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DEVELOPMENT OF "SURFACE" SHAPE FUNCTIONS ON THE BASIS OF INVARIANT APPROXIMATIONS TECHNIQUE

Mariya Howykowycz
Lviv Polytechnic National University, Ukraine
howykowycz@ieee.org

Abstract: Two objectives have been formulated: to guarantee the invariance of algebraic analogues of integral-differential equations corresponding to the invariance of the original integral-differential equations system and to include boundary conditions directly into approximating dependences (shape functions) describing the field inside finite elements pertaining to the border. The above problem has been solved using the technique of invariant approximation of functions. As it has been shown, this approach reduces the order of the original system of equations.

1. Introduction

The finite element method was introduced in electrical engineering calculations in 1970 and, since then, it has been applied to the simulation of a great variety of electromagnetic problems in static and transient state in two and three dimensions: electrostatics, magnetostatics, eddy current, wave propagation phenomena, etc. Nowadays, it is the basis of several commercial codes such us Ansys, Femlab, Magnet, MSC/Emas, Opera, etc. The development of different finite element methods and mathematical tools to analyze and numerically solve Maxwell equations has been one of the main directions of research during the past decades.

The fundamental idea of the finite elements method [1] is to subdivide a domain under consideration into small sub-domains called finite elements (FE). Sought scalar and vector functions are approximated within each finite element by simple functions called shape functions. A shape function is a continuous function defined over a single FE. The shape functions of individual FE's are combined into global shape functions, also called basis functions. Since first mathematical analysis of the method in the 1960's [2] it has been developed by introducing new shape functions [3] and rigorous analysis of their stability, accuracy, reliability, and adaptability. Nodal and edge finite elements are widely used but their properties do not provide the possibility to "embed" the discontinuities of electromagnetic field variables, caused by abrupt changes in electric conductivity and magnetic permeability, in their configuration. Vector field variables have a physical and mathematical identity that goes beyond their representation in any particular coordinate frame. By dividing the vector into three Cartesian parts, node-based elements fail to take this into account. For example, boundary conditions in electromagnetics often take the form of a specification of only the part of the vector function that is tangential to the boundary. With node-based elements, this physical constraint must be transformed into linear relationships between the Cartesian components what increases the number of equations and, consequently, the computational error.

The aim of our research is to develop such shape functions that automatically take into account boundary conditions on the edges of two-dimensional FE or on the faces of three-dimensional FE. Those shape functions are called "surface" functions because they are used for modeling the sub-domains adjacent to boundaries between regions with different electromagnetic properties.

2. Statement of problem

The main problematic tasks that are encountered by a researcher while applying FEM to electromagnetic field analysis are:

- Formulation of simplified assumptions
- The choice of the primary unknown in each subdomain (terms of certain scalar and magnetic potentials, terms of the electric field or the magnetic field)
- Definition of adequate boundary conditions
- The choice of FE for each subdomain.

Nodal and edge finite elements are widely used but their properties do not provide the possibility to "embed" the discontinuities of electromagnetic field variables, caused by abrupt changes in electric conductivity and magnetic permeability, in their configuration.

Inside **each nodal finite element**, a scalar or a vector function is approximated by a linear combination of shape functions associated with nodes. Within an element, a scalar function u is approximated as

$$u = \sum_{i=1}^{P} u_i S_i ,$$

where S_i is the nodal shape function corresponding to node *i*. The index *P* is the number of nodes in the element. The coefficient u_i - the degree of freedom - is the value of *u* at the node *i*.

A vector function $\vec{u}$ is treated simply as a combination of three scalar components, u_x, u_y and u_z in a Cartesian *x*, *y*, *z* coordinate system. Each node then has three degrees of freedom instead of one, and $\vec{u}$ is approximated as

$$\vec{u} = \sum_{i=1}^{P} \vec{u}_i S_i = \sum_{i}^{P} (u_{xi}\vec{i} + u_{yi}\vec{j} + u_{zi}\vec{k}) S_i,$$

where the coefficient $\vec{u}_i$ is the value of $\vec{u}$ at node *i*, and $\vec{u}_{xi}$, $\vec{u}_{yi}$, and $\vec{u}_{zi}$ are the three components of $\vec{u}_i$. When two elements share a node *i*, the nodal values $\vec{u}_i$ i at node *i* are set to be equal. Applying this procedure throughout a mesh makes the vector function **u** normally and tangentially continuous across all element interfaces. However, vectors are not simply triplets of numbers. They have a physical and mathematical identity that goes beyond their representation in any particular coordinate frame. By dividing the vector into three Cartesian parts, node-based elements fail to take this into account. For example, boundary conditions in electromagnetics often take the form of a specification of only the part of the vector function that is tangential to the boundary. With node-based elements, this physical constraint must be transformed into linear relationships between the Cartesian components what increases the number of equations and, consequently, the computational error.

Inside each **edge finite element**, a vector function is approximated by a linear combination of shape functions associated with edges. Within an element, a vector function **u** is approximated as

$$\vec{u} = \sum_{i=1}^{e} u_{ti} \vec{S}_{ei},$$

where the coefficient u_{ti} is the degree of freedom at edge *i* and $\vec{S}_{ei}$ is the edge shape function corresponding to edge *i*.

The line integral of $\vec{S}_{ei}$ along edge *i* equals unity, yielding that the line integral of $\vec{u}$ along edge *i* can be written as

$$\int_i \vec{u} d\vec{l} = \int_i u_{ti} \vec{S}_{ei} d\vec{l} = u_{ti}.$$

Thus, u_{ti} ti is the line integral of $\vec{u}$ along edge *i*, and the degrees of freedom, instead of being components of the vector function at element nodes, are to be interpreted as the line integrals of the approximated vector function along element edges. When two elements share an edge *i*, the degrees of freedom u_{ti} at edge *i* are set to be equal. Applying this procedure throughout a mesh makes the vector function $\vec{u}$ tangentially continuous across all element interfaces. The vector function thus constructed is not normally continuous.

The aim of our research is to develop such shape functions that automatically take into account boundary conditions on the edges of two-dimensional FE or on the faces of three-dimensional FE. Those shape functions are called "surface" functions because they are used for modeling the sub-domains adjacent to boundaries between regions with different electromagnetic properties. Within such "surface" element, a scalar function *u* is approximated as

$$u = \sum_{i=1}^{P-b} u_i S_i + \sum_{j=1}^{b} g_j S_{bj},$$

where the index *b* is the number of boundary conditions given for the discussed FE. The coefficient g_i is the value of imposed boundary condition at node *j* and S_{bj} is the "surface" shape function corresponding to edge *j*.

3. Development of "surface" elements

The distinctive features of proposed approach are:

- Application of invariant shape functions that preserve the tensor character of original differential operators after the transformation of the set of differential equations into the corresponding set of algebraic equations
- Development of special "surface" shape functions that allow us to take into account boundary conditions in implicit form what reduces the order of the set of algebraic equations.

Let Ω be a bounded open set in $\Re^n$, $\vec{r} \in \Omega$, $k \in \Im$, and assume that $u \in C^k(\Omega)$ can be extended from Ω to a continuous function on $\bar{\Omega}$, the closure of the set Ω, where *u* is the sought function described by partial differential equations supplemented by one of the following boundary conditions, with *g* denoting a given function defined on the boundary ∂Ω:

$u = g$ on ∂Ω (Dirichlet boundary condition);

$\partial u / \partial n = g$ on ∂Ω, where n denotes the unit outward normal vector to ∂Ω (Neumann boundary condition);

$\partial u / \partial n + \sigma u = g$ on ∂Ω, where $\sigma(\mathbf{x}) \geq 0$ on ∂Ω (Robin boundary condition).

In accordance with invariant approximations technique the sought function and its derivative can be represented within *m*-th finite element in the form

$$u[\vec{r}] = \vec{T}[\vec{r}]\mathbf{T}_m^{-1}\vec{U}_{m*}; \quad (1)$$

$$\partial u / \partial n = \vec{T}[\vec{r}]\vec{n}[\vec{r}]\mathbf{N}\mathbf{T}_m^{-1}\vec{U}_{m*}, \quad (2)$$

where $\vec{T}[\vec{r}]$ is Taylor's vector; $\mathbf{T}_m^{-1}$ is inverse Taylor's matrix for *m*-th FE; $\vec{U}_{m*}$ is the column of nodal values for *m*-th FE; $\vec{n}[\vec{r}]$ consists of cosines of the angles between unit outward normal vector to ∂Ω and a corresponding axis.

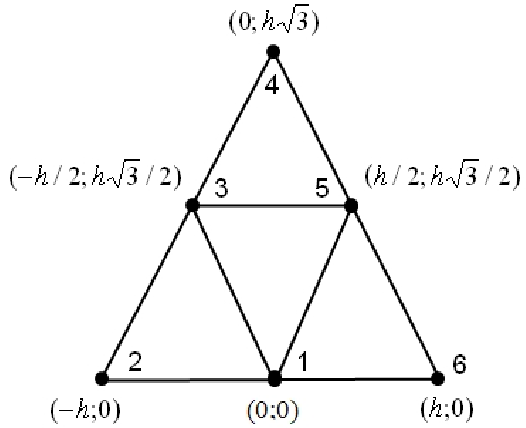
In case of, for example, Robin boundary condition given on one or more faces of a finite element constructed in the form of an invariant polyhedron we receive a set of equations

$$\vec{T}[\vec{r}_i]\vec{n}[\vec{r}_i]\mathbf{N}\mathbf{T}_m^{-1}\vec{U}_{m*} + \vec{T}[\vec{r}_i]\mathbf{T}_m^{-1}\vec{U}_{m*} = g_i\,,\ i = \overline{1,\ldots,s} \quad (3)$$

where s is the number of nodes located on the appropriate faces of the m-th finite element. The solution of (3) gives us the sought shape function comprising both nodal values in inside nodes and boundary conditions in surface nodes.

4. An illustrative example

Let us show the application of the proposed approach to a triangular FE.



We assume that the node with the coordinates $(h/2; h\sqrt{3}/2)$ belongs to a boundary and the value g of normal derivative of the sought function is given for this node:

$$du/dn = du/dx \cos(\widehat{\vec{n},\vec{i}}) + du/dy \cos(\widehat{\vec{n},\vec{j}}) = g. \quad (4)$$

For such mesh Taylor's vector for any point with radius-vector $\vec{r} = \vec{i}x + \vec{j}y$ has the form

$$\vec{T} = \begin{Vmatrix} 1 & x & y & x^2/2 & xy & y^2/2 \end{Vmatrix} \quad (5)$$

The Taylor's matrix for the FE looks like

$$\mathbf{T} = \begin{Vmatrix} 1 & x_1 & y_1 & x_1^2/2 & x_1y_1 & y_1^2/2 \\ 1 & x_2 & y_2 & x_2^2/2 & x_2y_2 & y_2^2/2 \\ 1 & x_3 & y_3 & x_3^2/2 & x_3y_3 & y_3^2/2 \\ 1 & x_4 & y_4 & x_4^2/2 & x_4y_4 & y_4^2/2 \\ 1 & x_5 & y_5 & x_5^2/2 & x_5y_5 & y_5^2/2 \\ 1 & x_6 & y_6 & x_6^2/2 & x_6y_6 & y_6^2/2 \end{Vmatrix} =$$

$$= \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -h & 0 & h^2/2 & 0 & 0 \\ 1 & -h/2 & h\sqrt{3}/2 & h^2/8 & -h^2\sqrt{3}/4 & 3h^2/8 \\ 1 & 0 & h\sqrt{3} & 0 & 0 & 3h^2/2 \\ 1 & h/2 & h\sqrt{3}/2 & h^2/8 & h^2\sqrt{3}/4 & 3h^2/8 \\ 1 & h & 0 & h^2/2 & 0 & 0 \end{Vmatrix} \quad (6)$$

Let's invert the Taylor's matrix using blockwise inversion:

$$\begin{Vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{Vmatrix}^{-1} =$$

$$= \begin{Vmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{Vmatrix}$$

As a result of corresponding consequence of mathematical operations we receive the inverse Taylor's matrix:

$$\mathbf{T}^{-1} = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{2h} & 0 & 0 & 0 & \frac{1}{2h} \\ \frac{-2}{\sqrt{3}h} & \frac{-1}{2\sqrt{3}h} & \frac{2}{\sqrt{3}h} & \frac{-1}{\sqrt{3}h} & \frac{2}{\sqrt{3}h} & \frac{-1}{2\sqrt{3}h} \\ \frac{-2}{h^2} & \frac{1}{h^2} & 0 & 0 & 0 & \frac{1}{h^2} \\ 0 & \frac{1}{\sqrt{3}h^2} & \frac{-2}{\sqrt{3}h^2} & 0 & \frac{2}{\sqrt{3}h^2} & \frac{-1}{\sqrt{3}h^2} \\ \frac{2}{3h^2} & \frac{1}{3h^2} & \frac{-4}{3h^2} & \frac{4}{3h^2} & \frac{-4}{3h^2} & \frac{1}{3h^2} \end{Vmatrix} \quad (7)$$

The expression of the sought function, in accordance with (1), within the standard nodal FE takes the form

$$\begin{aligned} u &= u_1 + (-\frac{1}{2}u_2 + \frac{1}{2}u_6)\frac{x}{h} + \\ &+ (-2u_1 - \frac{1}{2}u_2 + 2u_3 - u_4 + 2u_5 - \frac{1}{2}u_6)\frac{y}{\sqrt{3}h} + \\ &+ (-u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_6)\frac{x^2}{h^2} + \\ &+ (u_2 - 2u_3 + 2u_5 - u_6)\frac{xy}{\sqrt{3}h^2} + \\ &+ (u_1 + \frac{1}{2}u_2 - 2u_3 + 2u_4 - 2u_5 + \frac{1}{2}u_6)\frac{y^2}{3h^2} \end{aligned} \quad (8)$$

As we can see, the expression contains only nodal values without regard to boundary values.

The equation (3) can be written for the given example in the form

$$\vec{T}[\vec{r}_i]\vec{n}[\vec{r}_i]\mathbf{N}\mathbf{T}^{-1}\vec{U}_* = g \quad (9)$$

and in unfolded form:

$$\begin{Vmatrix} 1 & h/2 & h\sqrt{3}/2 & h^2/8 & h^2\sqrt{3}/4 & 3h^2/8 \end{Vmatrix} \bullet$$

$$(\vec{i}\sqrt{\mathbf{3}}/2+\vec{j}/2)\begin{Vmatrix} 0 & \vec{i} & \vec{j} & 0 & 0 & 0 \\ 0 & 0 & 0 & \vec{i} & \vec{j} & 0 \\ 0 & 0 & 0 & 0 & \vec{i} & \vec{j} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix} \bullet$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{2h} & 0 & 0 & 0 & \frac{1}{2h} \\ \frac{-2}{\sqrt{3}h} & \frac{-1}{2\sqrt{3}h} & \frac{2}{\sqrt{3}h} & \frac{-1}{\sqrt{3}h} & \frac{2}{\sqrt{3}h} & \frac{-1}{2\sqrt{3}h} \\ \frac{-2}{h^2} & \frac{1}{h^2} & 0 & 0 & 0 & \frac{1}{h^2} \\ 0 & \frac{1}{\sqrt{3}h^2} & \frac{-2}{\sqrt{3}h^2} & 0 & \frac{2}{\sqrt{3}h^2} & \frac{-1}{\sqrt{3}h^2} \\ \frac{2}{3h^2} & \frac{1}{3h^2} & \frac{-4}{3h^2} & \frac{4}{3h^2} & \frac{-4}{3h^2} & \frac{1}{3h^2} \end{Vmatrix} \begin{Vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Vmatrix} = g \qquad (10)$$

Thus

$$-2\sqrt{3}/(3h)u_1+\sqrt{3}/(3h)u_2-2\sqrt{3}/(3h)u_3+ \\ +\sqrt{3}/(6h)u_4+2\sqrt{3}/(3h)u_5+\sqrt{3}/(6h)u_6=g. \qquad (11)$$

The combination of standard nodal shape function (8) and the boundary condition (11) gives us the "surface" shape function that combines the nodal values and boundary condition value:

$$u=u_1+(-\frac{1}{2}u_2+\frac{1}{2}u_6)\frac{x}{h}+$$
$$+(-\frac{\sqrt{3}}{2}u_2+\frac{4\sqrt{3}}{3}u_3-\frac{\sqrt{3}}{2}u_4-\frac{\sqrt{3}}{3}u_6+gh)\frac{y}{h}+$$
$$+(-u_1+\frac{1}{2}u_2+\frac{1}{2}u_6)\frac{x^2}{h^2}+$$
$$+(\frac{2\sqrt{3}}{3}u_2-\frac{\sqrt{3}}{6}u_4-\frac{\sqrt{3}}{2}u_6+gh)\frac{xy}{h^2}+$$
$$+(-u_1+\frac{3}{2}u_2-4u_3+\frac{5}{2}u_4+u_6-\sqrt{3}gh)\frac{y^2}{3h^2} \qquad (11)$$

The proposed algorithm gives a researcher the possibility not to stick rigidly to existing shape function and construct most appropriate for a task "surface" shape functions with regard to given boundary conditions.

5. Conclusion

New shape functions that are invariant with respect to linear transformations of local and global coordinate frames and automatically satisfy boundary conditions on edges of two-dimensional and faces of three-dimensional FE's have been developed. Their utilization reduces the order of equation system and, subsequently, the computational error.

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ПОБУДОВА «ПОВЕРХНЕВИХ» ФУНКЦІЙ ФОРМИ НА ПІДСТАВІ МЕТОДОЛОГІЇ ІНВАРІАНТНИХ НАБЛИЖЕНЬ

Марія Говикович

Перед нами було поставлено три завдання: забезпечити інваріантність алгебричних аналогів інтегро-диференційних рівнянь, що відповідає інваріантності вихідної інтегро-диференційної системи рівнянь; врахувати граничні умови безпосередньо у апроксимаційних залежностях, що описують поле всередині скінченних елементів, дотичних до границі. Вказані завдання було розв'язано застосуванням методики інваріантного наближення функцій. У результаті показано, що такий підхід забезпечує зниження порядку вихідної системи рівнянь.

Mariya Howykowycz – graduated form Lviv Polytechnic Institute, Ukraine (major "Electrical Systems") and Ivan Franko National University of Lviv, Ukraine (majors "English" and "German"). She defended Ph. D. dissertation thesis at Ivan Franko National University, Ukraine, in Applied Mathematics.

She has been working at Lviv Polytechnic National University since 1992 as Associate Professor at the Department Electrical Engineering. IEEE member since 2002.