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Peculiarities of modelling of the gas motion process in pipeline

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Peculiarities of modeling the gas motion process in horizontal pipeline are considered. The mathematical model of the gas motion process in pipeline is modified to avoid the operations with the numbers of different orders and the instability of desired solution during the setting of boundary conditions on the one of the ends of pipeline. The iterative method of determination of the gas motion parameters based on the data which are set at one of the ends of pipeline is suggested.

Key words: pipeline, gas motion modelling, system of partial differential equations, mathematical model, Laplace transform.

Introduction. Depending on the put problems the systems of partial differential equations of varying degrees of complexity and of different orders are used to describe unsteady non-isothermal gas motion process in pipelines [1]. Different mathematical and computational difficulties arise during formulating and solving the problems of mathematical physics. In particular: the equations which describe processes are nonlinear; the coefficients of equations are dependent both on the coordinates and on the time; in computer terms it is necessary to perform the operations with the numbers of equal orders etc [2, 3]. In addition the boundary conditions are usually set on the both ends of the pipeline. Setting the boundary conditions on one of the ends that often happens in practice causes the considerable instability during the calculation of the desired solution.

The aim of the work is to modify the mathematical model of the gas motion process in pipeline to avoid the operations with the numbers of different orders and the instability of desired solution during the setting of boundary conditions on one of the ends of pipeline.

1. The formulation and solving the problem

In the case of isothermal approximation the system of interconnected partial differential equations are used for describing the gas motion in pipeline [1-3]

$$\frac{\partial p}{\partial x}+\frac{\lambda\rho\upsilon^2}{2D}+\frac{\partial(\rho\upsilon)}{\partial t}=0,\qquad \frac{\partial(\rho\upsilon)}{\partial x}+\frac{1}{c^2}\frac{\partial p}{\partial t}=0,\qquad p=\rho zRT\,, \tag{1}$$

here $p=p(x,t)$ is the pressure distribution along the pipeline; ρ is the gas density; υ is the gas motion velocity; λ is the coefficient of hydraulic resistance; T is the gas temperature; R is the gas constant; z is the coefficient of the gas compressibility; x is the movable coordinate $x \in [0, L]$; L is the length of pipeline; D is the inner diameter of pipeline; t is the time; c is the sound velocity in gas.

The first equation of the system of equations (1) is obtained in the case of change neglect of the gas density over time. Otherwise, the system of equations has the form

$$\frac{\partial p}{\partial x} + \frac{\lambda \rho \upsilon^2}{2D} + \rho \frac{\partial \upsilon}{\partial t} = 0, \quad \upsilon \frac{\partial \rho}{\partial x} + \rho \frac{\partial \upsilon}{\partial x} + \frac{1}{c^2}\frac{\partial p}{\partial t} = 0, \quad p = \rho z R T. \tag{2}$$

If we introduce the marking $\gamma = zRT$ the system (2) will be written

$$\gamma \frac{\partial \ln p}{\partial x} + \frac{\lambda \upsilon^2}{2D} + \frac{\partial \upsilon}{\partial t} = 0, \qquad \upsilon \frac{\partial \ln p}{\partial x} + \frac{\partial \upsilon}{\partial x} + \frac{\gamma}{c^2}\frac{\partial \ln p}{\partial t} = 0. \tag{3}$$

The system of equations (3) is written concerning the key functions p and υ. In practically important problems the pressure p is the value of order 10^6 and the velocity υ is the value of order 10 [1]. During the study of the system of equations (3) by means of numerical methods it should write it on such solving functions which will acquire the values of the same orders. Such change will significantly improve the stability of numerical methods which are used in the algorithm of solving. For this purpose use the equation $\ln p = f$.

In this case the system (3) will be written

$$\gamma \frac{\partial f}{\partial x} + \frac{\lambda \upsilon^2}{2D} + \frac{\partial \upsilon}{\partial t} = 0, \qquad \upsilon \frac{\partial f}{\partial x} + \frac{\partial \upsilon}{\partial x} + \frac{\gamma}{c^2}\frac{\partial f}{\partial t} = 0. \tag{4}$$

In order to find the analytical solution of the system (4) construct the following iterative algorithm. Let υ_c is the velocity value at the previous iteration step. Taking into account that the change of the gas motion velocity is minor we can write the system (4) in linearized form

$$\gamma \frac{\partial f}{\partial x} + \frac{\lambda \upsilon_c \upsilon}{2D} + \frac{\partial \upsilon}{\partial t} = 0, \qquad \upsilon_c \frac{\partial f}{\partial x} + \frac{\partial \upsilon}{\partial x} + \frac{\gamma}{c^2}\frac{\partial f}{\partial t} = 0. \tag{5}$$

In the stationary case the system (5) will look like

$$\gamma \frac{df}{dx} + \frac{\lambda \upsilon_c \upsilon}{2D} = 0, \qquad \upsilon_c \frac{df}{dx} + \frac{d\upsilon}{dx} = 0.$$

From the obtained system express velocity and pressure distribution along the pipeline in steady state for the given conditions at the pipeline inlet υ_0 and $f_0 = \ln p_0$:

$$\upsilon(x) = \upsilon_0 e^{\frac{\lambda \upsilon_c^2 x}{2\gamma D}}, \qquad f(x) = f_0 - \frac{\upsilon_0}{\upsilon_c}\left(e^{\frac{\lambda \upsilon_c^2 x}{2\gamma D}} - 1 \right).$$

The last two equations are initial conditions for the formulation of the problem in the unsteady case. Turn to the definition of the boundary conditions which we shall set at the pipeline inlet. If one need to change the magnitude of pressure on the value Δp then as follows from the experimental data analysis the dependence of boundary condition from the time is approximated good enough by the function $p(0,t)=$ $= p_{0n} + \Delta p e^{-\gamma_0 t}$; $\Delta p = p_0 - p_{0n}$; here p_0, p_{0n} are initial and eventual value of pressure; parameter γ is responsible for the transition rate from one stationary process of the gas motion to another. As for setting the speed limit at the pipeline inlet note the following.

It is evident that pressure and velocity correlate by the formula $\rho_0 q_0 = \rho q =$ $= pS\upsilon/(zRT)$ or $\upsilon = zRT\rho_0 q_0/(pS)$. If volumetric gas outlay under standard conditions at the pipeline inlet varies according to the formula $q(0,t) = q_{0n} + \Delta q e^{-\vartheta_0 t}$; $\Delta q = q_0 - q_{0n}$; here q_0, q_{0n} are initial and eventual value of volumetric gas outlay under standard conditions, and ϑ determines the transition rate from the one steady gas motion mode on another then

$$\upsilon(0,t) = \frac{zRT\rho_0}{S}\frac{q_0}{p_{0n}}\left(1 + \frac{\Delta q}{q_{0n}}e^{-\vartheta_0 t} + \frac{\Delta p}{p_{0n}}e^{-\gamma_0 t}\right).$$

The similar relationship between pressure and velocity of gas motion can be used on another end of pipeline

$$f(l,t) \approx \ln p_k + \frac{\Delta p_k}{p_k}e^{-\gamma_k t}, \quad \upsilon(l,t) = \frac{zRT\rho_0}{S}\frac{q_k}{p_{kn}}\left(1 + \frac{\Delta q_k}{q_{kn}}e^{-\vartheta_k t} + \frac{\Delta p_l}{p_{kn}}e^{-\gamma_k t}\right).$$

Here index *k* appertains to the end of pipeline. Let's get to solving the problem.

In Laplace transforms [4] the linearized system (5) has the form

$$\begin{cases} \gamma\dfrac{\partial \bar{f}(x,s)}{\partial x} + \dfrac{\lambda\upsilon_c\bar{\upsilon}(x,s)}{2D} + s\bar{\upsilon}(x,s) - \upsilon(x,0) = 0, \\ \upsilon_c\dfrac{\partial \bar{f}(x,s)}{\partial x} + \dfrac{\partial\bar{\upsilon}(x,s)}{\partial x} + \dfrac{\gamma}{c^2}s\bar{f}(x,s) - \dfrac{\gamma}{c^2}f(x,0) = 0. \end{cases} \tag{6}$$

Since the boundary conditions are nonzero then it is reasonable to solve the system of equations (6) in the form of Fourier series for cosines which we write so

$$\phi(x) = \mathrm{Re}\sum_{-\infty}^{\infty}\phi_n e^{-\frac{n\pi i x}{L}}, \quad x \in [0, L]. \tag{7}$$

In the last formula $i^2 = -1$ is the complex unit, and Re means real part of a complex number. The coefficients of the series (7) are defined by the formula $\phi_n = \dfrac{1}{2L}\int_0^L \phi(x)e^{\frac{n\pi i x}{L}}dx$.

If we use the function expansion (7) for the system (6) we shall obtain the following system

$$\begin{cases} \gamma \dfrac{f_n}{\nu_n} + \gamma\psi_{nf} + \dfrac{\lambda\upsilon_c\upsilon_n}{2D} + s\upsilon_n - \upsilon_{0n} = 0, \\ \upsilon_c \dfrac{f_n}{\nu_n} + \upsilon_c\phi_{nf} + \dfrac{\upsilon_n}{\nu_n} + \psi_{n\upsilon} + \dfrac{\gamma}{c^2} sf_n - \dfrac{\gamma}{c^2} f_{0n} = 0. \end{cases} \quad (8)$$

Introduce the notations

$$h_n(A,B,s) = \frac{s(s+\alpha)}{(s-s_1)(s-s_2)}, \quad z_s = \frac{1}{s^2 + Ks + M}, \quad \alpha = B/A .$$

Then the solution of the system (8) will have the form

$$f_n(s) = h_{1n}(s) + h_{2n}(s)\upsilon(0,s) + h_{3n}(s)\upsilon(L,s) + h_{4n}(s)f(0,s) + h_{5n}(s)f(L,s),$$

there

$$h_{1n}(s) = h_n(A,B,s),\ h_{2n}(s) = h_n(A_1,B_1,s),\ h_{3n}(s) = h_n(A_2,B_2,s),$$
$$h_{4n}(s) = h_n(A_3,B_3,s),\ h_{5n}(s) = h_n(A_4,B_4,s),\ d_{1n}(s) = z_s(Xs+Y),\ d_{2n}(s) = z_sY_1,$$
$$d_{3n}(s) = z_sY_2,\ d_{4n}(s) = z_sX_3s,\ d_{4n}(s) = z_sX_3s,\ d_{5n}(s) = z_sX_4s .$$

Here

$$A = H_1 + iH_2,\quad A_1 = \frac{c^2}{L\gamma},\quad A_2 = (-1)^{n+1}A_1,\quad A_3 = A_1\upsilon_c,\quad A_4 = (-1)^{n+1}A_3,$$

$$B = \beta_1H_1 + \beta_2H_4 + i(\beta_1H_2 - \beta_2H_3),\ B_1 = \frac{c^2\lambda\upsilon_c}{2DL\gamma},\ B_2 = (-1)^{n+1}B_1,\ B_3 = B_1 - i\beta_3,$$

$$B_4 = B_2 + i(-1)^n\beta_3,\quad X = H_3 + iH_4,\quad X_3 = \frac{\gamma}{L},\quad X_4 = (-1)^{n+1}X_3,$$

$$Y = \xi_1H_2 - \xi_2H_4 + i(\xi_2H_3 - \xi_1H_1),\quad Y_1 = -i\beta_3,\quad Y_2 = i(-1)^{n+1}Y_1,$$

there

$$\beta_1 = \frac{\lambda\upsilon_c}{2D},\quad \beta_2 = \frac{c^2n\pi}{L\gamma},\quad \beta_3 = \frac{c^2n\pi}{L^2},\quad \xi_1 = \frac{n\pi\gamma}{L},\quad \xi_2 = \beta_2\upsilon_c .$$

Since $p = e^f$ then $\upsilon = zRT\rho_0q_0e^{-f}/S$. If we denote $r_0 = zRT\rho_0q_0/S$, then $\upsilon = r_0e^{-f}$. Then

$$f_n(s) = h_{1n}(s) + h_{2n}(s)r_0e^{-f(0,s)} + h_{3n}(s)r_0e^{-f(L,s)} + h_{4n}(s)f(0,s) + h_{5n}(s)f(L,s), \quad (9)$$

$$\upsilon_n(s) = d_{1n}(s) + d_{2n}(s)r_0e^{-f(0,s)} + d_{3n}(s)r_0e^{-f(L,s)} +$$
$$+d_{4n}(s)f(0,s) + d_{5n}(s)f(L,s) . \quad (10)$$

The initial problem solution in the Laplace transforms is written in the form

$$f(x,s)=\mathrm{Re}\sum_{n=0}^{\infty} f_n(s)\exp\left(-\frac{n\pi ix}{L}\right)$$

and

$$\upsilon(x,s)=\mathrm{Re}\sum_{n=0}^{\infty} \upsilon_n(s)\exp\left(-\frac{n\pi ix}{L}\right)$$

or

$$f(x,s)=\mathrm{Re}\sum_{n=0}^{\infty}\Big(h_{1n}(s)+h_{2n}(s)r_0e^{-f(0,s)}+h_{3n}(s)r_0e^{-f(L,s)}+$$
$$+h_{4n}(s)f(0,s)+h_{5n}(s)f(L,s)\Big)\exp\left(-\frac{n\pi ix}{L}\right) \qquad (11)$$

and

$$\upsilon(x,s)=\mathrm{Re}\sum_{n=0}^{\infty}\Big(d_{1n}(s)+d_{2n}(s)r_0e^{-f(0,s)}+d_{3n}(s)r_0e^{-f(L,s)}+$$
$$+d_{4n}(s)f(0,s)+d_{5n}(s)f(L,s)\Big)\exp\left(-\frac{n\pi ix}{L}\right).$$

Write the equation (11) so

$$f(x,s)=\mathrm{Re}\sum_{n=0}^{\infty}\Big(h_{1n}(s)+h_{2n}(s)r_0e^{-f(0,s)}+h_{4n}(s)f(0,s)\Big)\exp\left(-\frac{n\pi ix}{L}\right)+$$
$$+r_0e^{-f(L,s)}\,\mathrm{Re}\sum_{n=0}^{\infty}h_{3n}(s)\exp\left(-\frac{n\pi ix}{L}\right)+f(L,s)\,\mathrm{Re}\sum_{n=0}^{\infty}h_{5n}(s)\exp\left(-\frac{n\pi ix}{L}\right). \qquad (12)$$

The iterative method of solving the problem consists in the following. We set the initial velocity approximation υ_c and calculate velocity and pressure value at the given point. The found velocity value is assigned to υ_c and the calculations of the unknown parameters are repeated until the difference between two consecutive approximations is less than the given value. This approach made a good showing in solving nonlinear differential equations and systems of differential equations.

2. Methods for determining pressure and velocity value at the pipeline outlet

1. If the value $f(0,s)$ is known then we can determine $f(L,s)$ from the equation (12). $f(L,s)$ is the pressure value in Laplace transforms at the pipeline outlet. Heaving determined $f(L,s)$ one can determine pressure and velocity distribution in Laplace

transforms. Determination of the value $f(L,s)$ results in solving the transcendental algebraic equation. Note that in order to find the final solution we need move from the transforms space to the originals space. Since we search $f(L,s)$ by means of approximate or numerical method then we need find the original by means of the similar method.

2. Consider another way of finding the value $f(L,s)$. In order to find desired solutions $f(x,t)$ and $\upsilon(x,t)$ we need move from transforms to originals in equations (9) and (10) [4]. The originals of values h_i and d_i are found from the correspondence tables [5]. Since the originals $f(0,t)$ and $\upsilon(0,t)$ are known then the originals of coefficients $f_n(s)$ and $\upsilon_n(s)$ are found by the convolution namely

$$f_n(t) = h_{1n}(t) + \int_0^t \left[h_{2n}(t-\tau)\upsilon(0,\tau) + h_{3n}(t-\tau)\upsilon(L,\tau) + \right.$$
$$\left. + h_{4n}(t-\tau)f(0,\tau) + h_{5n}(t-\tau)f(L,\tau)\right]d\tau \tag{13}$$

and

$$\upsilon_n(t) = d_{1n}(t) + \int_0^t \left[d_{2n}(t-\tau)\upsilon(0,\tau) + d_{3n}(t-\tau)\upsilon(L,\tau) + \right.$$
$$\left. + d_{4n}(t-\tau)f(0,\tau) + d_{5n}(t-\tau)f(L,\tau)\right]d\tau . \tag{14}$$

If we substitute the equations (13) and (14) in corresponding series we shall obtain the system of integral convolution type equations in originals. Note that the original of transform $h_n(A,B,s)$ is function $h_n(A,B,t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t}$. Here

$$\alpha_1 = \frac{\alpha + s_1}{\sqrt{K^2 - 4M}}, \quad \alpha_1 = -\frac{\alpha + s_2}{\sqrt{K^2 - 4M}},$$
$$s_1 = \frac{1}{2}\left(-K - \sqrt{K^2 - 4M}\right), \; s_2 = \frac{1}{2}\left(-K + \sqrt{K^2 - 4M}\right).$$

3. If we substitute $x = L$ in equation (13) we shall obtain nonlinear algebraic equation with one unknown $f(L,s)$

$$f(L,s) = \operatorname{Re}\sum_{n=0}^{\infty}(-1)^n \left(h_{1n}(s) + h_{2n}(s) r_0 e^{-f(0,s)} + h_{4n}(s) f(0,s)\right) +$$
$$+ r_0 e^{-f(L,s)} \operatorname{Re}\sum_{n=0}^{\infty}(-1)^n h_{3n}(s) + f(L,s) \operatorname{Re}\sum_{n=0}^{\infty}(-1)^n h_{5n}(s) .$$

Heaving solved the equation and moved to the original we shall find the behavior of pressure and velocity at the output end of pipeline.

Conclusion. The problem of modeling the gas motion process in horizontal pipeline is considered. The method for determining parameters of transitional processes of the gas motion at the pipeline outlet for the given input data at the pipeline inlet based on the found solution is suggested.

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Особливості моделювання процесу руху газу в трубопроводі

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Розглянуто особливості моделювання процесу руху газу в горизонтальному трубопроводі. Модифіковано математичну модель процесу руху газу в трубопроводі з метою уникнення операцій із числами різних порядків і нестійкості шуканого розв'язку у разі задання граничних умов на одному з кінців трубопроводу. Запропоновано ітераційний метод визначення параметрів руху газу на основі даних, виміряних на одному з кінців трубопроводу.

Особенности моделирования процесса движения газа в трубопроводе

Галина Пяныло, Олег Браташ

Рассмотрены особенности моделирования процесса движения газа в горизонтальном трубопроводе. Модифицирована математическая модель процесса движения газа в трубопроводе во избежание операций с числами различных порядков и неустойчивости искомого решения при задании граничных условий на одном из концов трубопровода. Предложен итерационный метод определения параметров движения газа на основе данных, измеренных на одном из концов трубопровода.

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