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SOME SINGULARITIES OF KERNELS OF LINEAR AR AND ARMA PROCESSES AND THEIR APPLICATION TO SIMULATION OF INFORMATION SIGNALS

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Abstract: Singularities of kernels of linear autoregressive (AR) processes and linear autoregressive--moving-average (ARMA) processes are considered. The paper contains some aspects of the theory of linear stochastic process and discusses a method of estimation and simulation of kernels of linear AR and ARMA processes. Linear stationary AR and ARMA processes, as well as linear AR and ARMA processes with periodic structures are considered. Characteristic functions of linear stationary AR and ARMA processes are presented. Characteristic functions of linear AR and ARMA processes with periodic structures are presented as well.

Key words: linear AR process, linear ARMA process, infinitely divisible random process, random process with periodic structures, kernels of linear AR and ARMA processes.

1. Introduction

Many kinds of information signals like vibrations, acoustic emission signals, control signals, etc., can be represented as a response of some linear system to white noise action [1–9]. In our paper we discuss singularities of kernels of linear autoregressive (AR) and linear autoregressive--moving-average (ARMA) processes. The proposed approach to problem solving is based on the theory of linear stochastic process [1, 5, 9] and it can be considered the development of the method of generating processes [9]. Let us discuss some aspects of the theory and classify the simulating processes.

Many processes in radiophysics, hydroacoustics, meteorology, power engineering, biomedical systems and, consequently, random functions describing such processes have characteristics recurring over time or space. The period of process variation is usually considered to be a time or space interval after which the variation of process characteristics fully recurs, though the realization of such random process may not have the same properties. For their mathematical simulation it is expedient, in this case, to use random processes with periodic structures [12–15].

We consider a problem of statistical simulation of discrete-time linear random processes (LRPs). One of the most interesting models of that class is a linear autoregressive process (AR-process) and linear autoregressive moving-average process (ARMA-process). We will show some methods of the process representation and formulate the simulation problem for this case.

2. Kernels of linear AR and ARMA processes

By definition, a linear stationary AR-process can be represented as

$$x_t = -\sum_{i=1}^{p} a_i x_{t-i} + V_t, \; t \in Z, \qquad (1)$$

where $\{a_j, a_j \neq 0, j = \overline{1, p}\}$ are real-valued autoregression parameters; p is the order of autoregression; Z is the set of integers; $V_t, t \in Z$, is a generating process. We consider the AR-process that is stochastically equivalent to the process (1) defined on a discrete equidistant lattice. As stated above, such AR-processes are identified as linear, and henceforce we deal only with such processes.

The linear stationary AR-process admits the Wold representation [15]

$$x_t = \sum_{t=1}^{\infty} j_{AR}(t) V_{t-t}, \qquad (2)$$

where $j_{AR}(t)$ is the kernel of the linear process [9]. It is assumed that

$$j_{AR}(0) = 1; \qquad \sum_{t=0}^{\infty} \left| j_{AR}(t) \right|^2 < \infty. \qquad (3)$$

The kernel $j_{AR}(t)$ is recursively connected with the autoregression parameters [4]

$$j_{AR}(0) = 1,$$

$$\text{if } p = 1 \quad j_{AR}(s) = -a_p j_{AR}(s-1) \text{ for } s = 1, 2, \mathbf{K} \qquad (4)$$

$$\text{if } p > 1 \quad j_{AR}(s) = -\sum_{j=1}^{s} a_j j_{AR}(s-j) \text{ for } s = \overline{1, p-1}$$

$$j_{AR}(s) = -\sum_{j=1}^{p} a_j j_{AR}(s-j) \text{ for } s = p, p+1, \mathbf{K}$$

where $j_{AR}(t)$ is the kernel of the linear stationary AR-process and is defined by the following expressions:

$$j_{AR}(t)=\begin{cases}0 & k<0\\ 1 & k=0\\ \sum_{t=1}^{k} a_t j_{AR}(k-t) & k=\overline{1,p-1},\\ \sum_{t=1}^{p} a_t j_{AR}(k-t) & k=\overline{p,p+1},\end{cases} \quad (5)$$

By definition, a linear stationary ARMA-process can be represented as

$$x_t=-\sum_{i=1}^{p} a_i x_{t-i}+V_t+\sum_{j=1}^{q} b_j V_{t-j}, \quad (6)$$

where $\{a_i, a_i \neq 0, i=\overline{1,p}\}$ are autoregressive parameters; p is the order of autoregression; $\{b_j, b_j \neq 0, j=\overline{1,q}\}$ are moving-average parameters; q is the order of moving-average; V_t is a generating process.

A linear stationary ARMA process can be also defined by the expression

$$x_t=V_t+\sum_{t=1}^{\infty} j_{ARMA}(t)V_{t-t}, \quad (7)$$

where $j_{ARMA}(t)$ is the kernel of the linear stationary ARMA process and is defined by the following expressions:

$$j_{ARMA}(k)=\begin{cases}0 & k<0\\ 1 & k=0 \qquad q\le p\\ b_k+\sum_{t=1}^{k} a_t j_{ARMA}(k-t) & k=\overline{1,p-1}\\ b_k+\sum_{t=1}^{p} a_t j_{ARMA}(k-t) & k=\overline{p,p+1}\end{cases} \quad (8)$$

It is shown in [4,7,8] that linear AR processes and linear ARMA processes are infinitely divisible processes, that is, their finite-dimensional characteristic functions (CFs) are infinitely divisible ones. Therefore, the CF can be represented in one of three canonical forms [10].

A linear autoregressive process $\{x_t, t\in Z\}$ with periodic structures determined over the set of integers $Z=\{...,-1,0,1,...\}$ is a random process with periodically varying autoregression parameters. It can be written down as follows [14]:

$$x_t+a_1(t-1)x_{t-1}+...+a_p(t-p)x_{t-p}=V_t, \quad (9)$$

where $a_1(t),...,a_p(t)$ are autoregression parameters alternating over time with the same period T, i.e. $a_1(t)=a_1(t+T),\ldots,a_p(t)=a_p(t+T)$ $\forall\ t \in Z$; p is the order of autoregression ($p>0$, $p\in Z$); $V_t, t\in Z$ is a discrete-time random process with independent values and infinitely divisible distribution law. Singularities of the processes were considered in [14]. The autoregressive process with periodically changing autoregression parameters can be also defined by the expression:

$$x_t=\sum_{t=1}^{\infty} j_{AR}(t,t)V_{t-t} \quad (10)$$

where $j_{AR}(t,t)$, is a real-valued or complex-valued function which is bounded with respect to both arguments, i.e. $|j_{AR}(t,t)|<K$ $(K<\infty)$. For the kernel $j_{AR}(t,t)$ of the process (9) the following equality is fulfilled:

$$j_{AR}(t,t)=j_{AR}(t,t+T),\ T>0 \quad (11)$$

A linear ARMA process $\{x_t, t\in Z\}$ with periodic structures determined over the set of integers $Z=\{...,-1,0,1,...\}$ is a random process with periodically varying autoregression parameters and periodically varying moving-average parameters

$$\begin{aligned}&x_t+a_1(t-1)x_{t-1}+...+a_p(t-p)x_{t-p}=V_t+\\&+b_1(t-1)V_{t-1}+...+b_q(t-q)V_{t-q},\end{aligned} \quad (12)$$

where $a_1(t),...,a_p(t)$ are autoregression parameters alternating over time with the same period T, i.e. $a_1(t)=a_1(t+T),\ldots,a_p(t)=a_p(t+T)$, $\forall\ t \in Z$; p is the order of autoregression ($p>0$, $p\in Z$); $b_1(t),...,b_q(t)$ are moving-average parameters alternating over time with the same period T, i.e. $b_1(t)=b_1(t+T),\ldots$ $\forall\ t \in Z$; q is the order of moving-average ($q>0$, $q\in Z$); $V_t, t\in Z$ is a discrete-time random process with independent values and infinitely divisible distribution law.

The linear ARMA process with periodically changing autoregression parameters and moving-average parameters can be also specified in the form:

$$x_t=\sum_{t=1}^{\infty} j_{ARMA}(t,t)V_{t-t}, \quad (13)$$

where $j_{ARMA}(t,t)$ is a real-valued or complex-valued function which is bounded with respect to both arguments, i.e. $|j_{AR}(t,t)|<K$ $(K<\infty)$. For the kernel $j_{AR}(t,t)$ of the process (12) the following equality is fulfilled:

$$j_{AR}(t,t) = j_{AR}(t,t+T),\ T>0 \tag{14}$$

The kernel $j_{ARMA}(t,t)$ of a linear ARMA process with periodic structures as well as the kernel $j_{AR}(t,t)$ of a linear AR process have periodic structure in terms of t:

$$j_{ARMA}(t,t) = j_{ARMA}(t,t+T),\ T>0. \tag{15}$$

Thus, linear AR and ARMA processes with autoregression parameters and moving-average parameters that cyclically vary over time with the same period $T>0$, generated by a random process with independent values and infinitely divisible distribution law, can be represented by a linear random process with discrete time and periodic kernel (in terms of t).

3. Characteristic functions of linear AR and ARMA processes

A process x_t is assumed to be strictly stationary and satisfy the ergodic theorem [7,8].

The characteristic function (CF) of the process x_t has the Kolmogorov representation and its one-dimensional logarithm can be defined as follows:

$$\ln f_x(u,t) = \ln f_x(u,1) = im_x u + \int_{-\infty}^{\infty} \left\{e^{iux} - 1 - iux\right\} \frac{dK_x(x)}{x^2},$$

where parameter m_x and spectral functions of jumps $K_x(x)$ determine uniquely the characteristic function.

The logarithm of the one-dimensional characteristic function of the linear stationary autoregression process can be written in the form

$$\ln f_x(u,t) = \ln f_x(u,1) = im_z u \sum_{t=-\infty}^{\infty} j(t) + \\ + \sum_{t=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^{iuxj(t)} - 1 - iuxj(t)\right) \frac{dK_z(x)}{x^2}$$

where parameters m_V and $K_z(x)$ determine the characteristic function of the generative process V_t, while $j(t)$ is the kernel of the linear random process x_t. The parameters m_x and m_V as well as the Poisson spectra of the jumps $K_x(x)$ and $K_z(x)$ are interrelated as follows

$$m_x = m_z \sum_{t=0}^{\infty} j(t), \tag{16}$$

$$K_x(x) = \int_{-\infty}^{\infty} R_j(x,y) dK_V(y)$$

where $R_j(x,y)$ is so-called transformation kernel, which is invariant with respect to the generative process V_t and uniquely defined by the coefficients $\{a_j, a_j \neq 0, j=\overline{1,p}\}$ or $\{a_i, a_i \neq 0, i=\overline{1,p}\}$ and $\{b_j, b_j \neq 0, j=\overline{1,q}\}$.

Singularities of the kernel $R_j(x,y)$ are discussed in [7,8,15]. An inverse kernel $R_j^{-1}(x,y)$ exists and the inverse integral transform exists as well:

$$K_V(y) = \int_{-\infty}^{\infty} R_j^{-1}(x,y) dK_x(x) \tag{17}$$

Some applications require finding statistical characteristics of the generating process V_t if autoregression parameters $\{a_j, a_j \neq 0, j=\overline{1,p}\}$ or $\{a_i, a_i \neq 0, i=\overline{1,p}\}$ and moving-average parameters $\{b_j, b_j \neq 0, j=\overline{1,q}\}$ are known. The statistical characteristics of the observed linear AR process or linear ARMA process are known. Occasionally, such a problem is referred to as an inverse problem [6, 7, 8, 15].

It is known [11] that any stochastically continuous process with independent increments can be represented as a sum of two stochastically independent components, that is, Gaussian and Poisson ones, which not necessarily occur simultaneously. These components are defined as processes of the Gaussian and Poison types. The first type comprises the homogeneous (Wiener) and non-homogeneous Gaussian processes with independent increments. Simple Poisson processes, renewal processes and their linear combinations, as well as generalized Poisson processes with independent increments belong to the second type. Being generated by one of the mentioned processes, linear random processes (LRP) possess some typical properties. This statement may be extended to the random process with discrete time. This fact is taken as a basic principle of the classification and simulation algorithms.

The logarithm of the characteristic function of a linear AR process with the periodic kernel has the following form [14]:

$$\ln f_x(u,t) = \\ = ik_{V1} u \sum_{t=0}^{\infty} f_{AR}(t,t) - 0.5u^2 k_{V2} \sum_{t=0}^{\infty} f_{AR}{}^2(t,t) + \\ + \sum_{t=0}^{\infty} \int_{-\infty}^{\infty} \left\{\exp[iuxf_{AR}(t,t)] - 1 - \frac{iuxf_{AR}(t,t)}{1+x^2}\right\} dL(x)$$

where k_{V1}, k_{V2} are the first and second cumulants of a generating process V_t, respectively; $j_{AR}(t,t)$ is the kernel of the linear AR process, $L(x)$ is the Poisson spectrum of jumps in the Levy formula of the generating random process V_t.

The logarithm of the characteristic function of a linear ARMA process with the periodic kernel has the same form.

4. Conclusion

The proposed methods make it possible to find kernels and characteristic functions of linear AR processes and linear ARMA processes. The characteristic functions could be used for development of novel classification algorithms for information signals of power equipment diagnostic systems.

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ДЕЯКІ ОСОБЛИВОСТІ ЯДЕР ЛІНІЙНИХ AR ARMA ПРОЦЕСІВ І ЇХ ВИКОРИСТАННЯ ДЛЯ МОДЕЛЮВАННЯ ІНФОРМАЦІЙНИХ СИГНАЛІВ

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Розглянуто особливості ядер лінійних авторегресивних (AR) випадкових процесів та лінійних авторегресивних з плаваючим середнім (ARMA) випадкових процесів. Стаття містить деякі положення теорії лінійних випадкових процесів, обговорюються методи оцінки і моделювання ядер лінійних AR та ARMA процесів. Розглядаються лінійні стаціонарні AR та ARMA процеси і лінійні AR та ARMA процеси з періодичними структурами та їх характеристичні функції.

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