УДК 517.5

# Summation of trigonometric series by methods with power multiplier

Halyna Ivasyk

PhD., Lviv Polytechnic National University, S. Bandera str., 12, 79013, Lviv, Ukraine,
e-mail: ivasyk_g@yandex.ua

*The methods of power summation of factors, degree of factors which are arbitrary powers of summation indices are reviewed. It is shown that by Poisson-Abel method only those series can be summarized the order of member increase of which is proportional to the exponent depending on the summation index. Gauss-Weierstrass method and other power factors methods can be also applied to the series the terms of which increase in proportion to the exponential dependence of the indices summation.*

**Keywords:** trigonometric series, generalized sum of the series, the method of Gauss-Weierstrass of series summation, Poisson-Abel method of series summation.

**Introduction.** Methgods of series summation based on the mathematical aparatus of averaging integral operators are formulated in the works [1, 2].

Power multiplier methods of and Gauss-Weierstrass [2-5] with the factors having, respectively, the first and the second degree of sequence numbers of the series members (summation indeces) are powerful enough from the point of view of summing up the series with the growing number of members.

This paper reviews power multiplier methods of series summation, degrees of factors of which are arbitrary powers of summation indeces. It is shown that by Poisson-Abel method can be summarized only the series the members of which have the order of growth proportional to the exponent depending on the index. Gauss-Weierstrass method and other power multiplier methods (power facors methods) can also be applied to the series whose members grow in proportion to the exponential dependence of the index summation.

## 1. Generalized sum of trigonometric series

Here are the main characteristics of averaging operators [2].

1. Let the function $f(x)$ such that the function $f(x)\left(1+|x|\right)^{-\lambda} \in L^1\left(E\right)$ be integrated by Lebesgue where $\lambda > 1$; $E = \left\{x : |x| < \infty\right\}$, and $\omega(x) \in L^1\left(E\right)$ is the function that satisfies the conditions

$$|\omega(x)|\left(1+|x|\right)^{\lambda} \le M < \infty\,, \qquad \int\limits_{-\infty}^{\infty} \omega(x)\,dx = 1\,. \tag{1}$$

Then, at each point of the continuity of the function $f(x)$ the equality is true

$$f(x) = \lim_{\sigma\to+0} \int\limits_{-\infty}^{\infty} f(t)\frac{1}{\sigma}\omega\left(\frac{t-x}{\sigma}\right)dt\,. \tag{2}$$

where $\{\sigma\}$ is positive numerical set with the point of condensation $\sigma = 0$.

Let us consider trigonometric series of periodical function $f(x) \in L^1[-\pi;\pi]$,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\,, \tag{3}$$

where $a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx\,dx$; $b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx\,dx$.

Applying the averaging operator to the function $f(x)$ and developing it into the series with the fixed value $\sigma \ne 0$, we'll get the series

$$f_\sigma(x) = \int\limits_{-\infty}^{\infty} f(t)\frac{1}{\sigma}\omega\left(\frac{t-x}{\sigma}\right)dt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \varphi_n(\sigma)\left(a_n \cos nx + b_n \sin nx\right), \tag{4}$$

where $\varphi_n(\sigma) = \int\limits_{-\infty}^{\infty} \omega(x)\cos(\sigma n x)dx$.

If the conditions of statement 1 are true, then at each point of continuity of the function $f(x)$ marginal equation (2) is true, which with (4) takes the form

$$f(x) = \lim_{\sigma\to+0}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \varphi_n(\sigma)\left(a_n \cos(nx) + b_n \sin(nx)\right)\right]. \tag{5}$$

2. The equality (5) defines a generalized sum, and the sequence $\{\varphi_n(\sigma)\}$ with $\sigma \to +0$ is the series summation method (3).

Traditionally for theoretical and practical studies of convergence of series they use methods of power factors $\left\{\varphi_n = \rho^{n^\nu}\right\}$, $\nu > 0,\ \rho \to 1-0$. Prominent place in mathematical analysis is occupied by Poisson-Abel method $\left\{\varphi_n(\sigma) = \rho^n\right\}$ with the kernel of averaging operator $\omega(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, where $\rho = e^{-\sigma}$, $\rho \to 1-0$, and the method of Gauss-Weierstrass $\left\{\varphi_n(\sigma) = \rho^{n^2}\right\}$ with the kernel $\omega(x) = \frac{1}{2\sqrt{\pi}}\exp\left(-x^2/4\right)$, where

$\rho = e^{-\sigma^2}$ , $\rho \to 1-0$ . Nuclear features of these methods are infinitely differentiated functions which satisfy the conditions (1).

The similar limit equality to (5) for the derivative of the function is true.

3. If the periodical function $f(x) \in L^1[-\pi;\pi]$ has the derivative of k order in the point $x$, then the next equality is true

$$f^{(k)}(x) = \lim_{\rho \to 1-0} \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^{n^\nu} n^k \left[ a_n \cos\left( nx + \frac{k\pi}{2} \right) + b_n \sin\left( nx + \frac{k\pi}{2} \right) \right] \right\}.$$

Note that averaging operators have a clear expressions only in these two cases.

## 2. Sequences of generalized partial sums of series

By entering generalized partial sum of the series (4), the limit equality (5) can be written as double limit equality

$$f(x) = \lim_{\rho \to 1-0} \lim_{N \to \infty} \left[ \frac{a_0}{2} + \sum_{n=1}^{N} \rho^{n^\nu} \left( a_n \cos nx + b_n \sin nx \right) \right]. \quad (6)$$

Then, $f(x)$ is generalized sum of the series (3) in the point $x$, if for any arbitrarily small number $\varepsilon > 0$ there exists number $N$ and there exists number $\rho_N$, $\lim_{N \to \infty} \rho_N = 1$, such that for all $n \geq N$ and $\rho = \rho_N$ the following inequality is true

$$\left| f(x) - \left[ \frac{a_0}{2} + \sum_{k=1}^{n} \rho_N^{k^\nu} \left( a_k \cos(kx) + b_k \sin(kx) \right) \right] \right| \leq \varepsilon . \quad (7)$$

So, approximate numerical value of the generalized sum (6) is the value of variants of the corresponding sequence (dependent on two parameters) at sufficiently large value of number $N$ and very close to one parameter $\rho = \rho(N)$. In this regard, there appears the challenge of choosing the numerical values of the parameters of the variant which provide the least error of the corresponding approximate value of the series sum. No less important is the problem of the choice of indicator $\nu \geq 1$, defining the general factors of power summation method $\left\{ \varphi_n = \rho^{n^\nu} \right\}$. The choice of the values of these parameters depends on the order of ascending of coefficients of the series (3) and the points in which the generalized sum of series is determined.

According to Abel [2] we will transform the series in the formula (4) under the condition $0 < \rho < 1$ and taking into consideration of the formulas

$$S_n^c(x) = \sum_{k=0}^{n} \cos(kx) = \frac{\sin\left[ (n+1)x/2 \right]}{\sin(x/2)} \cos\frac{nx}{2},$$

$$S_n^s(x) = \sum_{k=0}^{n} \sin(kx) = \frac{\sin\left[ (n+1)x/2 \right]}{\sin(x/2)} \sin\frac{nx}{2}.$$

We will get

$$f_\sigma(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\rho^{n^\nu}\left(a_n\cos x+b_n\sin x\right)=-\frac{a_0}{2}+\sum_{n=0}^{\infty}\rho^{n^\nu}\left(a_n\cos x+b_n\sin x\right)=$$
$$=-\frac{a_0}{2}+\sum_{n=0}^{\infty}\left\{\left[a_n\rho^{n^\nu}-a_{n+1}\rho^{(n+1)^\nu}\right]S_n^c(x)+\left[b_n\rho^{n^\nu}-b_{n+1}\rho^{(n+1)^\nu}\right]S_n^s(x)\right\}. \qquad (8)$$

Let us explore the convergence of the series (8) if for large values of the indices of summing its coefficients have the estimates

$$a_n=\mathrm{O}\left(n^m r^n\right),\quad b_n=\mathrm{O}\left(n^m r^n\right), \qquad (9)$$

where $r\ge 1$; $m\ge 0$.

The estimates $a_n=\mathrm{O}\left(n^m\right)$, $b_n=\mathrm{O}\left(n^m\right)$ have the coefficients of the series which are derivatives of any orders from trigonometric Fourier series, the estimates (9) correspond to derivatives of the power series.

Let us estimate residual of series (8), considering its two parts

$$\Lambda_N^s=\sum_{n=N}^{\infty}\left[n^m r^n\rho^{n^\nu}-(n+1)^m r^{n+1}\rho^{(n+1)^\nu}\right]S_n^s(x),$$
$$\Lambda_N^c=\sum_{n=N}^{\infty}\left[n^m r^n\rho^{n^\nu}-(n+1)^m r^{n+1}\rho^{(n+1)^\nu}\right]S_n^c(x). \qquad (10)$$

The sequence $\left\{\varphi_n(\rho,r,m)=n^m r^n\rho^{n^\nu}\right\}_{n=0}^{\infty}$ is not monotonically decreasing. Number of the largest member of the sequence is determined by the necessary condition of extremum of the function that sets the general member of this sequence

$$n\ln r+\nu n^\nu\ln\rho+m=0. \qquad (11)$$

Let us write the numbers of the largest members of the sequence separately for cases $r=1$ and $m=0$

$$N_0=N_m(\rho)=\left[\left(\ln\frac{1}{\rho^{\nu/m}}\right)^{-1/\nu}\right],\quad N_0=N_r(\rho)=\left[\left(\ln\frac{1}{\rho^{\nu/\ln r}}\right)^{-1/(\nu-1)}\right].$$

*Remark 1.* In the case of estimates of series coefficients $a_n=\mathrm{O}\left(n^m\right)$, $b_n=\mathrm{O}\left(n^m\right)$ sequence $\left\{\varphi_n(\rho,1,m)\right\}$, starting from the number $N_0=N_m(\rho)$, monotonically decreases and tends to zero for all $\nu>0$, in the case of estimates $a_n=\mathrm{O}\left(r^n\right)$, $b_n=\mathrm{O}\left(r^n\right)$, starting from the number $N_0=N_r(\rho)$, monotonically decreases and tends to zero only for $\nu>1$.

From the equality (11) we will find inverse relationship of the parameter $\rho$ from the number of the biggest variant of the sequence $N_0$,

$$\rho(N_0) = e^{-\frac{N_0 \ln r + m}{\nu N_0^{\nu}}} = r^{-\frac{1}{\nu N_0^{\nu-1}}} e^{-\frac{m}{\nu N_0^{\nu}}} . \qquad (12)$$

The parameters $N$ and $\rho$ or according to (12) $N$ and $N_0$, $N > N_0$, are chosen in the expression (10) independently. By entering variable coefficient $\alpha \to \infty$ using formula $N = \alpha N_0$ and (20), we will find the estimate for the remains

$$\left|\Lambda_N^i\right| = \left|\sum_{n=N}^{\infty}\left[n^m r^n \rho^{n^{\nu}} - (n+1)^m r^{n+1} \rho^{(n+1)^{\nu}}\right] S_i(x)\right| \le$$

$$\le \sum_{n=N}^{\infty}\left[n^m r^n \rho^{n^{\nu}} - (n+1)^m r^{n+1} \rho^{(n+1)^{\nu}}\right] \left|S_i(x)\right| \le$$

$$\le A(x) \sum_{n=N}^{\infty}\left[n^m r^n \rho^{n^{\nu}} - (n+1)^m r^{n+1} \rho^{(n+1)^{\nu}}\right] S_i = A(x) N^m r^N \rho^{N^{\nu}} =$$

$$= A(x)(\alpha N_0)^m r^{\alpha N_0} r^{-\frac{\alpha^{\nu} N_0}{\nu}} e^{-\frac{m\alpha^{\nu}}{\nu}} = A(x) N_0^m \alpha^m e^{-\frac{m\alpha^{\nu}}{\nu}} r^{-\frac{N_0}{\nu}\left(\alpha^{\nu} - \nu\alpha\right)} . \qquad (13)$$

*Remark 2.* If for the coefficients of the series (3) the estimates $a_n = O\left(n^m\right)$, $b_n = O\left(n^m\right)$ are true, then remainder (13) tends to zero when $\alpha \to \infty$. If the estimates (9) are true, then remainder (13) tends to zero when $\alpha \to \infty$ under the condition $\alpha^{\nu} - \nu\alpha > 0$ or $\nu > 1$.

Based on the estimate of the remainder (13) we have necessary condition for existing of generalized sum of the series.

*Theorem.* Let generalized sum of series exist (3), ie the limit equality (5) be true. Then implementation of the inequality (7) is provided by the choice of values $N_0$, $N > \alpha N_0$, at the same time, if for the coefficients of series (3) the estimates are true $a_n = O\left(n^m\right)$, $b_n = O\left(n^m\right)$, then we put $\nu > 0$, $\alpha > 1$ and parameter $\rho$ is calculated by the formula (12) when $r = 1$; if the estimates (9) are true, then we put $\nu > 1$, $\alpha > \nu^{\nu-1}$ and parameter $\rho$ is calculated by the formula (12).

*Remark 3.* The maximum value of sequence variants $\left\{\varphi_n(\rho, r, m)\right\}_{n=0}^{\infty}$ significantly increases with parameter $\rho$ approaching one. Therefore we can not reach sufficient precision of calculations of generalized sum with parameter $\rho$ approaching one.

*Example 1.* Let us consider trigonometric Fourier series of the function $f(x) = x/2$,

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) , \quad |x| < \pi .$$

The derivative of this function of $k$-th $(k \geq 2)$ order is equal to zero. We will find approximate value of generalized sum of derivative of the sixth order from the considered series

$$f^{(6)}(x) \approx f^{(6)}_{\rho,N}(x) = \sum_{n=1}^{N} (-1)^{n+1} \rho^{n^2} n^5 \sin(nx) .$$

If $\rho = 0{,}999$, $N = 200$, then $f^{(6)}(0{,}5) \approx 1 \cdot 10^{-10}$, $f^{(6)}(3) \approx 1 \cdot 10^{-10}$.

*Example 2*. Trigonometric series of the function

$$f(r,\varphi) = \operatorname{Im}\frac{1}{1-z} = \frac{r\sin\varphi}{1+r^2-2r\cos\varphi} = \sum_{n=1}^{\infty} r^n \sin(n\varphi) ,$$

where $z = re^{i\phi}$, converges (in the classic sense of the sum) in the circle $r < 1$, $0 \leq \phi < 2\pi$ and diverges in the point $(r = 1{,}1;\ \varphi = \pi/4)$. The function $f(r,\varphi) = \operatorname{Im}\left[1/(1-z)\right]$ in this point takes the value $f(1{,}1;\ \pi/4) = 1{,}188$. We will find approximate value of the generalized sum of series in this point using formula $f(r,\varphi) \approx f_{\rho,N}(r,\varphi) =$ $= \sum_{n=1}^{N} r^n \rho^{n^2} \sin(n\varphi)$. If $\rho = 0{,}9999$ and $N = 1500$, then $f_{\rho,N}(1{,}1;\ \pi/4) = 1{,}189$.

**Conclusions.** Since the coefficients of the divergent (in the classic sense) series increase with increasing values of their serial numbers, it is important to find effective methods of finding the generalized sum of this series. Adding smaller units of multipliers to the expression of coefficients (by which methods of power factors are realized) and implementating sufficient conditions for the existence of the generalized sums of series provide construction algorithm for finding numerical value of this sum. Given examples confirm the possibility of calculating the generalized sum with the precision large enough. However, all numerical algorithms are not stable enough since the computation of generalized sum of series is accompanied with performing arithmetic operations with big numbers (coefficients values).

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## Підсумовування тригонометричних рядів методами степеневих множників

Галина Івасик

*Досліджено підсумовування розбіжних тригонометричних рядів методами степеневих множників. Показано, що методом Пуассона-Абеля можна підсумовати тільки ряди, порядок зростання членів яких пропорційний степеневій залежності від номера підсумовування. Метод Вейєрштрасса-Гаусса та інші методи степеневих множників можна застосувати також до рядів, члени яких зростають пропорційно показниковій залежності від номера підсумовування.*

## Суммирование тригонометрических рядов методами степенных множителей

Галина Ивасык

*Исследовано суммирование расходящихся тригонометрических рядов методами степенных множителей. Показано, что методом Пуассона-Абеля могут быть просуммированы только ряды, порядок возрастания членов которых пропорциональный степенной зависимости от номера суммирования. Метод Вейерштрасса-Гаусса и другие методы степенных множителей могут использоваться также для рядов, члены которых возрастают пропорционально показательной зависимости от номера суммируемости.*

**Представлено професором М. Сухорольським**

Отримано 08.09.15