\in\{0,1\}^{t-1}$ there exists such $w\in\mathrm{Aut}\,T$ that $\overline{G_{s,t,z}}=w\overline{G_{s,t,y}}w^{-1}$. The following theorem improves that result.

**Theorem 4.** *Suppose that $s\neq 1$ and $t\neq 2$ simultaneously.*

- *If $s\geqslant 2$ and $t\geqslant s+2$ or if $s\geqslant 3$ and $t=s+1$. Then for all possible words $y=y_1\dots y_{t-1}$ we have $\overline{G_{s,t,x}}=\overline{G_{s,t,y}}$.*
- *If $s=1$ and $t\geqslant 3$ then for all possible words $y=y_1\dots y_{t-1}$ where $x_{t-1}=y_{t-1}$ we have $\overline{G_{s,t,x}}=\overline{G_{s,t,y}}$.*
- *If $s=2,t=3$ then all 4 groups $\overline{G_{2,3,x}}$ for this case are conjugate.*

*Proof.* All groups in this theorem are self-similar groups of finite type therefore it is enough to prove the statement for pattern groups.

For the case $s=2$, $t=3$ using GAP we can see that there are 4 different groups $\overline{G_{2,3,x}}$ as sets. Pink [7] already prove that all $\overline{G_{2,3,x}}$ are conjugate. But we show some example of conjugator for $\overline{G_{2,3,00}}$ and $\overline{G_{2,3,01}}$.

We find using GAP such elements $f_1,f_2\in\mathrm{Aut}\,T^{[5]}$ and $g_1,g_2,g_3,g_4\in G_{2,3,00}|_{T^{[4]}}$ that

$$f_1=(g_1f_2|_{T^{[4]}},g_2f_2|_{T^{[4]}})\quad\text{and}\quad f_2=(g_3f_1|_{T^{[4]}},g_4f_1|_{T^{[4]}}),$$
$$G_{2,3,00}|_{T^{[5]}}=f_1G_{2,3,01}|_{T^{[5]}}f_1^{-1}\quad\text{and}\quad G_{2,3,00}|_{T^{[5]}}=f_2G_{2,3,01}|_{T^{[5]}}f_2^{-1}.$$

Then we can choose such elements $\widehat{g_1}, \widehat{g_2}, \widehat{g_3}, \widehat{g_4} \in G_{2,3,00}$ that $\widehat{g_i}|_{T^{[4]}} = g_i$. Therefore we can define elements

$$\widehat{f_1} = (\widehat{g_1}\widehat{f_2}, \widehat{g_2}\widehat{f_2}) \quad \text{and} \quad \widehat{f_2} = (\widehat{g_3}\widehat{f_1}, \widehat{g_4}\widehat{f_1}).$$

For $\widehat{h} = (h_1, h_2)\sigma^i \in \overline{G_{2,3,01}}$, $i \in \{0,1\}$ the following holds

$$\begin{aligned} \widehat{f_1}(h_1,h_2)\sigma^i\widehat{f_1}^{-1} &= (\widehat{g_1}\widehat{f_2}h_1\widehat{f_2}^{-1}\widehat{g_{1+i}}^{-1}, \widehat{g_2}\widehat{f_2}h_2\widehat{f_2}^{-1}\widehat{g_{2-i}}^{-1})\sigma^i,\\ \widehat{f_2}(h_1,h_2)\sigma^i\widehat{f_2}^{-1} &= (\widehat{g_3}\widehat{f_1}h_1\widehat{f_1}^{-1}\widehat{g_{3+i}}^{-1}, \widehat{g_4}\widehat{f_1}h_2\widehat{f_1}^{-1}\widehat{g_{4-i}}^{-1})\sigma^i. \end{aligned}$$

As $\overline{G_{2,3,00}}$ is self-similar group of finite type then it follows that

$$\widehat{f_1}\overline{G_{2,3,01}}\widehat{f_1}^{-1} = \overline{G_{2,3,00}} \quad \text{and} \quad \widehat{f_2}\overline{G_{2,3,01}}\widehat{f_2}^{-1} = \overline{G_{2,3,00}}.$$

One possible choice for $\widehat{g_i}$ is

$$\widehat{g_1} = 1, \quad \widehat{g_2} = a_2a_3^3a_1, \quad \widehat{g_3} = a_3^2a_2, \quad \widehat{g_4} = a_3a_2a_3^2a_2a_1.$$

We exclude case $s = 2$, $t = 3$ from further consideration.

Let us to fix the following elements

$$a'_{s+1} = \begin{cases} (a'_s, a'_t), & \text{if } y_s = 0,\\ (a'_t, a'_s), & \text{otherwise.} \end{cases}$$

$$a'_i = \begin{cases} (a'_{i-1}, 1), & \text{if } y_{i-1} = 0,\\ (1, a'_{i-1}), & \text{otherwise.} \end{cases}$$

In our proof we use a notation for the Kronecker delta

$$\delta_{ij} = \begin{cases} 0, & \text{if } i = j,\\ 1, & \text{if } i \neq j. \end{cases}$$

Then we have for $2 \leqslant i \leqslant t$:

$$\begin{aligned} a_i &= a_1^{x_{i-1}}(a_{i-1}, a_t^{\delta_{i,s+1}})a_1^{x_{i-1}},\\ a'_i &= a_1^{y_{i-1}}(a'_{i-1}, (a'_t)^{\delta_{i,s+1}})a_1^{y_{i-1}}. \end{aligned}$$

We define function $\phi$ on the set of all words in the alphabet $\{a_1, \ldots, a_{t-1}\}$

$$\phi : \{a_1, \ldots, a_{t-1}\}^* \to \{a_1, \ldots, a_t\}^*$$

recursively as follows

$$\begin{aligned}\phi(\varnothing) &= \varnothing,\\ \phi(a_i) &= a_1^{x_i} a_{i+1} a_1^{x_i}, \quad 1 \leqslant i \leqslant t-1,\\ \phi(\alpha\beta) &= \phi(\alpha)\phi(\beta), \quad \text{where } \alpha, \beta \in \{a_1, \ldots, a_{t-1}\}^*.\end{aligned}$$

Note that $a_i^0$ is the empty word therefore $\phi(a_i^0) = 1$.

We define function $\psi(w)$ as element from the group $G_{s,t,x}$ that correspond to the word $w \in \{a_1, \ldots, a_t\}^*$, note that $\psi(\varnothing) = 1$. Functions $\psi$ and $\phi$ are homomorphisms. Let us to show that the following properties hold for all words $w \in \{a_1, \ldots, a_{t-1}\}^*$

$$\psi(\phi(w)) = (\psi(w), a_t^{|w|_s}), \qquad |\psi(\phi(w))|_1 = 0.$$

We prove these by induction on length of the word. For the first one we have

$$\begin{aligned}\psi(\phi(\varnothing)) &= \psi(\varnothing) = 1 = (\psi(\varnothing), 1),\\ \psi(\phi(a_i)) &= \psi(a_1^{x_i} a_{i+1} a_1^{x_i}) = a_1^{x_i} a_{i+1} a_1^{x_i}\\ &= a_1^{x_i} a_1^{x_i} (a_i, a_t^{\delta_{i+1,s+1}}) a_1^{x_i} a_1^{x_i} = (a_i, a_t^{|a_i|_s}),\\ \psi(\phi(a_i w)) &= \psi(\phi(a_i)\phi(w)) = \psi(\phi(a_i))\psi(\phi(w)) = (a_i\psi(w), a_t^{|a_i|_s + |w|_s}).\end{aligned}$$

For the second one we have

$$\begin{aligned}|\psi(\phi(\varnothing))|_1 &= |1|_1 = 0,\\ |\psi(\phi(a_i))|_1 &= |a_1^{x_i} a_{i+1} a_1^{x_i}|_1 = 0, \quad 1 \leqslant i \leqslant t-1,\\ |\psi(\phi(a_i w))|_1 &= |\psi(\phi(a_i))|_1 + |\psi(\phi(w))|_1 = 0.\end{aligned}$$

Without limiting the generality it is enough to show that

$$\langle a_1', \ldots, a_t' \rangle|_{T^{[t+1]}} < \langle a_1, \ldots, a_t \rangle|_{T^{[t+1]}}.$$

We construct such elements from $G_{s,t,x}$ that act on first $t+1$ levels as $a_i'$. So we define

$$\begin{aligned}c_1 &= \varnothing, & c_i &= a_1^{y_{i-1}} \phi(c_{i-1}) a_1^{x_{i-1}}, \quad 2 \leqslant i \leqslant t,\\ d_1 &= a_1' = a_1, & d_i &= \psi(c_i) a_i \psi(c_i)^{-1}, \quad 2 \leqslant i \leqslant t.\end{aligned}$$

Note that $c_i \in \{a_1, \ldots, a_{t-1}\}^*$ for $1 \leqslant i \leqslant t$ . We prove the following

$$d_i = a_1^{y_{i-1}} (d_{i-1}, a_t^{\delta_{i,s+1}}) a_1^{y_{i-1}}$$

by induction on $i$.

$$\begin{aligned}
c_2 &= a_1^{y_1} a_1^{x_1},\\
d_2 &= a_1^{y_1} a_1^{x_1} a_2 a_1^{x_1} a_1^{y_1} = a_1^{y_1}(a_1, a_t^{\delta_{2,s+1}}) a_1^{y_1}.\\
d_i &= \psi(c_i) a_i \psi(c_i)^{-1} = a_1^{y_{i-1}} \psi(\phi(c_{i-1})) a_1^{x_{i-1}} a_i (a_1^{y_{i-1}} \psi(\phi(c_{i-1})) a_1^{x_{i-1}})^{-1}\\
&= a_1^{y_{i-1}} (\psi(c_{i-1}) a_{i-1} \psi(c_{i-1})^{-1}, a_t^{\delta_{i,s+1}}) a_1^{y_{i-1}} = a_1^{y_{i-1}} (d_{i-1}, a_t^{\delta_{i,s+1}}) a_1^{y_{i-1}}.
\end{aligned}$$

We have $d_{s+1} = a_1^{y_s}(d_s, a_t) a_1^{y_s}$. But we need an element

$$\overline{d_{s+1}} = a_1^{y_s}(d_s, d_t) a_1^{y_s} \in G_{s,t,x}.$$

We construct this element for each case.

- Case $s \geqslant 2$ and $t \geqslant s+2$ or if $s \geqslant 3$ and $t = s+1$. There exist such $b_1 \in \langle a_1, a_2\rangle$ and $b_2 \in \langle a_1, a_2, a_3\rangle$ that $[d_s, a_t^{b_1}] = 1$ and $b_2 = (b_1, 1)$. If $s \geqslant 2$ and $t \geqslant s+2$ then

  $$b_1 = a_1^{x_{t-1}+y_{s-1}+1}.$$

  If $s \geqslant 3$ and $t = s+1$ then

  $$b_1 = a_1^{x_{t-1}} \psi(\phi(a_1^{(x_{t-2}+y_{s-2}+1) \bmod 2})) a_1^{y_{s-1}}.$$

  For both cases

  $$b_2 = \psi(\phi(b_1)).$$

  Then we compute the following

  $$\begin{aligned}
  b_3 &= a_1^{y_s} \psi(\phi(c_t))^{a_1 b_2} a_1^{y_s} = a_1^{y_s}((a_t^r)^{b_1}, \psi(c_t)) a_1^{y_s}, r = |\psi(c_t)|_s,\\
  \overline{d_{s+1}} &= b_3 d_{s+1} b_3^{-1} = b_3(a_1^{y_s}(d_s, a_t) a_1^{y_s}) b_3^{-1}\\
  &= a_1^{y_s}((a_t^r)^{b_1} d_s ((a_t^r)^{b_1})^{-1}, \psi(c_t) a_t \psi(c_t)^{-1}) a_1^{y_s} = a_1^{y_s}(d_s, d_t) a_1^{y_s}.
  \end{aligned}$$

- Case $s = 1$ and $t \geqslant 3$ and $x_{t-1} = y_{t-1}$. Therefore

  $$\begin{aligned}
  |\psi(c_t)|_1 &= |\psi(a_1^{y_{t-1}} \phi(c_{t-1}) a_1^{x_{t-1}})|_1 = (y_{t-1} + x_{t-1}) \bmod 2 = 0,\\
  b_3 &= a_1^{y_s} \psi(\phi(c_t))^{a_1} a_1^{y_s} = a_1^{y_s}(a_t^{|c_t|_1}, \psi(c_t)) a_1^{y_s} = a_1^{y_s}(1, \psi(c_t)) a_1^{y_s},\\
  \overline{d_{s+1}} &= b_3 d_{s+1} b_3^{-1} = b_3(a_1^{y_s}(d_s, a_t) a_1^{y_s}) b_3^{-1}\\
  &= a_1^{y_s}((d_s, \psi(c_t) a_t \psi(c_t)^{-1}) a_1^{y_s} = a_1^{y_s}(d_s, d_t) a_1^{y_s}.
  \end{aligned}$$

We prove that $d_i|_{T^{[t]}} = a'_i|_{T^{[t]}}$ for $1 \leqslant i \leqslant t$ by induction on $i$. Base is true because $d_1 = a'_1$. We have for $2 \leqslant i \leqslant t$, $i \neq s+1$ the following

$$\begin{aligned} d_i|_{T^{[t]}} &= a_1^{y_{i-1}}|_{T^{[t]}}(d_{i-1}|_{T^{[t-1]}}, 1|_{T^{[t-1]}})a_1^{y_{i-1}}|_{T^{[t]}} \\ &= a_1^{y_{i-1}}|_{T^{[t]}}(a'_{i-1}|_{T^{[t-1]}}, 1|_{T^{[t-1]}})a_1^{y_{i-1}}|_{T^{[t]}} = a'_i|_{T^{[t]}}, \\ d_{s+1}|_{T^{[t]}} &= a_1^{y_s}|_{T^{[t]}}(d_s|_{T^{[t-1]}}, a_t|_{T^{[t-1]}})a_1^{y_s}|_{T^{[t]}} \\ &= a_1^{y_s}|_{T^{[t]}}(a'_s|_{T^{[t-1]}}, 1|_{T^{[t-1]}})a_1^{y_s}|_{T^{[t]}} = a'_{s+1}|_{T^{[t]}}. \end{aligned}$$

Therefore

$$\begin{aligned} d_i|_{T^{[t+1]}} &= a_1^{y_{i-1}}|_{T^{[t+1]}}(d_{i-1}|_{T^{[t]}}, 1|_{T^{[t]}})a_1^{y_{i-1}}|_{T^{[t+1]}} \\ &= a_1^{y_{i-1}}|_{T^{[t+1]}}(a'_{i-1}|_{T^{[t]}}, 1|_{T^{[t]}})a_1^{y_{i-1}}|_{T^{[t+1]}} = a'_i|_{T^{[t+1]}}, \\ \overline{d_{s+1}}|_{T^{[t+1]}} &= a_1^{y_s}|_{T^{[t+1]}}(d_s|_{T^{[t]}}, d_t|_{T^{[t]}})a_1^{y_s}|_{T^{[t+1]}} \\ &= a_1^{y_s}|_{T^{[t+1]}}(a'_s|_{T^{[t]}}, a'_t|_{T^{[t]}})a_1^{y_s}|_{T^{[t+1]}} = a'_{s+1}|_{T^{[t+1]}}. \end{aligned}$$

So $\langle a'_1, \ldots, a'_t\rangle|_{T^{[t+1]}} < \langle a_1, \ldots, a_t\rangle|_{T^{[t+1]}}$. □

### 3.1. Case $s = 1$ and $t = 2$

Here we consider the profinite closure $\overline{G} = \overline{G_{1,2,0}}$ of the infinite dihedral group $G = G_{1,2,0}$ generated by following automorphisms

$$a_1 = (1, 1)\sigma, \qquad a_2 = (a_1, a_2).$$

Note that $a_1^{-1} = a_1$ and $a_2^{-1} = a_2$. Let us to compute $a_1a_2$ and $a_2a_1$

$$a_1a_2 = (a_2, a_1)\sigma, \qquad a_2a_1 = (a_1, a_2)\sigma.$$

Then for all integer $k$ we have

$$\begin{aligned} (a_1a_2)^k &= (a_1a_2)^{2\lfloor \frac{k}{2} \rfloor + k \bmod 2} = ((a_1a_2)^2)^{\lfloor \frac{k}{2} \rfloor}(a_1a_2)^{k \bmod 2} \\ &= (a_2a_1, a_1a_2)^{\lfloor \frac{k}{2} \rfloor}(a_2^{k \bmod 2}, a_1^{k \bmod 2})\sigma^{k \bmod 2}. \end{aligned}$$

Therefore

$$(a_1a_2)^k = ((a_1a_2)^{-\lfloor \frac{k}{2} \rfloor}a_2^{k \bmod 2}, (a_1a_2)^{\lfloor \frac{k}{2} \rfloor}a_1^{k \bmod 2})\sigma^{k \bmod 2}. \qquad (3)$$

**Lemma 9.** *For all integer $n \geqslant 0$ the following equality holds*

$$\prod_{v \in T^n} l_0(((a_1a_2)^k)_{(v)}) = \begin{cases} 1, & \text{if } 2^n \mid k \text{ and } 2^{n+1} \nmid k, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Let us prove this by induction on $n$. One can directly check the equality for $n = 0, 1$. We assume that Lemma holds for $n-1$ where $n \geqslant 2$. If $2^n \mid k$ and $2^{n+1} \nmid k$ then $k = k_0 2^n$ for some odd $k_0$ and by equality 3

$$(a_1a_2)^k = ((a_1a_2)^{-k_0 2^{n-1}}, (a_1a_2)^{k_0 2^{n-1}}).$$

So in this case we have

$$\prod_{v\in T^n} l_0(((a_1a_2)^k)_{(v)}) \\ = \prod_{v\in T^{n-1}} l_0(((a_1a_2)^{-k_0 2^{n-1}})_{(v)}) \cdot \prod_{v\in T^{n-1}} l_0(((a_1a_2)^{k_0 2^{n-1}})_{(v)}) = 1 \cdot 1 = 1.$$

Let us to consider other cases. As $n \geqslant 2$ and $a_1$ act trivially on all levels except first we have for all $v \in T^{n-1}$

$$l_0(((a_1a_2)^k)_{(1v)}) = l_0(((a_1a_2)^{\lfloor \frac{k}{2} \rfloor} a_1^{k \bmod 2})_{(v)}) = l_0(((a_1a_2)^{\lfloor \frac{k}{2} \rfloor})_{(v)}).$$

If $\left\lfloor \frac{k}{2} \right\rfloor \neq k_0 2^{n-1}$ for all odd $k_0$ then

$$\prod_{v\in T^n} l_0(((a_1a_2)^k)_{(v)}) = \prod_{v\in 0T^{n-1}} l_0(((a_1a_2)^k)_{(v)}) \cdot \prod_{v\in T^{n-1}} l_0(((a_1a_2)^{\lfloor \frac{k}{2} \rfloor})_{(v)}) \\ = \prod_{v\in 0T^{n-1}} l_0(((a_1a_2)^k)_{(v)}) \cdot 0 = 0.$$

If $\left\lfloor \frac{k}{2} \right\rfloor = k_0 2^{n-1}$ for some odd integer $k_0$ then $k = k_0 2^n + 1$ (because now we consider $k \neq k_0 2^n$)

$$l_0(((a_1a_2)^k)_{(0v)}) = l_0(((a_1a_2)^{-k_0 2^{n-1}} a_2)_{(v)}) = l_0(((a_1a_2)^{-k_0 2^{n-1}} a_2a_1a_1)_{(v)}) \\ = l_0(((a_1a_2)^{-k_0 2^{n-1}-1} a_1)_{(v)}) = l_0(((a_1a_2)^{-k_0 2^{n-1}-1})_{(v)}).$$

Then

$$\prod_{v\in T^n} l_0(((a_1a_2)^k)_{(v)}) \\ = \prod_{v\in T^{n-1}} l_0(((a_1a_2)^{-k_0 2^{n-1}-1})_{(v)}) \cdot \prod_{v\in 1T^{n-1}} l_0(((a_1a_2)^k)_{(v)}) = 0.$$

□

**Lemma 10.** *For all $g \in G$ and all integer $n \geqslant 2$ the following congruence holds*

$$l_0(g_{(01^{n-1})}) + l_0(g_{(1^n)}) + \prod_{v\in T^{n-1}} l_0(g_{(v)}) \equiv 0 \mod 2. \tag{4}$$

*Proof.* Every element of the group $G$ is the product $(a_1a_2)^k a_1^{k_1}$ where $k$ is some integer and $k_1 \in \{0,1\}$. As $a_1$ act not trivially on the first level only it is enough to prove congruence for $(a_1a_2)^k$ for all integer $k$.

We use the following notations: for every integer $n \geqslant 0$ and every integer $k$ we denote

$$\alpha_{n+1}^k = l_0(((a_1a_2)^k)_{(01^n)}),$$
$$\beta_n^k = l_0(((a_1a_2)^k)_{(1^n)}).$$

From the equality 3 we conclude that for all $n \geqslant 1$ and all integer $k$

$$\beta_0^k = k \bmod 2,$$
$$\alpha_n^k = l_0(((a_1a_2)^{-\lfloor\frac{k}{2}\rfloor} a_2^{k \bmod 2})_{(1^{n-1})}),$$
$$\beta_n^k = l_0(((a_1a_2)^{\lfloor\frac{k}{2}\rfloor} a_1^{k \bmod 2})_{(1^{n-1})}).$$

Number of $a_1$ in the expression $(a_1a_2)^{\lfloor\frac{k}{2}\rfloor} a_1^{k \bmod 2}$ module 2 is equal to the number $\beta_1^k$

$$\beta_1^k = \left(\left\lfloor\frac{k}{2}\right\rfloor + k\right) \bmod 2.$$

As $a_1$ act not trivially on the first level only then we can omit last $a_1$ for $n \geqslant 2$

$$\beta_n^k = l_0(((a_1a_2)^{\lfloor\frac{k}{2}\rfloor} a_1^{k \bmod 2})_{(1^{n-1})}) = l_0(((a_1a_2)^{\lfloor\frac{k}{2}\rfloor})_{(1^{n-1})}) = \beta_{n-1}^{\lfloor\frac{k}{2}\rfloor}.$$

Then we have

$$\beta_n^k = \beta_{n-1}^{\lfloor\frac{k}{2}\rfloor} = \ldots = \beta_1^{\lfloor\frac{k}{2^{n-1}}\rfloor} = \left(\left\lfloor\frac{k}{2^n}\right\rfloor + \left\lfloor\frac{k}{2^{n-1}}\right\rfloor\right) \bmod 2 \quad \text{for } n \geqslant 1.$$

Now we compute $\alpha_n^k$ for all integer $k$ and all $n \geqslant 2$

$$\alpha_n^k = l_0(((a_1a_2)^{-\lfloor\frac{k}{2}\rfloor} a_2^{k \bmod 2})_{(1^{n-1})}) = l_0(((a_1a_2)^{-\lfloor\frac{k}{2}\rfloor - k \bmod 2})_{(1^{n-1})})$$

Note that $\left\lfloor\frac{k}{2}\right\rfloor + k \bmod 2 = \left\lceil\frac{k}{2}\right\rceil$. Then

$$\alpha_n^k = \beta_{n-1}^{-\lceil\frac{k}{2}\rceil} = \left(-\left\lceil\frac{k}{2^n}\right\rceil - \left\lceil\frac{k}{2^{n-1}}\right\rceil\right) \bmod 2 = \left(\left\lceil\frac{k}{2^n}\right\rceil + \left\lceil\frac{k}{2^{n-1}}\right\rceil\right) \bmod 2.$$

Now congruence 4 looks like the following

$$\left\lceil\frac{k}{2^n}\right\rceil + \left\lceil\frac{k}{2^{n-1}}\right\rceil + \left\lfloor\frac{k}{2^n}\right\rfloor + \left\lfloor\frac{k}{2^{n-1}}\right\rfloor + \prod_{v \in T^{n-1}} l_0(((a_1a_2)^k)_{(v)}) \equiv 0 \mod 2.$$

We have by the properties of floor and ceil that

$$\left\lceil \frac{k}{2^n} \right\rceil - \left\lfloor \frac{k}{2^n} \right\rfloor = \begin{cases} 0, & \text{if } 2^n \mid k, \\ 1, & \text{otherwise.} \end{cases}$$

$$\left\lceil \frac{k}{2^{n-1}} \right\rceil - \left\lfloor \frac{k}{2^{n-1}} \right\rfloor = \begin{cases} 0, & \text{if } 2^{n-1} \mid k, \\ 1, & \text{otherwise.} \end{cases}$$

If $2^n \mid k$ then $2^{n-1} \mid k$ and by the Lemma 9 congruence 4 looks like the following

$$\left\lceil \frac{k}{2^n} \right\rceil + \left\lfloor \frac{k}{2^n} \right\rfloor + \left\lceil \frac{k}{2^{n-1}} \right\rceil + \left\lfloor \frac{k}{2^{n-1}} \right\rfloor \equiv 0 + 0 \equiv 0 \mod 2.$$

If $2^n \nmid k$ and $2^{n-1} \nmid k$ then by the Lemma 9 congruence 4 looks like the following

$$\left\lceil \frac{k}{2^n} \right\rceil + \left\lfloor \frac{k}{2^n} \right\rfloor + \left\lceil \frac{k}{2^{n-1}} \right\rceil + \left\lfloor \frac{k}{2^{n-1}} \right\rfloor \equiv 1 + 1 \equiv 0 \mod 2.$$

If $2^n \nmid k$ and $2^{n-1} \mid k$ then by the Lemma 9 congruence 4 looks like the following

$$\left\lceil \frac{k}{2^n} \right\rceil + \left\lfloor \frac{k}{2^n} \right\rfloor + \left\lceil \frac{k}{2^{n-1}} \right\rceil + \left\lfloor \frac{k}{2^{n-1}} \right\rfloor + 1 \equiv 1 + 0 + 1 \equiv 0 \mod 2.$$

□

**Theorem 5.** *We suppose that element* $g = (g_0, g_1)\sigma^{i_\varnothing} \in G|_{T^{[n]}}$ *and* $g' = (g'_0, g'_1)\sigma^{i_\varnothing} \in \operatorname{Aut} T^{[n+1]}$ *where* $g'_0, g'_1 \in G|_{T^{[n]}}$, $g'_0|_{T^{[n-1]}} = g_0$ *and* $g'_1|_{T^{[n-1]}} = g_1$ *for integer* $n \geqslant 3$. *Then* $g' \in G|_{T^{[n+1]}}$ *if and only if the following congruence holds*

$$l_0(g'_{(01^{n-1})}) + l_0(g'_{(1^n)}) + \prod_{v \in T^{n-1}} l_0(g_{(v)}) \equiv 0 \mod 2. \tag{5}$$

*Proof.* Necessary condition follow from the Lemma 10.

Let us prove the sufficient condition. We know (see [7]) that $|G|_{T^{[n-1]}}| = 2^n$ for $n \geqslant 3$ and it imply that $|(\mathrm{St}(n-1))_n| = 2$ for $n \geqslant 3$. So there are 4 possible ways how we can construct $g'$ by given $g$. Group $G$ is level transitive and it imply that there are such elements $h_0, h_1$ in $\mathrm{St}(n-1)|_{T^{[n]}}$ that $l_0((h_0)_{(1^{n-1})}) = 0$ and $l_0((h_1)_{(1^{n-1})}) = 1$. It imply that all possibilities for the pair $(l_0(g'_{(01^{n-1})}), l_0(g'_{(1^n)}))$ are realized and pairs $(l_0(g'_{(01^{n-1})}), l_0(g'_{(1^n)}))$ are in one-to-one correspondence to $g'$ constructed in such way. For the fixed $g$ the product $\prod_{v\in T^{n-1}} l_0(g_{(v)})$ is also fixed. Then there are only two pairs $(l_0(g'_{(01^{n-1})}), l_0(g'_{(1^n)}))$ that satisfy congruence 5. As $|\mathrm{St}(n)|_{T^{[n+1]}}| = 2$ then there are two possibilities for the element $g'$ that belong to $G|_{T^{[n+1]}}$. So element $g'$ that corresponds to the pair $(l_0(g'_{(01^{n-1})}), l_0(g'_{(1^n)}))$ which satisfy congruence 5 belongs to $G|_{T^{[n+1]}}$. □

Note that group $G|_{T^{[2]}}$ is $\operatorname{Aut} T^{[2]}$. For all elements $g \in G|_{T^{[3]}}$ one can directly check following congruences

$$l_0(g_{(0)}) \equiv l_0(g_{(10)}) + l_0(g_{(11)}) \mod 2,$$
$$l_0(g_{(1)}) \equiv l_0(g_{(00)}) + l_0(g_{(01)}) \mod 2.$$

This congruences together with Lemma 10 and Theorem 5 give us full explicit description of groups $G|_{T^{[n]}}$ for $n \geqslant 3$.

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